CEN352
Digital Signal Processing

By

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Lecture No. 10

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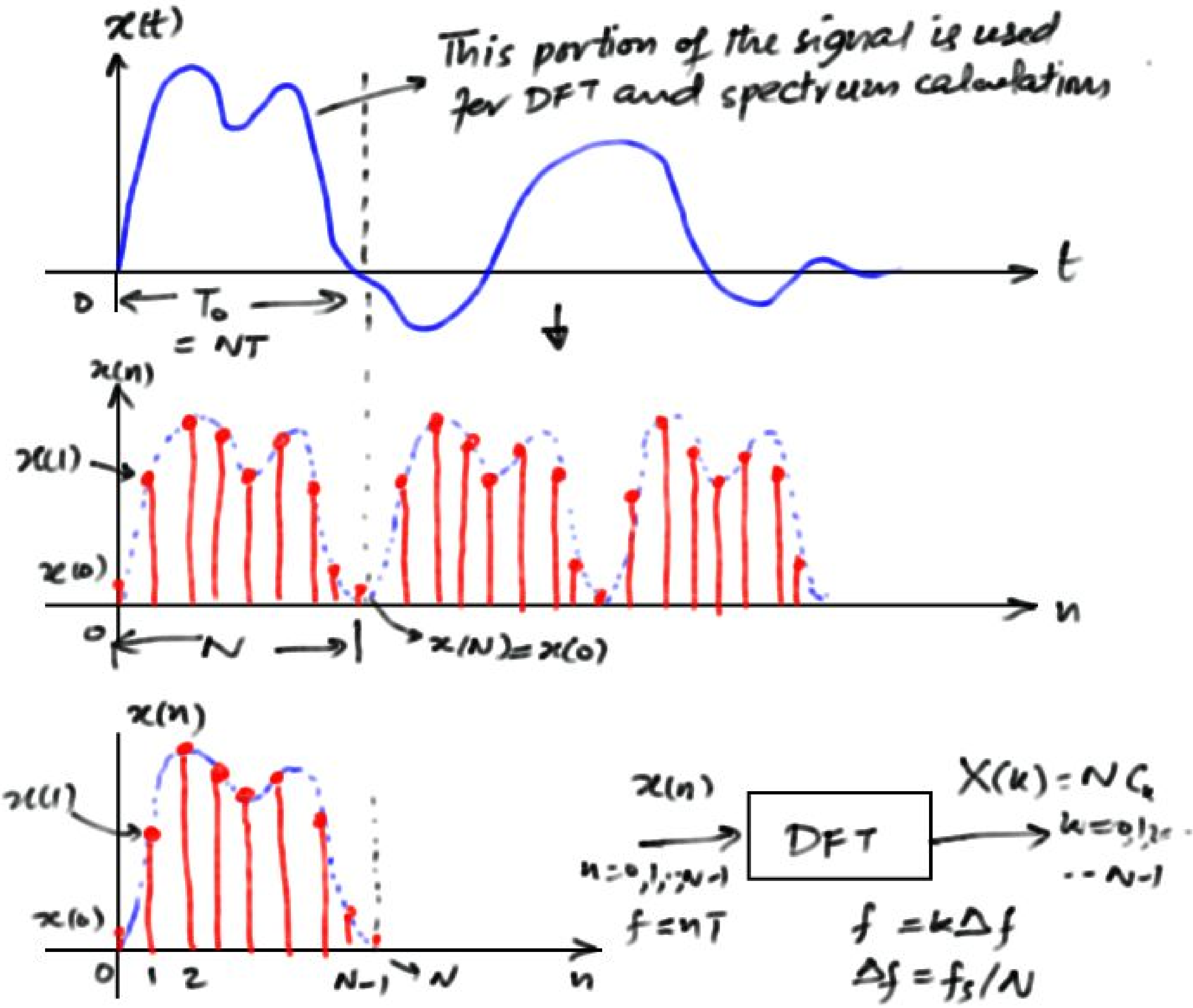
Discrete Fourier Transform Formulas



We first assume that data samples are acquired by digitizing a continuous signal for a duration of T seconds. Next, a periodic signal $x(n)$ is obtained by copying the acquired N data samples. Note that, we assume continuity between the N data sample frames. (This is not true in practice. We will tackle this problem later).

Treating now the discrete-time signal as a periodic signal with N samples within a period, the Fourier series coefficients can be written as:

$$X(k) = N c_k = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi k n}{N}}, \quad k=0,1,2,\cdots,N-1$$

where X(k) constitutes the DFT coefficients. It should be noticed that the factor of N is a constant and it does not affect the relative magnitudes of the DFT coefficients.

DFT Definition:

Given a sequence $x(n)$, $0 \leq n \leq N-1$, its DFT is defined as:

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} = \sum_{n=0}^{N-1} x(n) W_N^{kn} \quad -(4.7)$$

$$\text{for } k=0,1,2,\cdots,N-1$$

Equation (4.7) can be expanded as:

$$X(k) = x(0) W_N^{k0} + x(1) W_N^{k1} + x(2) W_N^{k2} + \cdots + x(N-1) W_N^{k(N-1)} \quad -(4.8)$$

$$\text{for } k=0,1,2,\cdots,N-1$$

where the factor W_N (called the twiddle factor) is defined as

$$W_N = e^{-j2\pi/N} = \cos\left(\frac{2\pi}{N}\right) + j\sin\left(\frac{2\pi}{N}\right) \quad -(4.9)$$

The inverse DFT is given by

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{-j2\pi kn/N} = \frac{1}{N}\sum_{k=0}^{N-1} X(k) W_N^{-kn} \quad -(4\cdot10)$$

for $n = 0, 1, \cdots, N-1$

It can be expanded as:

$$x(n) = \frac{1}{N}\Big(X(0) W_N^{-0n} + X(1) W_N^{-1n} + X(2) W_N^{-2n} + \cdots + X(N-1) W_N^{-(N-1)n} \quad -(4\text{-}11)$$

for $n = 0, 1, 2, \cdots, N-1$

We use MATLAB® functions fft() and ifft() to compute the DFT coefficients and the inverse DFT.

Example No. 4.2

Given a sequence $x(n)$ for $0 \le n \le 3$, where

$$x(0) = 1, \ x(1) = 2, \ x(2) = 3, \ x(3) = 4$$

a. Evaluate its DFT, $X(k)$.

Solution

a. Here, $N = 4$ and

$$W_N = W_4 = e^{-j2\pi/N} = e^{-j\pi/2}$$

Now, the DFT coefficients are given by

$$X(k) = \sum_{n=0}^{3} x(n) W_4^{kn}$$

$$\Rightarrow X(k) = x(0) W_4^{0} + x(1) W_4^{k} + x(2) W_4^{2k} + x(3) W_4^{3k}$$

Thus for $k=0$,

$$X(0) = x(0)e^{-0} + x(1)e^{-j0} + x(2)e^{-j0} + x(3)e^{-j0}$$

$$= x(0) + x(1) + x(2) + x(3)$$

$$= 1 + 2 + 3 + 4 = 10$$

For $k=1$,

$$X(1) = x(0)e^{-j\frac{\pi}{2}(0)} + x(1)e^{-j\frac{\pi}{2}(1)} + x(2)e^{-j\frac{\pi}{2}(2)} + x(3)e^{-j\frac{\pi}{2}(3)}$$

$$= x(0) + x(1)\left(\cos\frac{\pi}{2} - j\sin\frac{\pi}{2}\right)$$
$$+ x(2)\left(\cos\pi - j\sin\pi\right) + x(3)\left(\cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}\right)$$

$$= 1 + 2(-j) + 3(-1) + 4(j)$$

$$= -2 + 2j$$

For $k=2$,

$$X(2) = \sum_{n=0}^{3} x(n)e^{-j\pi n} = x(0)e^{0} + x(1)e^{-j\pi} + x(2)e^{-j2\pi} + x(3)e^{-j3\pi}$$

$$= 1(1) + 2(\cos\pi - j\sin\pi) + 3(\cos 2\pi - j\sin 2\pi)$$
$$+ 4(\cos 3\pi - j\sin 3\pi)$$

$$= 1 + 2(-1) + 3(1) + 4(-1)$$

$$= 1 - 2 + 3 - 4$$

$$= -2$$

And for 3

$$X(3) = \sum_{n=0}^{3} x(n)e^{-j\frac{3\pi}{2}n} = x(0)e^{0} + x(1)e^{-j\frac{3\pi}{2}} + x(2)e^{-j3\pi} + x(3)e^{-j\frac{9\pi}{2}}$$

$$= 1 + 2\left(\cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}\right) + 3(\cos 3\pi - j\sin 3\pi)$$
$$+ 4\left(\cos\frac{9\pi}{2} + j\sin\frac{9\pi}{2}\right)$$

$$= 1 + 2(j) + 3(-1) + 4(-j)$$

$$= -2 - 2j$$

This can be easily verified via MATLAB as:

```
>> X = fft([1 2 3 4])

X = 10.0000  -2.0000+2.0000i  -2.0000  -2.000-2.000i
```

Example 4.3

Using the DCT coefficients $X(k)$ for $0 \le k \le 3$ computed in Example 4.2.

a- Evaluate its inverse DFT to determine the time domain sequence $x(n)$.

Solution

In this case

$$N = 4, \quad W_4 = e^{-j2\pi/N} \Rightarrow W_4^{-1} = e^{-j\pi/2}$$

To obtain the inverse DFT we use the formula

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn}$$

which is simplified to:

$$x(n) = \frac{1}{4} \sum_{k=0}^{3} X(k) W_4^{-kn}$$

Thus, for $n = 0$,

$$x(0) = \frac{1}{4} \sum_{k=0}^{3} X(k) W_4^{-k(0)} = \frac{1}{4} \sum_{k=0}^{3} X(k)$$

$$= \frac{1}{4}(10 - 2 + 2j - 2 - 2 - 2j) = \frac{1}{4}(10-6) = \frac{4}{4}$$

$$= 1$$

For $n = 1$

$$x(1) = \frac{1}{4} \sum_{k=0}^{3} X(k) W_4^{-k}$$

$$\Rightarrow x(1) = \frac{1}{4}\left(X(0)e^{j0} + X(1)e^{j\frac{\pi}{2}} + X(2)e^{j\pi} + X(3)e^{j\frac{3\pi}{2}}\right)$$

$$= \frac{1}{4}\left(10(1) + (-2+j2)(j) + (-2)(-1) + (-2-j2)(-j)\right)$$

$$= \frac{1}{4}\left(10 - 2j - 2 + 2 + 2j - 2\right) = \frac{8}{4} = 2$$

For $n = 3$

$$x(3) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{jk\pi} = \frac{1}{4}\left(X(0)e^{0} + X(1)e^{j\pi} + X(2)e^{j2\pi} + X(3)e^{j3\pi}\right)$$

$$= \frac{1}{4}\left(10(1) + (-2+j2)(-1) + (-2)(1) + (-2-j2)(-1)\right)$$

$$= \frac{1}{4}\left(10 + 2 - j2 - 2 + 2 + j2\right) = \frac{12}{4} = 3$$

And for $n = 4$

$$x(4) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{jk\frac{3\pi}{2}}$$

$$= \frac{1}{4}\left(X(0)e^{0} + X(1)e^{j\frac{3\pi}{2}} + X(2)e^{j3\pi} + X(3)e^{j\frac{9\pi}{2}}\right)$$

$$= \frac{1}{4}\left(10(1) + (-2+j2)(-j) + (-2)(-1) + (-2-j2)(j)\right)$$

$$= \frac{1}{4}\left(10 + 2j + 2 + 2 - 2j + 2\right) = \frac{16}{4} = 4$$

We can verify this using the MATLAB function ifft() as:

```
>> x = ifft([10  -2+2j  -2  -2-2j])

x = 1   2   3   4
```

Example 4.4

Given a digital signal $x(n)$, for $0 \le n \le 3$ as:

$$x(0)=1,\ x(1)=2,\ x(2)=3 \text{ and } x(3)=4$$

and the four DFT coefficients $X(k)$, for $0 \le k \le 3$ in the frequency domain as:

$$X(0)=10,\ X(1)=-2+j2,\ X(2)=-2,\ X(3)=-2-j2$$

If the sampling rate is 10 Hz,

a) Determine the sampling period, time-index and the sampling time instant for a digital sample $x(3)$ in time domain.

b) Determine the frequency resolution, frequency bin number, and mapped frequency for each of the DFT coefficients $X(1)$ and $X(3)$ in the frequency domain.

Solution

(a) In the time domain, we have the sampling period calculated as:

$$T = 1/f_s = \frac{1}{10} = 0.1 \text{ second}$$

For the sample $x(3)$, the time index is 3, therefore the sampling time instant for this sample will be

$$t = nT = (3)(0.1)$$

$$= 0.3 \text{ second}.$$

(b) In the frequency domain, since

$N = 4$,

therefore, the frequency resolution is:

$$\Delta f = f_s/4 = 10/4 = 2.5 \text{ Hz}.$$

The frequency bin number for $X(1)$ is $k=1$; its corresponding frequency is

$$f = \frac{k f_s}{N} = \frac{(1)10}{4} = 2.5 \text{ Hz}.$$

Similarly, for $X(3)$, the frequency bin number is 3; its corresponding frequency is:

$$f = \frac{k f_s}{N} = \frac{(3)(10)}{4} = 7.5 \text{ Hz}.$$

Amplitude Spectrum and Power Spectrum

One of the applications of DFT is the conversion of a finite-element digital signal $x(n)$ into the spectrum in frequency domain.

- First, the digital sequence $x(n)$ is obtained by sampling the analog signal $x(t)$ and truncating the sampled signal with a data window of length $T_0 = NT$, where T is the sampling period and N is the number of data points. The time for the data window is

$$T_0 = NT \quad \text{— (4.16)}$$

As a result we get the truncated sequence $x(n)$, with a range $n = 0, 1, 2, \cdots, N-1$, as:

$$x(0), x(1), x(2), \cdots, x(N-1) \quad \text{— (4.17)}$$

- Next we apply the DFT to obtain

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \text{ for } k = 0, 1, 2 \cdots, N-1 \quad \text{— (4.18)}$$

where $W_N = e^{-j2\pi/N}$

Each calculated DFT coefficient $X(k)$ is a complex number. It is not convenient to plot these coefficients versus the frequency index.

⟶ Magnitude & Phase are calculated.

We define the **Amplitude Spectrum** as

$$A_k = \frac{1}{N}\left|X(k)\right| = \frac{1}{N}\sqrt{\left(Real\left[X(n)\right]\right)^2 + \left(Imag\left[X(n)\right]\right)^2}$$

$$k = 0, 1, 2, \cdots, N-1 \qquad \text{—}(4.19)$$

- This amplitude spectrum is sometimes converted to a **One-sided Amplitude Spectrum** by doubling the amplitudes in Equation (4.19) keeping the original DC term at $k=0$. Thus

$$\bar{A}_k = \begin{cases} \frac{1}{N}\left|X(0)\right|, & k=0 \\ \frac{2}{N}\left|X(k)\right|, & k=1,2,\cdots,N/2 \end{cases} \qquad \text{—}(4.20)$$

We can map the frequency bin k to its corresponding frequency as

$$f = \frac{kf_s}{N} \qquad \text{—}(4.21)$$

- Correspondingly, the phase spectrum is given by

$$\phi_k = \tan^{-1}\left(\frac{Imag\left[X(k)\right]}{Real\left[X(k)\right]}\right), \quad k=0,1,\cdots,N-1 \qquad \text{—}(4.22)$$

- Besides the amplitude spectrum, the power spectrum is also used. The DFT **Power Spectrum** is defined as:

$$P_k = \frac{1}{N^2}\left|X(k)\right|^2 = \frac{1}{N^2}\left\{\left(Real\left[X(k)\right]\right)^2 + \left(Imag\left[X(k)\right]\right)^2\right\}$$

$$k = 0, 1, 2, \cdots, N-1 \qquad \text{—}(4.23)$$

Similarly, One-Sided Power Spectrum is given by

$$\overline{P_k} = \begin{cases} \frac{1}{N^2}|X(0)|^2, & k=0 \\ \frac{2}{N^2}|X(k)|^2, & k=1,2,\cdots,\frac{N}{2} \end{cases} \quad \text{—(4.24)}$$

and

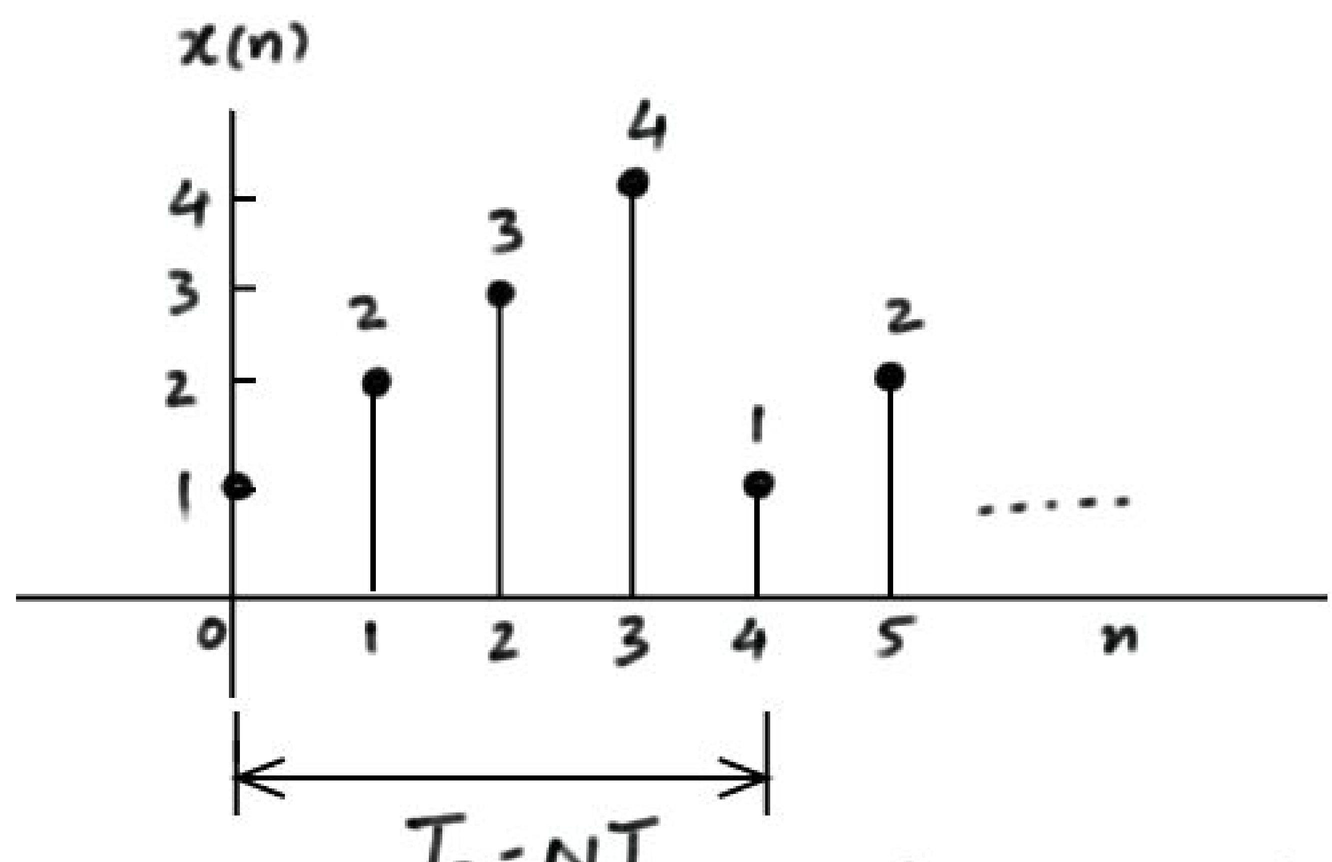
$$f = \frac{kf_s}{N} \quad \text{—(4.25)}$$

- It can be noted that the frequency resolution (which denotes the frequency spacing between DFT coefficients in the frequency domain) is defined as

$$\Delta f = \frac{f_s}{N} \text{ (Hz)} \quad \text{—(4.26)}$$

Example 4.5

Consider the sequence



a) Assuming that $f_s = 100$ Hz, compute the amplitude spectrum, phase spectrum and power spectrum.

Solution

We have N=4, and from Example 4.1, the DFT coefficients are given by

$$X(0)=10,\ X(1)=-2+j2,\ X(2)=-2,\ X(3)=-2-j2$$

The amplitude spectrum, phase spectrum and power density spectrum are given by

- For $k=0$, $f=kf_s/N=0$ Hz

$$A_0=\frac{1}{4}|X(0)|=\frac{10}{4}=2.5,\quad \varphi_0=\tan^{-1}\left(\frac{\text{Imag}[X(0)]}{\text{Real}[X(0)]}\right)=0^\circ$$

$$P_0=\frac{1}{4^2}|X(0)|^2=\frac{100}{16}=6.25$$

- For $k=1$, $f=\frac{kf_s}{N}=\frac{1\times 100}{4}=25$ Hz

$$A_1=\frac{1}{4}|X(1)|=\frac{\sqrt{4+4}}{4}=0.7071,\quad \varphi_1=\tan^{-1}\left(\frac{\text{Imag}[X(1)]}{\text{Real}[X(0)]}\right)=135^\circ$$

$$P_0=\frac{1}{4^2}|X(1)|^2=\frac{4+4}{16}=0.5$$

- For $k=2$, $f=\frac{kf_s}{N}=\frac{2\times 100}{4}=50$ Hz

$$A_2=\frac{1}{4}|X(2)|=\frac{\sqrt{(-2)^2+(0)^2}}{4}=\frac{2}{4}=0.5$$

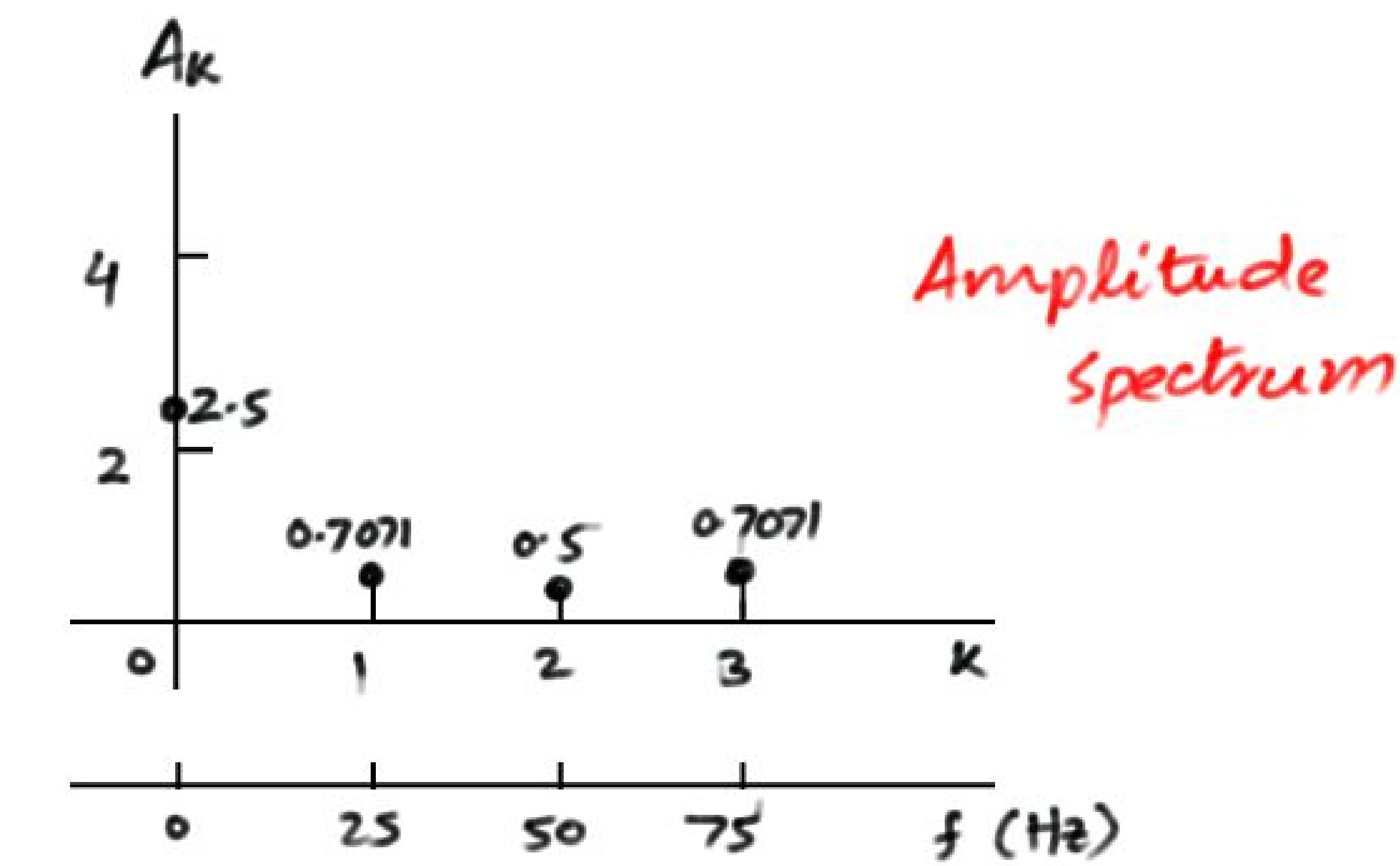
$$\varphi_2=\tan^{-1}\left(\frac{\text{Imag}[X(2)]}{\text{Real}[X(2)]}\right)=\tan^{-1}\left(\frac{0}{-2}\right)=180^\circ,\quad P_2=0.25$$

- For $k=3$, $f=\frac{kf_s}{N}=\frac{3\times 100}{4}=75$ Hz

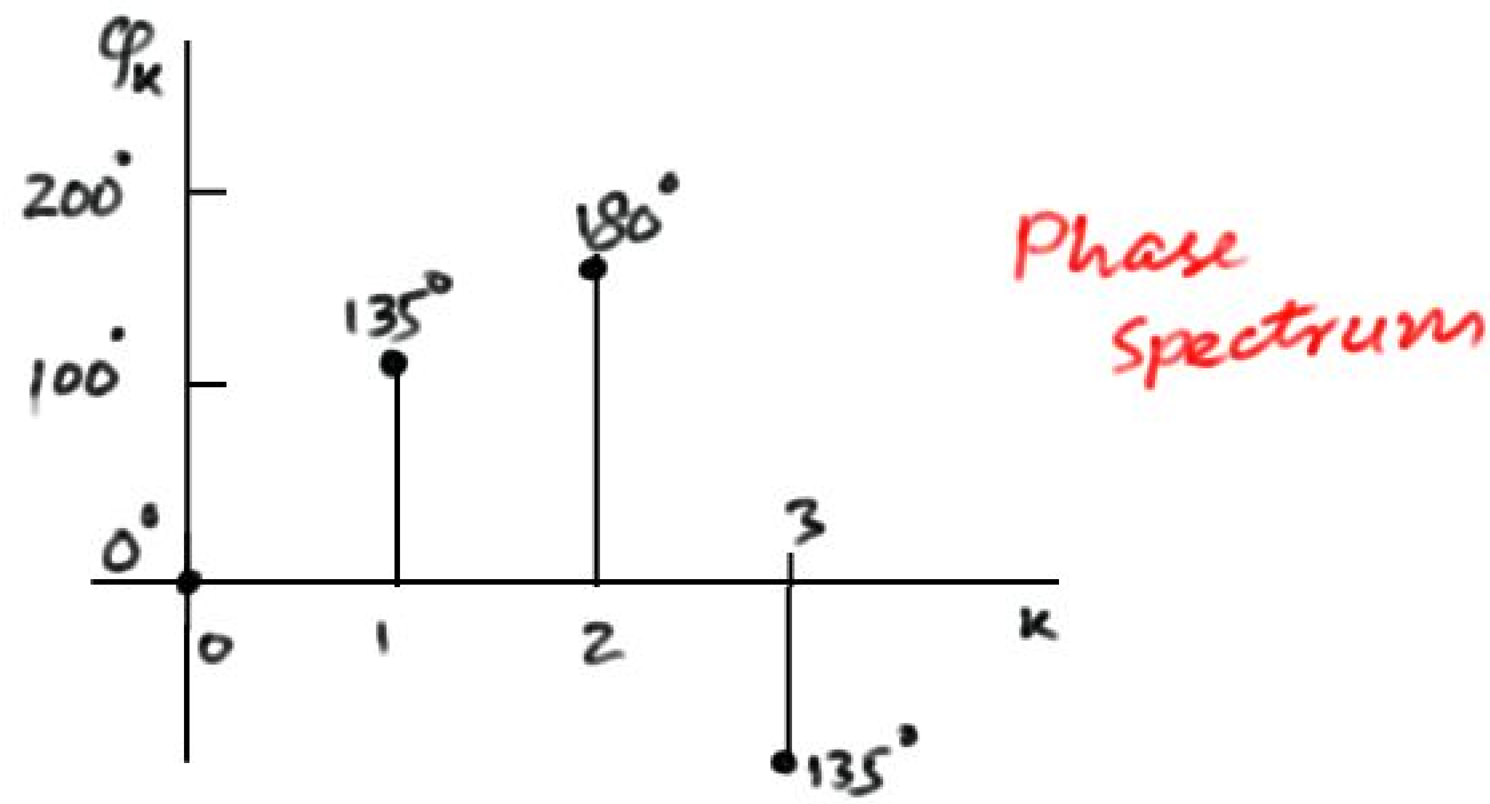
$$A_3=\frac{1}{4}|X(3)|=\frac{\sqrt{4+4}}{4}=0.5$$

$$\varphi_3=\tan^{-1}\left(\frac{\text{Imag}[X(3)]}{\text{Real}[X(3)]}\right)=\tan^{-1}\left(\frac{-2}{-2}\right)=-135^\circ$$

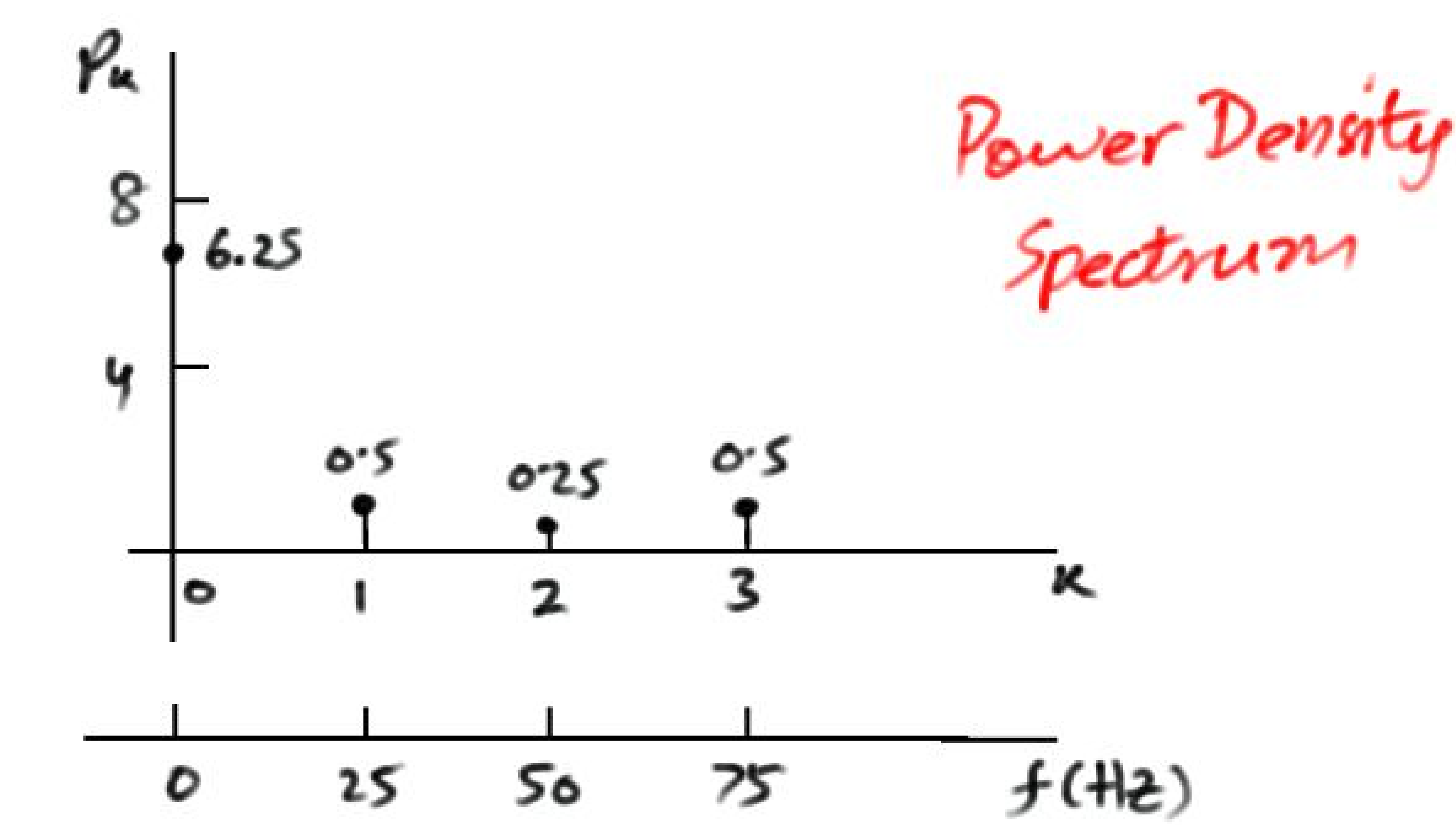
$$P_3=\frac{1}{4^2}|X(3)|^2=\frac{4+4}{16}=0.5$$



Amplitude spectrum
A_k
4
2
2.5
0.7071
0.5
0.7071
0
1
2
3
k
0
25
50
75
f (Hz)



Phase spectrum
φ_k
200°
100°
0°
135°
180°
3
135°
0
1
2
k



Power Density Spectrum
P_k
8
6.25
4
0.5
0.25
0.5
0
1
2
3
k
0
25
50
75
f (Hz)

We can very easily find the one-sided amplitude spectrum and one-sided power spectrum as:

$\bar{A}_0 = 2.5$, $\bar{A}_1 = 1.4141$, $\bar{A}_2 = 1$

and

$\bar{P}_0 = 6.25$, $\bar{P}_1 = 2$, $\bar{P}_2 = 1$ } upto $\frac{N}{2}$ terms

One-sided amplitude spectrum is plotted below:

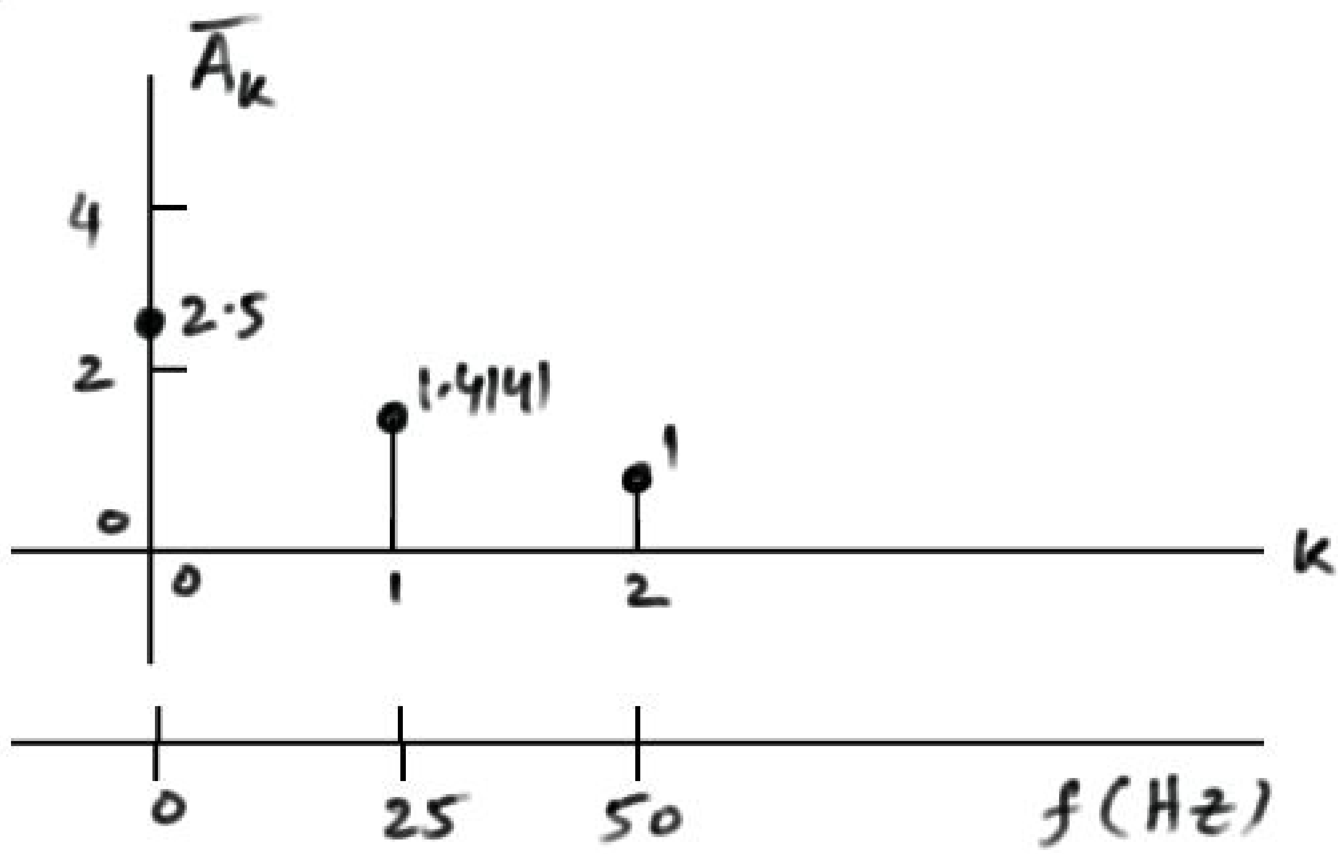


Figure 4.10 One-Sided Amplitude Spectrum