CEN352
Digital Signal Processing

By

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Lecture No. 4

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Sampling Theorem

Time Domain   Frequency Domain

We first state the sampling theorem in time-domain.

"The sampling theorem gaurentees that an analog signal can be in theory perfectly recovered as long as the sampling rate is at least twice as large as the highest-frequency component of the analog signal to be sampled".

$$f_s \geq 2f_{max}$$

f_s: Sampling rate/frequency

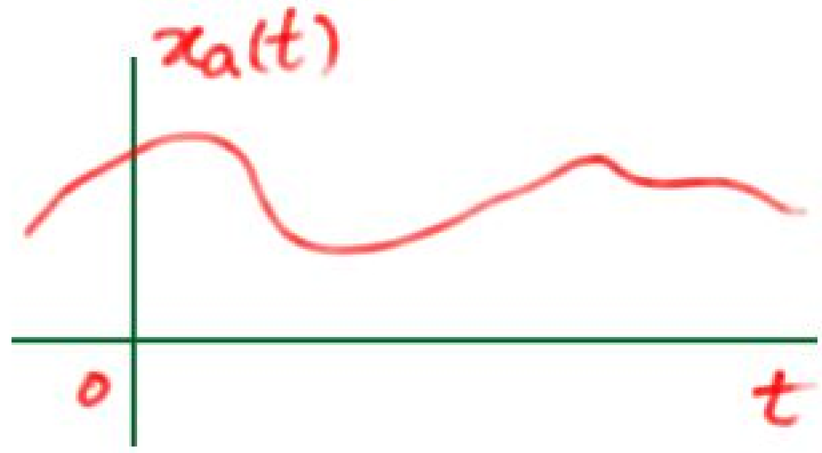
f_{max}: Maximum-frequency component of the analog signal.

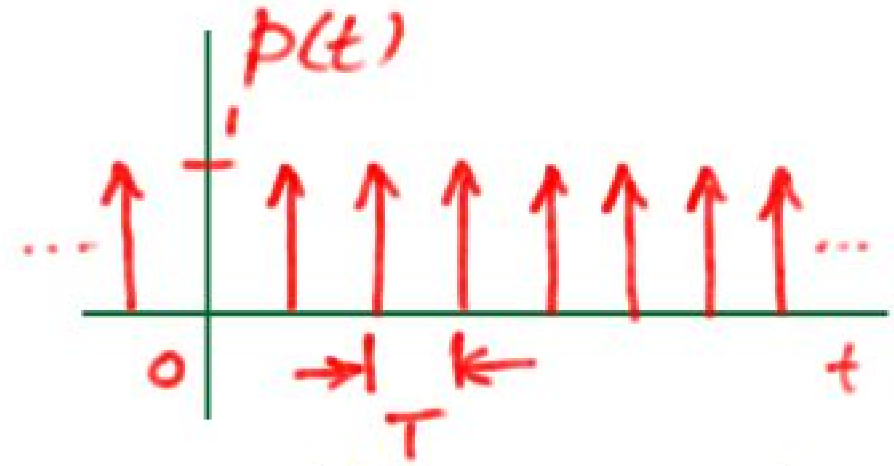
For example, to sample a speech signal containing frequencies upto 4kHz, the minimum sampling rate is chosen to be at least 8kHz ($\Rightarrow f_s \geq 2 \times 4$kHz).

If a sampling rate of $f_s < 2f_{max}$ is chosen, this would lead to aliasing. As a result, we will not be able to reconstruct the original signal from the sampled signal.

Development of the Sampling Theorem in the Frequency Domain

Consider a continuous-time/analog signal $x_a(t)$ and a pulse train (periodic with period T) $p(t)$ as shown below:





Then the sampled/discrete-time signal is given by

$$x_s(t) = x_a(t)\,p(t) \qquad -(1)$$

Taking Fourier transform of both sides:

$$\mathcal{F}\{x_s(t)\} = \mathcal{F}\{x_a(t)\,p(t)\} \qquad -(2)$$

where

$$\mathcal{F}\{x(t)\} = X(j\omega) = \int_{-\infty}^{+\infty} x(t)\,e^{-j\omega t}\,dt$$

According to the multiplication property of the Fourier transform:

$$\mathcal{F}\{x(t)\,y(t)\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)\,Y(j(\omega-\theta))\,d\theta$$

Fourier transform of the product of two signals in time domain

Convolution of the Fourier transforms of the two signals in the frequency domain

$$= \frac{1}{2\pi} X(j\omega) * Y(j\omega)$$

In our case

$$\mathcal{F}\{p(t)\cdot x_a(t)\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} P(j\theta)\, X_a(j(\omega-\theta))\,d\theta \qquad \text{— (3)}$$

Now lets work out $P(j\omega) = ?$ The pulse-train $p(t)$ can be written as:

$$p(t) = \sum_{n=-\infty}^{+\infty} \delta(t-nT) \qquad \text{— (4)}$$

For a periodic signal $x(t)$ (which could be expressed as a Fourier series), the Fourier transform is given by

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \xleftrightarrow{\mathcal{F}} X(j\omega) = \sum_{k=-\infty}^{+\infty} a_k 2\pi\delta(\omega - k\omega_0)$$

Therefore, in case of the pulse train (Eq. 4)

$$a_n = \frac{1}{T}\int_T p(t) e^{-jn\omega_s t} dt = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jn\omega_s t} dt = \frac{1}{T}$$

$$\Rightarrow P(j\omega) = \sum_{n=-\infty}^{+\infty} \frac{1}{T}\cdot 2\pi\,\delta(\omega - n\omega_s)$$

or

$$P(j\theta) = \frac{2\pi}{T}\sum_{n=-\infty}^{+\infty} \delta(\theta - n\omega_s) \qquad \text{— (5)}$$

Substituting in equation (2)

$$X_s(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{2\pi}{T}\sum_{n=-\infty}^{+\infty} \delta(\theta - n\omega_s) X_a(j(\omega-\theta))\,d\theta$$

$$= \frac{1}{T}\sum_{n=-\infty}^{+\infty}\int_{-\infty}^{+\infty} \delta(\theta - n\omega_s)\, X_a(j\omega - j\theta)\,d\theta$$

$$= \frac{1}{T}\sum_{n=-\infty}^{+\infty} X_a(j\omega - jn\omega_s) \qquad \text{— (6)}$$

$$X_s(j\omega) = \frac{1}{T}\sum_{n=-\infty}^{+\infty} X_a(j\omega - nj\omega_s)$$

or

$$\boxed{X_s(f) = \frac{1}{T}\sum_{n=-\infty}^{+\infty} X_a(f - nf_s)} \quad \text{—(7)}$$

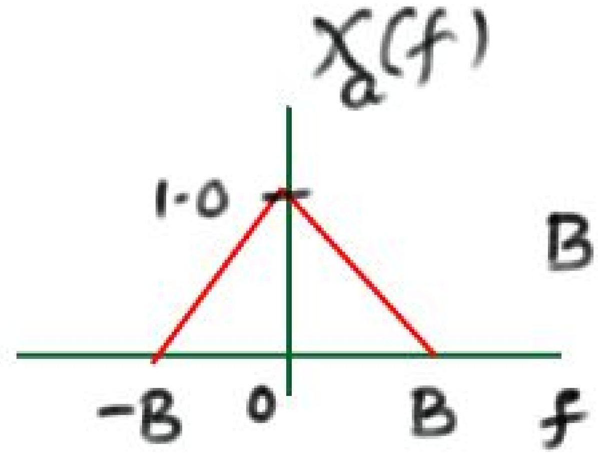
Expanding it, we can write:

$$X_s(f) = \cdots + \frac{1}{T}X_a(f+f_s) + \frac{1}{T}X_a(f) + \frac{1}{T}X_a(f-f_s) + \cdots \quad \text{—(8)}$$

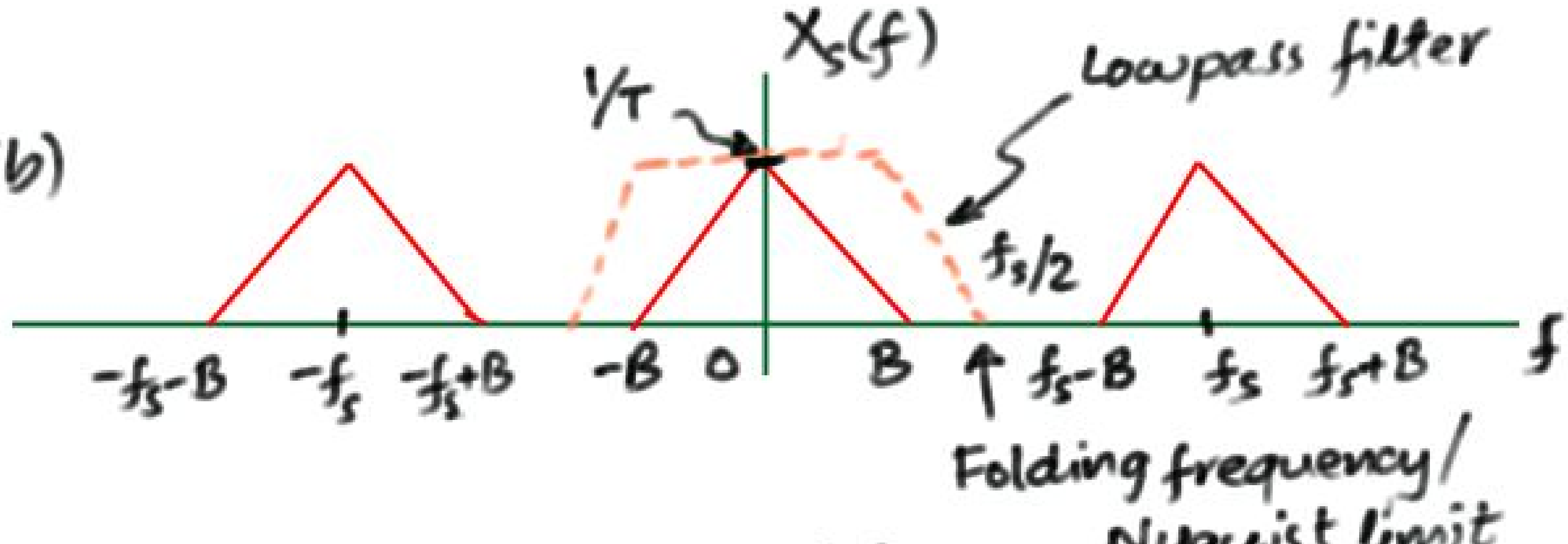
Equation (8) indicates that the sampled signal spectrum ($X_s(f)$) is the sum of the scaled original spectrum ($\frac{1}{T}X_a(f)$) and copies of its shifted versions ($\frac{1}{T}X_a(f-f_s)$, $\frac{1}{T}X_a(f+f_s)$, ...) called 'replicas' or 'images'.

Suppose that the spectrum of the original signal is as shown in Fig(a) on next page. The sampled spectrum according to the equation 8 is plotted in Fig(b) (next page), this is one of the three possible situations (whereby the replicas are well separated). The other two situations are plotted in Fig(c) and Fig(d), where the base-band spectrum and its replicas are just connected and overlap, respectively.

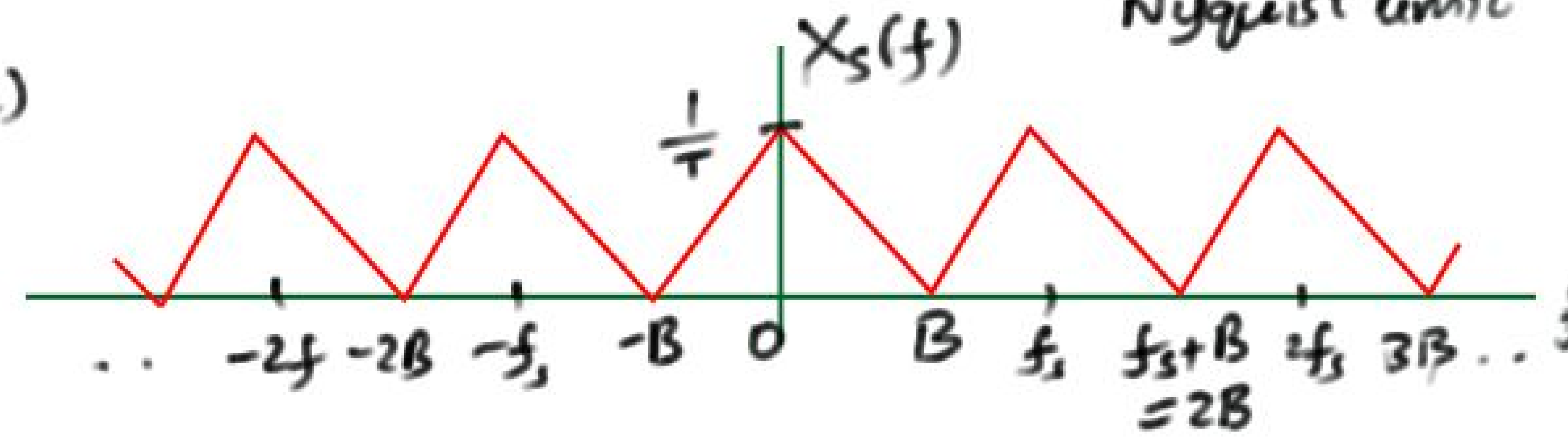
Fig(a)



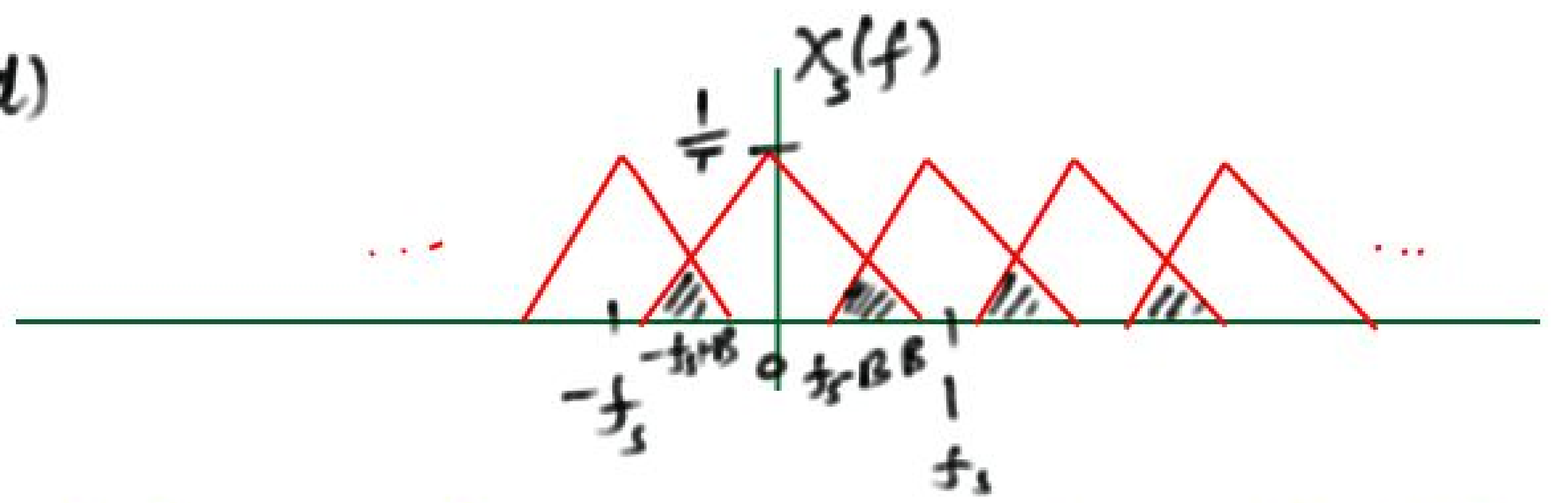
Fig(b)



Fig(c)



Fig(d)



If applying a lowpass reconstruction filter to obtain exact reconstruction of the original signal spectrum, the follow condition must be satisfied

$$f_s - f_{max} \geq f_{max}$$

$$\Rightarrow \boxed{f_s \geq 2 f_{max}}$$

or $$\omega_s \geq 2\omega_{max}$$

The Shannon Sampling Theorem

This fundamental conclusion is well known as the "Shannon sampling theorem"

For a uniformly sampled DSP system, an analog signal can be perfectly recovered as long as the sampling rate is at least twice as large as the highest-frequency component of the analog signal to be sampled.

Two key points:

(1) Sampling theorem establishes a minimum sampling rate for a given band-limited analog signal with highest frequency component f_{max}:

$$f_s \geq 2 f_{max}$$

(2) Half of the sampling frequency $f_s/2$ is usually called the Nyquist frequency (Nyquist limit) or folding frequency.