PHYS 502

Lecture 6: Bessel Functions

Dr. Vasileios Lempesis

Introduction

- Bessel functions appear in problems involving vibrations and heat conductions in regions with circular symmetry.
- They are named after the German mathematician and astronomer Friedrich Wilhelm Bessel who first used them to describe three body motion.
- They are solutions of the following differential equation:

$$x^2 y_\nu'' + x y_\nu' + (x^2 - \nu^2) y_\nu = 0, \qquad \nu \geq 0$$

The constant ν determines the order of the Bessel Functions and can take any real value. For cylindrical problems it is integer while for spherical is half-integer.

Bessel Functions of First Kind

Bessel functions of the first kind, denoted as J_ν , are solutions of Bessel's differential equation that are finite at the origin ($x = 0$) for integer or positive ν, and diverge as x approaches zero for negative non-integer ν. It is possible to define the function by its Taylor series expansion around x = 0 as follows:

$$J_\nu(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(n+s)!}\left(\frac{x}{2}\right)^{n+2s} = \frac{x^n}{2^n n!} - \frac{x^{n+2}}{2^{n+2}(n+1)!} + \dots$$

This relation holds for $\nu < 0$. **If ν is integer** then we may show that

$$J_{-n}(x) = (-1)^n J_n(x)$$

Bessel functions of first kind are finite at x=0 for all real values of ν.

Bessel Functions of First Kind

The generating function

The Bessel functions can be obtained with the help of the so called *generating* function:

$$g(x,t) = e^{(x/2)(t-1/t)}$$

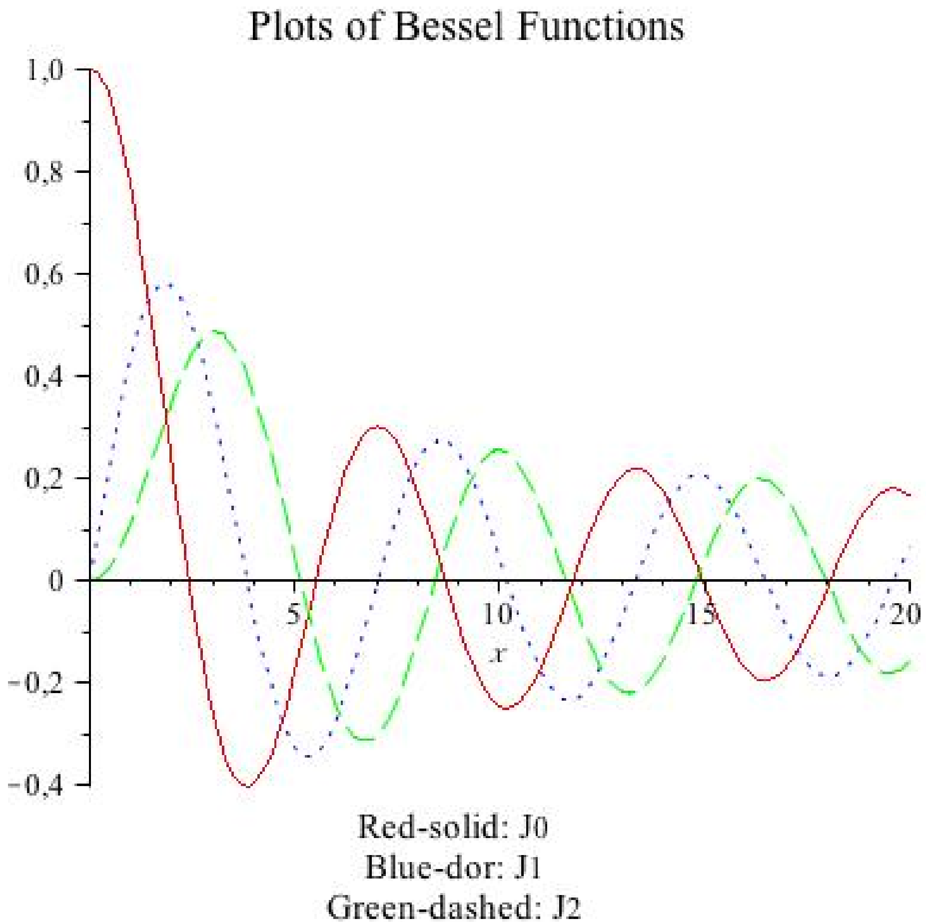
If we try to expand this function as a Laurent series we obtain

$$e^{(x/2)(t-1/t)} = \sum_{n=-\infty}^{\infty} J_n(x) t^n$$

Hint: Generating function may define only Bessel functions of integral order.

Bessel Functions of First Kind

plots



Hint: The zeroes of the Bessel functions are not equidistant!

Bessel Functions of First Kind

The recurrence function

For Bessel functions of first kind we can prove the following relations:

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x)$$

$$J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x)$$

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$$

Bessel Functions of First Kind

The integral representation function

Using the generating function we may prove a particularly useful and powerful way of representing the Bessel functions with the help of integrals.

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)\,d\theta, \qquad n = 0,\ 1,\ 2,\ 3,\ \ldots$$

Bessel Functions of First Kind

Orthogonality

If we divide Bessel's equation by *x*, it becomes self-adjoint and, therefore, by Sturm-Liouiville theory the solutions are expected to be orthogonal - if we can arrange to have appropriate boundary conditions satisfied. To take care of the boundary conditions for a finite interval [0, *a*], we introduce parameters *a* and $a_{\nu m}$ into the argument of J_ν *to get* $J_\nu(a_{\nu m}\rho/a)$. In this case we can show that

$$\int_0^a J_\nu\left(a_{\nu m}\frac{\rho}{a}\right)J_\nu\left(a_{\nu n}\frac{\rho}{a}\right)\rho d\rho = 0 \qquad \int_0^a\left[J_\nu\left(a_{\nu m}\frac{\rho}{a}\right)\right]^2\rho d\rho = \frac{a^2}{2}\left[J_{\nu+1}\left(a_{\nu m}\right)\right]$$

Bessel Functions of First Kind

Bessel series

- If we assume that the set of Bessel functions $J_\nu(\alpha_{\nu m}\rho / a)$ (ν fixed, $m = 1, 2, 3, \ldots$) is complete, then any well behaved but otherwise arbitrary function $f(\varrho)$ may be expanded in a Bessel series (Bessel-Fourier series):

$$f(\rho) = \sum_{m=1}^{\infty} c_{mn} J_\nu(\alpha_{\nu m}\rho / a), \qquad 0 \le \rho \le a, \qquad \nu > \text{-}1$$

$$c_{\nu m} = \frac{2}{a^2 \left[J_{\nu+1}(\alpha_{\nu m}) \right]^2} \int_0^a f(\rho) J_\nu(\alpha_{\nu m}\rho / a) \rho d\rho$$

Bessel Functions of First Kind

Potential Applications

- Problems involving electric fields, vibrations, heat conduction, optical diffraction plus others involving cylindrical or spherical symmetry.
- Transient heat conduction in a thin wall
- Steady heat flow in a circular cylinder of finite length.

Bessel Functions of Second Kind
Neumann Functions-a

The Neumann function is defined as follows:

$$N_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}$$

For nonintegral ν the above function clearly satisfies Bessel's equation, for it is a linear combination of known solutions. However if ν is integer the definition is given by

$$N_n(x) = \frac{1}{\pi}\left[\frac{\partial J_\nu(x)}{\partial \nu} - (-1)^n \frac{\partial J_{-\nu}(x)}{\partial \nu}\right]_{\nu=n}$$

Bessel functions of 2nd kind satisfy the same recurrence relations as the first kind.

Bessel Functions of Second Kind
Neumann Functions-b

- With the help of this function the most general solution of the Bessel equation can be written as

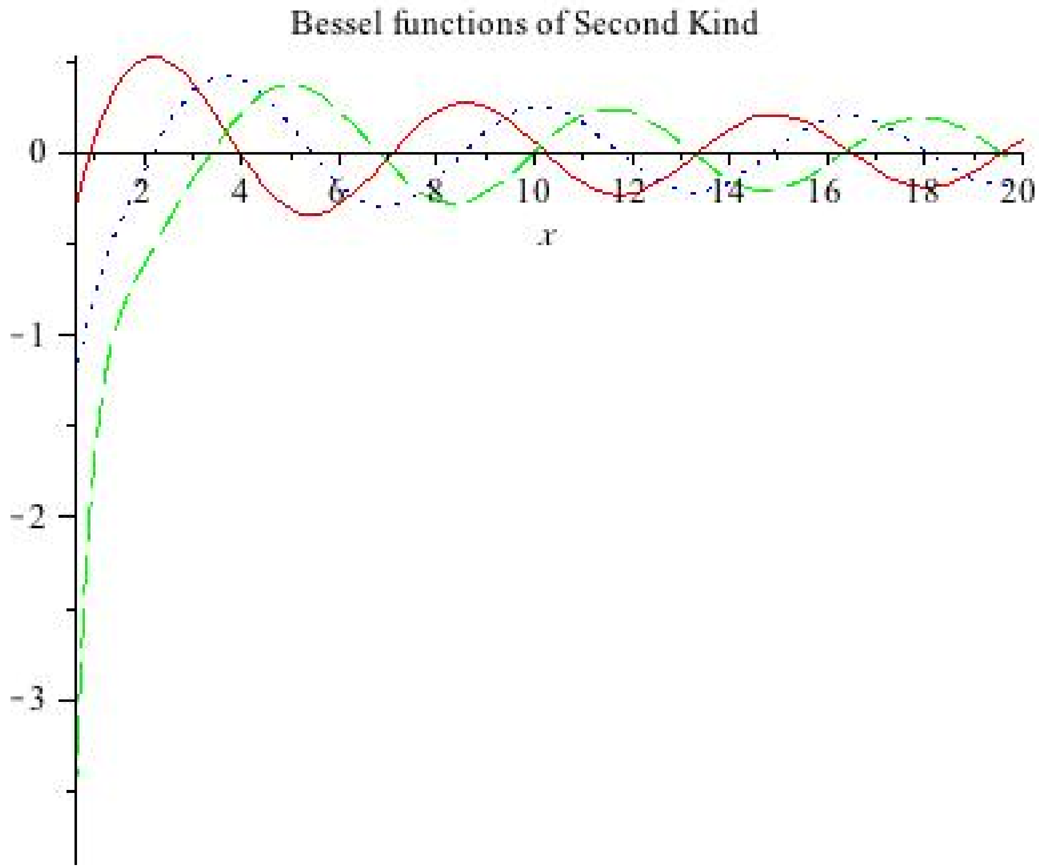
$$y(x) = AJ_\nu(x) + BN_\nu(x)$$

with A and B as arbitrary constants determined by boundary conditions.

- This is the only choice when ν *is an integer*. In this case we know that $J_{-n}(x) = (-1)^n J_n(x)$ thus the two Bessel functions are linearly dependent.
- When ν is non-integer the above equation is redundant and we may write

$$y(x) = AJ_\nu(x) + BJ_{-\nu}(x)$$

Bessel Functions of Second Kind



Solid Red: Y_0
Blue Dot: Y_1
Green Dash: Y_2

Bessel Functions of Third Kind
Hankel Functions

In Bessel functions theory we introduce the so-called Hankel functions, which are defined as follows:

$$H_\nu^{(1)}(x) = J_\nu(x) + iN_\nu(x) \qquad x > 0$$

$$H_\nu^{(2)}(x) = J_\nu(x) - iN_\nu(x) \qquad x > 0$$

Because of the linear independence of the Bessel function of the first and second kind, the Hankel functions provide an alternative pair of solutions to the Bessel differential equation.

Hankel functions satisfy the same recurrence relations as the first and second kind Bessel functions. Why?

Bessel Functions
Modified Bessel Functions-a

- Modified Bessel functions are found as solutions to the modified Bessel equation

$$x^2 y_\nu'' + x y_\nu' - (x^2 - \nu^2) y_\nu = 0$$

which transforms to the original equation if x is replaced by ix. To avoid dealing with complex solutions in practical applications we have introduced the modified Bessel functions.

$$I_\nu(x) \equiv i^{-\nu} J_\nu\left(ix\right)$$

$$I_\nu(x) = \sum_{s=0}^{\infty} \frac{1}{s!\left(n+s\right)!} \left(\frac{x}{2}\right)^{\nu+2s}$$

Bessel Functions
Modified Bessel Functions-b

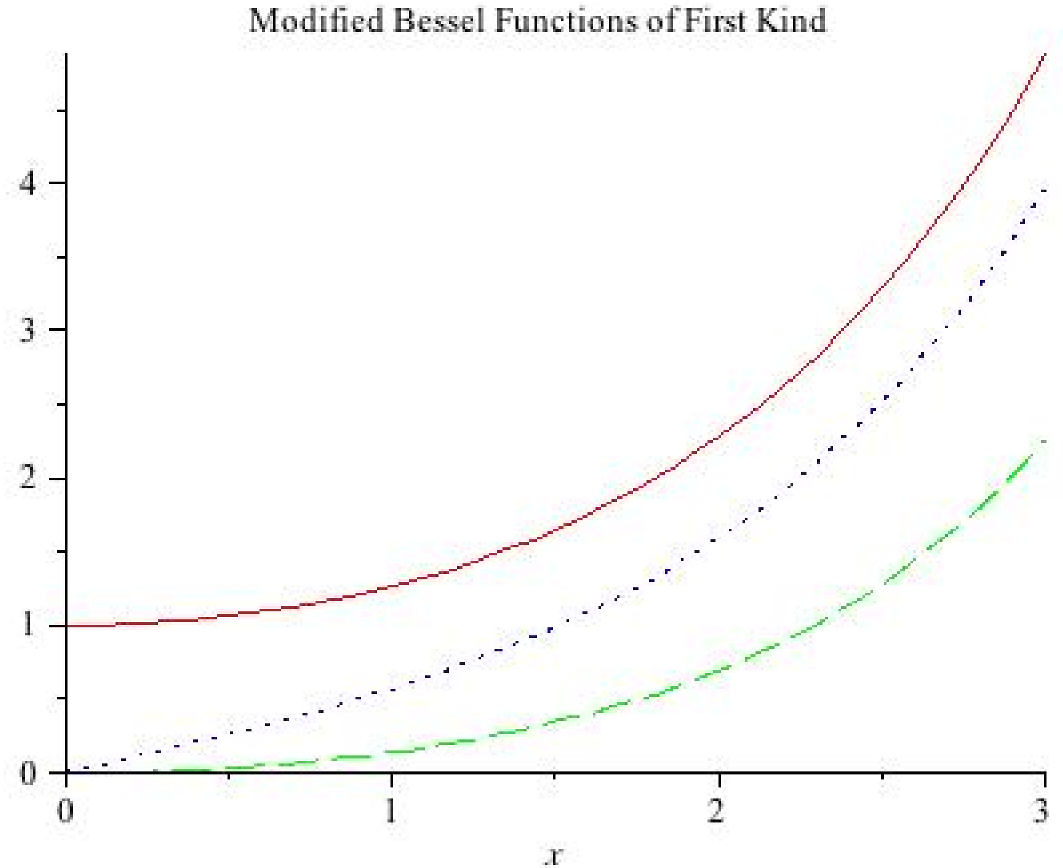
- The general solution of the modified Bessel function is expressed as follows with the help of the first and second order modified Bessel functions:

$$y(x) = CI_\nu(x) + DK_\nu(x) \qquad x > 0$$

- A solution for non-integer orders of ν is given by the modified Bessel functions of the second kind

$$K_\nu(x) = \frac{\pi}{2}\frac{I_{-\nu}\left(x\right) - I_\nu\left(x\right)}{\sin\nu\pi}$$

Modified Bessel Functions of 1st Kind



Red solid: I_0
Blue dot: I_1
Green dash: I_2

Modified Bessel Functions
Recurrence relations

- We may show the following recurrence relation for the modified Bessel functions of first kind

$$I_{\nu-1}(x) - I_{\nu+1}(x) = \frac{2\nu}{x} I_{\nu}(x)$$

$$I_{\nu-1}(x) + I_{\nu+1}(x) = 2I_{\nu}'(x)$$

Modified Bessel Functions of 2nd Kind

Potential Applications

- Modified Bessel functions appear less frequently in applications, but can be found in transmission line studies, non-uniform beams, and the statistical treatment of a relativistic gas in statistical mechanics
- Problems involving the displacement of a vibrating membrane.
- Heat conduction in an annular fin of rectangular cross section attached to a circular base.

Bessel Functions
Zeroes of Bessel Functions

- The zeroes, or roots, of the Bessel functions are the values of x where value of the Bessel function becomes zero. Frequently the roots are given by in tabulated formats.
- Bessel functions of first and second kind have an infinite number of roots as x goes to infinity.
- The modified Bessel functions of the first kind have only one root at $x=0$, and the modified Bessel function of the second kind functions do not have zeroes.

Spherical Bessel Functions

When we try to solve the Helmholtz equation in spherical coordinates, the radial equation which comes out leads us to define the so called spherical Bessel equations:

$$j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+1/2}(x), \quad n_n(x) = \sqrt{\frac{\pi}{2x}} N_{n+1/2}(x) = \left(-1\right)^{n+1} \sqrt{\frac{\pi}{2x}} J_{-n-1/2}(x)$$

$$h_n^{(1)}(x) = \sqrt{\frac{\pi}{2x}} H_{n+1/2}^{(1)}(x) = j_n(x) + in(x)$$

$$h_n^{(2)}(x) = \sqrt{\frac{\pi}{2x}} H_{n+1/2}^{(2)}(x) = j_n(x) - in(x)$$

Spherical Bessel Functions Orthogonality

- Spherical Bessel functions obey the following orthogonality relation:

$$\int_0^a j_n\left(\alpha_{np}\frac{\rho}{a}\right) j_n\left(\alpha_{nq}\frac{\rho}{a}\right)\rho^2 d\rho = \frac{a^3}{2}\left[j_{n+1}\left(\alpha_{np}\right)\right]^2 \delta_{pq}$$

- Where α_{np} and α_{np} are roots of j_n