1. **The z-Plane Pole Zero Plot and Stability**

**The z-plane pole-zero plot is a very useful tool to analyze a digital system. This graphical technique allows us to gather information about various characteristics of a digital system, including its stability.**

**Outside of the unit Circle**

**Inside of the unit Circle**

$× - $ **pole**

$ - $ **pole**

$$Im\left(z\right)$$

$$Re\left(z\right)$$

**1**

**The pole-zero plot has the following features:**

1. **The horizontal axis is the real part of the variable z, and the vertical axis represents the imaginary part of the variable z.**
2. **The z-plane is divided into two parts by a unit circle.**
3. **Each pole is marked on the z-plane with the cross symbol**$ ×$ **and each zero is marked with the zero (small circle) symbol** O**.**

**Example 8**

**Given the following transfer function**

$$H\left(z\right)=\frac{\left(z^{-1}-0.5z^{-2}\right)}{\left(1+1.2z^{-1}+0.45z^{-2}\right)}$$

**Convert it into its pole-zero form and plot the poles and zeros.**

**Solution**

**We first multiply the numerator and denominator by**$ z^{2}$ **to obtain the transfer function whose numerator and the denominator polynomials have the positive powers of**$ z$**, as follows**

$$H\left(z\right)=\frac{\left(z^{-1}-0.5z^{-2}\right) z^{2}}{\left(1+1.2z^{-1}+0.45z^{-2}\right) z^{2}}=\frac{z-0.5}{z^{2}+1.2z+0.45}$$

**Putting the numerator polynomial equal to zero and then finding the roots, gives us the zeros of the transfer function,**

$$z-0.5=0$$

**Therefore, we get**$ z\_{1}=0.5$ **as the root.**

**Now, setting the denominator polynomial equal to zero and find the roots, gives us the poles of the transfer function,**

$$z^{2}+1.2z+0.45=0$$

$$z=\frac{-1.2\pm \sqrt{\left(1.2\right)^{2}-4\left(1\right)(0.45)}}{2(1)}=\frac{-1.2\pm \sqrt{1.44-1.8}}{2}=\frac{-1.2\pm \sqrt{-0.36}}{2}=\frac{-1.2\pm j0.6}{2}=-0.6\pm j0.2$$

**Therefore, the poles are**$ p\_{1}=-0.6+j0.2$ **and**$ p\_{2}=-0.6-j0.2$**. The transfer function can now be written in the pole-zero form as**

$$H\left(z\right)=\frac{\left(z-0.5\right)}{\left(z+0.6-j0.2\right)\left(z+0.6+j0.2\right)}$$

**0.3**

$$Im\left(z\right)$$

$$Re\left(z\right)$$

**1**

**- 0.3**

**0.5**

**- 0.6**

**The pole-zero plot for Example 1 is shown above in the figure.**

**Stability of the System and its Pole Zero Diagram**

1. **If the outermost pole(s) of the z-transfer function**$ H\left(z\right)$ **describing the DSP system is(are) inside the unit circle on the z-plane pole-zero plot, then the system is stable.**
2. **If the outermost pole(s) of the z-transfer function**$ H\left(z\right)$ **is (are) outside of the unit circle on the z-plane pole-zero plot, the system is unstable.**
3. **If the outermost pole(s) is(are) first-order pole(s) of the z-transfer function**$ H\left(z\right)$ **and on the unit circle on the z-plane pole-zero plot, then the system is marginally stable.**
4. **If the outermost pole(s) is(are) multiple-order pole(s) of the z-transfer function**$ H\left(z\right)$ **and on the unit circle on the z-plane pole-zero plot, then the system is unstable.**
5. **The zeros do not affect the system stability.**

**As discussed before, the following facts apply to a stable system**

1. **If the input to the system is bounded, then the output of the system will also be bounded (BIBO), or the impulse response of the system will go to zero in a finite number of steps.**
2. **An unstable system is one in which the output of the system will grow without bound due to any bounded input, initial condition, or noise, or its impulse response will grow without bound.**
3. **The impulse response of a marginally stable system stays at a constant level or oscillates between two finite values.**

**Example 9**

**The following transfer functions describe digital systems**

1. $H\left(z\right)=\frac{\left(z + 0.5\right)}{\left(z-0.5\right)\left(z^{2} + z + 0.5\right)}$
2. $H\left(z\right)=\frac{\left(z^{2} + 0.25\right)}{\left(z-0.5\right)\left(z^{2} + 3 z + 2.5\right)}$
3. $H\left(z\right)=\frac{\left(z + 0.5\right)}{\left(z-0.5\right)\left(z^{2} + 1.4141 z + 1\right)}$
4. $H\left(z\right)=\frac{\left(z^{2} + z + 0.5\right)}{\left(z-1\right)^{2}\left( z +1\right)\left( z-0.6\right)}$

**For each, sketch the z-plane pole-zero plot and determine the stability status for the digital system.**

**Solution**

**Part (a): Putting the numerator polynomial equal to zero and then finding the roots, gives us the zeros of the transfer function,**

$$z+0.5=0$$

**Therefore, we get**$ z\_{1}=-0.5$ **as the root.**

**Now, setting the denominator polynomial equal to zero and find the roots, gives us the poles of the transfer function,**

$$\left(z-0.5\right)\left(z^{2} + z + 0.5\right)=0$$

**This leads to**

$$z-0.5=0$$

**and**

$$z=\frac{-1\pm \sqrt{\left(1\right)^{2}-4\left(1\right)(0.5)}}{2(1)}=\frac{-1\pm \sqrt{1-2}}{2}=\frac{-1\pm \sqrt{-1}}{2}=\frac{-1.0\pm j}{2}=-0.5\pm j0.5$$

**Therefore, the poles are**$ p\_{1}=0.5$**,** $p\_{2}=-0.5+j0.5$ **and**$ p\_{3}=-0.5-j0.5$**. The magnitudes of these poles are**

$$\left|p\_{1}\right|=0.5 $$

$$\left|p\_{2}\right|=\left|-0.5+j0.5\right|=\sqrt{\left(-0.5\right)^{2}+(0.5)^{2}}=0.707$$

$$\left|p\_{3}\right|=\left|-0.5-j0.5\right|=\sqrt{\left(-0.5\right)^{2}+(-0.5)^{2}}=0.707$$

**It can be noticed that the magnitudes of all the poles are less than 1 (i.e. they are inside the unit circle on z-plane pole-zero plot). Therefore, the system is stable. This is shown in the following figure.**

**0.5**

$$Im\left(z\right)$$

$$Re\left(z\right)$$

**1**

**- 0.5**

**- 0.5**

**0.5**

**Part (b): Putting the numerator polynomial equal to zero and then finding the roots, gives us the zeros of the transfer function,**

$$z^{2}+0.25=0$$

$$⟹ z^{2}=-0.25$$

$$⟹ z=\sqrt{-0.25}=\pm j 0.5$$

**Therefore, we get**$ z\_{1}=+j0.5$ **and**$ z\_{2}=-j0.5$ **as the roots.**

**Now, setting the denominator polynomial equal to zero and find the roots, gives us the poles of the transfer function,**

$$\left(z-0.5\right)\left(z^{2} +3 z + 2.5\right)=0$$

**This leads to**

$$z-0.5=0$$

**and**

$$z=\frac{-3\pm \sqrt{\left(3\right)^{2}-4\left(1\right)(2.5)}}{2(1)}=\frac{-3\pm \sqrt{9-10}}{2}=\frac{-3\pm \sqrt{-1}}{2}=\frac{-3\pm j}{2}=-1.5\pm j0.5$$

**Therefore, the poles are**$ p\_{1}=0.5$**,** $p\_{2}=-1.5+j0.5$ **and**$ p\_{3}=-1.5-j0.5$**. The magnitudes of these poles are**

$$\left|p\_{1}\right|=0.5 $$

$$\left|p\_{2}\right|=\left|-1.5+j0.5\right|=\sqrt{\left(-1.5\right)^{2}+(0.5)^{2}}=1.5811$$

$$\left|p\_{3}\right|=\left|-1.5-j0.5\right|=\sqrt{\left(-1.5\right)^{2}+(-0.5)^{2}}=1.5811$$

**It can be noticed that the magnitudes of the two poles are greater than 1 (i.e. they are outside the unit circle on z-plane pole-zero plot).**

**Therefore, the system is unstable. This is shown in the following figure.**

**0.5**

$$Im\left(z\right)$$

$$Re\left(z\right)$$

**1**

**- 0.5**

**- 1.5**

**0.5**

1. **Digital Filter Frequency Response**

**Example 10**

**Given the following digital system with a sampling rate of 8000 Hz,**

$$y\left(n\right)=0.5 x\left(n\right)+0.5x\left(n-1\right)$$

**Determine the frequency response of the system.**

**Solution**

**Taking the z-transform of the both sides of the difference equation, we get**

$$Y\left(z\right)=0.5 X\left(z\right)+0.5 z^{-1} X\left(z\right)$$

$$⟹ Y\left(z\right)=\left(0.5 +0.5 z^{-1}\right) X\left(z\right)$$

**Therefore, the transfer function of the system us given by**

$$H\left(z\right)=\frac{Y\left(z\right)}{X\left(z\right)}=\left(0.5 +0.5 z^{-1}\right) $$

**To find out the frequency response of the system, we replace**$ z$ **with**$ e^{jΩ}$**. This leads to**

$$H\left(e^{jΩ}\right)=\left(0.5 +0.5 e^{-jΩ}\right)$$

**This can be written as**

$$H\left(e^{jΩ}\right)=0.5 +0.5 cos\left(Ω\right)-j0.5sin\left(Ω\right)$$

**Therefore, the magnitude frequency response and phase response are given by**

$$\left|H\left(e^{jΩ}\right)\right|=\left|0.5 +0.5 cos\left(Ω\right)-j0.5sin\left(Ω\right)\right|=\sqrt{\left(0.5 +0.5 cos\left(Ω\right)\right)^{2}+\left(-0.5sin\left(Ω\right)\right)^{2}}$$

**And**

$$∠H\left(e^{jΩ}\right)=tan\left(\frac{-0.5sin\left(Ω\right)}{0.5 +0.5 cos\left(Ω\right)}\right)$$

**This is an example of a lowpass filter.**

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**Example 11**

**Given the following digital system with a sampling rate of 8000 Hz,**

$$y\left(n\right)= x\left(n\right)-0.5y\left(n-1\right)$$

**Determine the frequency response of the system.**

**Solution**

**Taking the z-transform of the both sides of the difference equation, we get**

$$Y\left(z\right)= X\left(z\right)-0.5 z^{-1} Y\left(z\right)$$

$$⟹ Y\left(z\right)+0.5 z^{-1} Y\left(z\right)= X\left(z\right)$$

$$⟹ \left(1+0.5 z^{-1}\right) Y\left(z\right)= X\left(z\right)$$

**Therefore, the transfer function of the system us given by**

$$H\left(z\right)=\frac{Y\left(z\right)}{X\left(z\right)}=\frac{1}{\left(1+0.5 z^{-1}\right)} $$

**To find out the frequency response of the system, we replace**$ z$ **with**$ e^{jΩ}$**. This leads to**

$$H\left(e^{jΩ}\right)=\frac{1}{\left(1+0.5 e^{-jΩ}\right)}$$

**This can be written as**

$$H\left(e^{jΩ}\right)=\frac{1}{\left(1+0.5 cos\left(Ω\right)-j0.5sin\left(Ω\right)\right)}$$

**Therefore, the magnitude frequency response and phase response are given by**

$$\left|H\left(e^{jΩ}\right)\right|=\left|\frac{1}{\left(1+0.5 cos\left(Ω\right)-j0.5sin\left(Ω\right)\right)}\right|=\frac{1}{\sqrt{\left(1 +0.5 cos\left(Ω\right)\right)^{2}+\left(-0.5sin\left(Ω\right)\right)^{2}}}$$

**And**

$$∠H\left(e^{jΩ}\right)=-tan\left(\frac{-0.5sin\left(Ω\right)}{1 +0.5 cos\left(Ω\right)}\right)$$

**This is an example of a highpass filter.**

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1. **Basic Types of Filtering**

**Ideal Filter Characteristics**

$$\left|H\left(e^{jΩ}\right)\right|$$

$$-π$$

$$π$$

$$-ω\_{c}$$

$$ω\_{c}$$

**0**

**1**

$$B$$

Ideal Lowpass

$$\left|H\left(e^{jΩ}\right)\right|$$

$$-π$$

$$π$$

$$-ω\_{c}$$

$$ω\_{c}$$

**0**

**1**

Ideal Highpass

$$\left|H\left(e^{jΩ}\right)\right|$$

$$-π$$

$$π$$

$$ω\_{1} ω\_{0} ω\_{2}$$

**0**

**1**

Ideal Bandpass

$$-ω\_{1} -ω\_{0} -ω\_{2}$$

$$\left|H\left(e^{jΩ}\right)\right|$$

$$-π$$

$$π$$

$$-ω\_{c}$$

$$ω\_{c}$$

**0**

**1**

Ideal Bandstop

Ideal Allpass

$$\left|H\left(e^{jΩ}\right)\right|$$

$$-π$$

$$π$$

**0**

**1**

**Filters are generally classified according to their frequency characteristics as (1) lowpass, (2) highpass, (3) bandpass, (4) bandstop, and (5) allpass filters. The magnitude frequency responses of these filters in ideal cases are shown in the figures above. For example, in case of an ideal lowpass filter, all the frequencies of the input signal in the range from**$-ω\_{c}$ **to**$ ω\_{c}$ **are allowed to pass, while the other frequencies in the input signal are stopped. The frequency**$ ω\_{c}$ **is called the cutoff frequency of the lowpass filter in this case.**

**The ideal filters are generally not causal. They are also unstable. These filters are not physically realizable.**

**% plots the magnitude frequency response and phase response**

**clear all**

**close all**

**clc**

**%**

**% Case (a)**

**[h w] = freqz([1], [1 -0.5], 1024);**

**phi = 180\*unwrap(angle(h))/pi;**

**figure**

**subplot(2,1,1), plot(w, abs(h)); grid on;**

**xlabel('Frequency (radians)'); ylabel('Magnitude');**

**subplot(2,1,2), plot(w, phi); grid on;**

**xlabel('Frequency (radians)'); ylabel('Phase (degrees)');**

**%**

**% Case (b)**

**[h w] = freqz([1 -0.5], [1], 1024);**

**phi = 180\*unwrap(angle(h))/pi;**

**figure**

**subplot(2,1,1), plot(w, abs(h)); grid on;**

**xlabel('Frequency (radians)'); ylabel('Magnitude');**

**subplot(2,1,2), plot(w, phi); grid on;**

**xlabel('Frequency (radians)'); ylabel('Phase (degrees)');**

**%**

**% Case (c)**

**[h w] = freqz([0.5 0 -0.32], [1 -0.5 0.25], 1024);**

**phi = 180\*unwrap(angle(h))/pi;**

**figure**

**subplot(2,1,1), plot(w, abs(h)); grid on;**

**xlabel('Frequency (radians)'); ylabel('Magnitude');**

**subplot(2,1,2), plot(w, phi); grid on;**

**xlabel('Frequency (radians)'); ylabel('Phase (degrees)');**

**%**

**% Case (d)**

**[h w] = freqz([1 -0.9 0.81], [1 -0.6 0.36], 1024);**

**phi = 180\*unwrap(angle(h))/pi;**

**figure**

**subplot(2,1,1), plot(w, abs(h)); grid on;**

**xlabel('Frequency (radians)'); ylabel('Magnitude');**

**subplot(2,1,2), plot(w, phi); grid on;**

**xlabel('Frequency (radians)'); ylabel('Phase (degrees)');**

$$H\left(z\right)=\frac{z}{\left(z-0.5 \right)}$$



$$H\left(z\right)=1-0.5 z^{-1}$$

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$$H\left(z\right)=\frac{0.5z^{2}-0.32}{z^{2}-0.5 z+0.25}$$

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$$H\left(z\right)=\frac{1-0.9 z^{-1}+0.81 z^{-2}}{1-0.6 z^{-1}+0.36 z^{-2}}$$

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