PHYS 502

Lecture 8: Legendre Functions

Dr. Vasileios Lempesis

Introduction

- Legendre functions or Legendre polynomials are the solutions of Legendre's differential equation that appear when we separate the variables of Helmholtz' equation, Laplace equation or Schrodinger equation using spherical coordinate.
- They are solutions of the following differential equation:

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_n(x)}{dx}\right] + n(n+1)P_n(x) = 0$$

•**The constant *n* is an integer (n=0,1,2,…).**

• $P_n(x)$ **converge only when** $|x| < 1$

Legendre Functions

* Legendre functions (or polynomials) is a power series solution of Legendre differential equation about the origin ($x = 0$).

•The series solution should converge to be terminated in order to meet the physical requirements. The series solution converges when $|x| < 1$ satisfies.

• The Legendre polynomials can be expressed as Rodrigues' Formula:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n$$

The degree of Legendre polynomial depends on the value of n, such as

Legendre Polynomial

plots

$$P_0(x) = 1$$

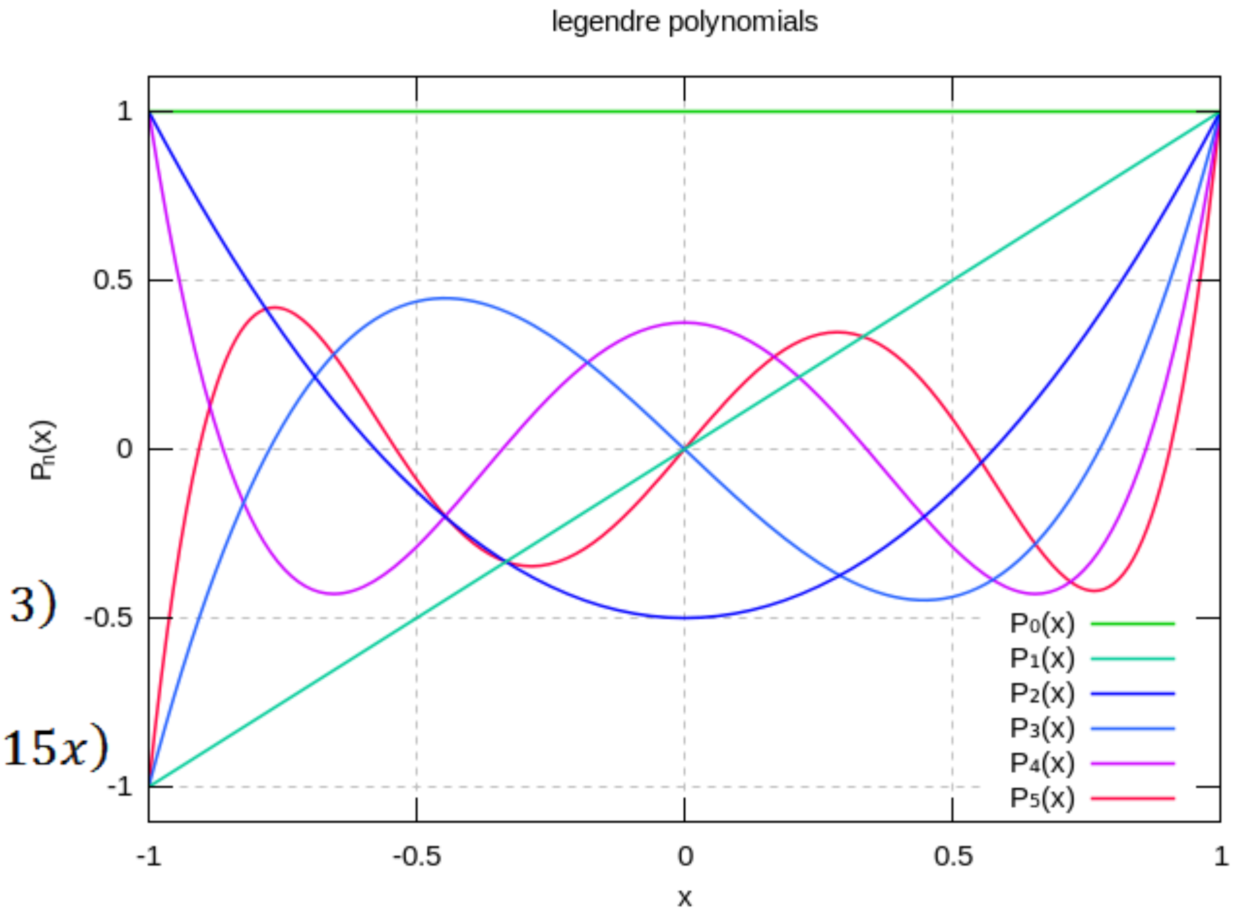
$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$



Legendre Polynomial

The generating function

The *generating* function of Legendre polynomials:

$$g(t,x) = (1 - 2xt + t^2)^{-1/2}$$

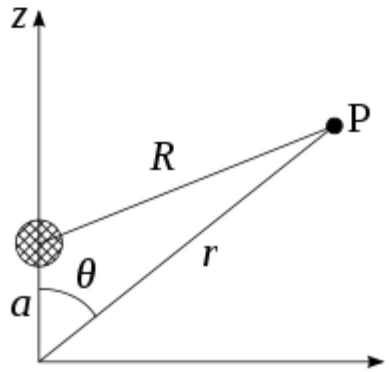
Which has an important application in **electric multipole expansions**. If we expand this function as a binomial series if $|t| < 1$ we obtain

$$g(t,x) = \sum_{n=0}^{\infty} P_n(x)\, t^n$$

Legendre Polynomial

The generating function (example)

- Consider an electric charge q placed on the z-axis at $z = 0$. the electrostatic potential of the charge at point P is



$$\emptyset = k\frac{q}{|\vec{R}|}$$

- If we express the electrostatic potential in terms of spherical polar coordinates.

$$\frac{1}{|\vec{R}|} = (r^2 - 2ar\cos(\theta) + a^2)^{-1/2} = \frac{1}{r}\left(1 - 2\frac{a}{r}\cos(\theta) + \left(\frac{a}{r}\right)^2\right)^{-1/2}$$

In case of $\frac{a}{r} < 1$ we can express the electrostatic potential as series of Legendre polynomial

$$\emptyset = kq\sum_{n=0}^{\infty} P_n(\cos(\theta))\left(\frac{a}{r}\right)^n$$

Legendre Polynomials

The recurrence function

the Legendre polynomials obey the three term recurrence relations:

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x)$$

$$P'_{n+1}(x) - P'_{n-1}(x) = (2n+1)P_n(x)$$

$$(1-x^2)P'_n(x) = nP_{n-1}(x) - nxP_n(x)$$

But these relations are valid for $n = 1,2,3,\ldots$

Legendre Polynomials

Special Properties

- Some special values

$$P_n(1) = 1$$

$$P_n(-1) = (-1)^n$$

and

$$\left.\begin{aligned} P_{2n}(0) &= (-1)^n \frac{(2n-1)!!}{(2n)!!} \\ P_{2n+1}(0) &= 0 \end{aligned}\right\} \quad \text{for } n = 0,1,2,\ldots$$

Where $n!! = \begin{cases} n(n-2)(n-4)\ldots 1 \text{ if } n \, is \, odd \\ n(n-2)(n-4)\ldots 2 \text{ if } n \, is \, even \\ 1 \qquad\qquad\qquad\qquad \text{if } n = 0 \end{cases}$ (called double factorial)

Legendre Polynomials

Special Properties

- The Parity property:(with respect to $x = 0 \;, \theta = \pi/2$)

$$P_n(-x) = (-1)^n P_n(x)$$

$$P_n(\cos(\pi - \theta)) = (-1)^n P_n(\cos(\theta))$$

If *n* is odd the parity of the polynomial is odd, but if it is even the parity of the polynomial is even.

- Upper and lower Bounds for $P_n(\cos(\theta))$

$$|P_n(\cos(\theta))| \leq P_n(1) = 1$$

Legendre Polynomials

Orthogonality

* Legendre's equation is self-adjoint. Which satisfies Sturm-Liouiville theory where the solutions are expected to be orthogonal to satisfying certain boundary conditions. Legendre polynomials are a set of orthogonal functions on (-1,1).

$$\int_{-1}^{1} P_n(x)P_m(x)dx = \frac{2}{2n+1}\delta_{n,m}$$

where

$$\delta_{n,m} = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases}$$

Legendre Polynomials

legendre Series

- According to Sturm-Liouville theory that Legendre polynomial form a complete set.

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$$

- The coefficients a_n are obtained by multiplying the series by $P_m(x)$ and integrating in the interval [-1,1]

$$a_n = \frac{2m+1}{2} \int_{-1}^{1} f(x) P_m(x) dx = 0$$

Legendre Polynomials

Potential Applications

- Electrostatic potential, and gravitational potential involving spherical symmetry.

$$U(\vec{r}_1 - \vec{r}_2) \propto \frac{1}{|\vec{r}_1 - \vec{r}_2|} = \frac{1}{r_>} \sum_{n=0}^{\infty} \left(\frac{r_<}{r_>}\right)^n P_n(\cos(\theta))$$

where

. $\left.\begin{matrix} r_> = |r_1| \\ r_< = |r_2| \end{matrix}\right\} for\ |r_1| > |r_2|$ and $\left.\begin{matrix} r_> = |r_2| \\ r_< = |r_1| \end{matrix}\right\} for\ |r_2| > |r_1|$

Legendre Polynomials

Alternate Definitions of Legendre Polynomials

- Schaefli Integral:

Schaefli integral is obtained by using Cauchy's integral

$$f(z) = \frac{1}{2\pi i}\oint \frac{f(t)}{t-z}dt \quad \text{with} \quad f(z) = (z^2-1)^n$$

by differentiating $f(z)$ n times with respect to z and multiplying by $1/2^n n!$, then we can get

$$P_n(z) = \frac{2^{-n}}{2\pi i}\oint \frac{(t^2-1)^n}{(t-z)^{n+1}}dt$$

with the contour enclosing the point $t = z$. Schaefli integral is useful to define $P_\nu(z)$ for nonintegral ν.

Associated Legendre Functions

If Laplace's or Helmholtz's equations are separated in spherical polar coordinates we get associated Legendre equation (with $x = \cos\theta$):

$$\frac{d}{dx}\left[(1-x^2)\frac{dP_n^m(x)}{dx}\right]+\left[n(n+1)-\frac{m^2}{1-x^2}\right]P_n^m(x)=0$$

The new integer m associated with another polar angle φ. The solutions of this equation are called associated Legendre functions

$$P_n^m(x)=(1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x)$$

If the separation only has the azimuthal part then *m* = 0, the equation become Legendre equation.

Associated Legendre Functions

Properties

- Since the highest power of x in $P_n(x)$ is x^n then $m \leq n$ (or the m-fold differentiation will drive our function to zero).
- The integer m can be negative or positive and take the values

  $-n \leq m \leq n$.
- The relation between $P_n^m(x)$ and $P_n^{-m}(x)$ is given by

$$P_n^{-m}(x) = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m(x)$$

- The generating function for the associated Legendre function

$$\frac{(2m)!\,(1-x^2)^{m/2}}{2^m m!\,(1-2tx+t^2)^{m+1/2}} = \sum_{s=0}^{\infty} P_{s+m}^m(x) t^s$$

Associated Legendre Functions

Recurrence Relations

Because of the existence of two indices, we have wide variety of recurrence relation:

$$P_n^{m+1} - \frac{2mx}{(1-x^2)^{1/2}} P_n^m + [n(n+1) - m(m-1)] P_n^{m+1} = 0$$

$$(2n+1)xP_n^m = (n+m)P_{n-1}^m + (n-m+1)P_{n+1}^m$$

$$(2n+1)(1-x^2)^{1/2} P_n^m = P_{n+1}^{m+1} - P_{n-1}^{m+1}$$
$$= (n+m)(n+m-1)P_{n-1}^{m-1} + (n-m+1)(n-m+2)P_{n+1}^{m-1}$$

$$(1-x^2)^{1/2} P_n^{m\prime} = \frac{1}{2} P_n^{m+1} - \frac{1}{2}(n+m)(n-m+1)P_n^{m-1}$$

Associated Legendre Functions

Parity

- Associated Legendre function satisfies the parity relation as $x \to -x$ about the origin. Because P_n^m *m*-fold differentiation yields a factor of $(-1)^m$:

$$P_n^m(-x) = (-1)^{n+m} P_n^m(x)$$

- From the parity property and the defintion of $P_n^m(x)$

  we can deduce that:

$$P_n^m(\pm 1) = 0 \quad \text{for } m \neq 0$$

Associated Legendre Functions

Orthogonality

- The orthogonality of associated Legendre function depends on the values of two indices *n* and *m*:
- If *m* is the same for both functions:

$$\int_{-1}^{1} P_p^m(x) P_q^m(x) dx = \frac{2}{2q+1} \cdot \frac{(q+m)!}{(q-m)!} \delta_{p,q}$$

- If *n* is the same for both functions:

$$\int_{-1}^{1} P_n^m(x) P_n^k(x) (1-x^2)^{-1} dx = \frac{(n+m)!}{m(n-m)!} \delta_{m,k}$$

where $(1-x^2)^{-1}$ is the weighting factor.

Associated Legendre Functions

Spherical Harmonics

- The Schrödinger equation for central force

$$\nabla^2\psi + k^2 f(x)\psi = 0$$

if use the spherical polar coordinates

$$\left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{\mathbb{L}^2}{r^2} + k^2\right]\psi(r,\theta,\varphi) = 0$$

If the wavefunction expressed as $\psi(r,\theta,\varphi) = R(r)Y_n^m(\theta,\varphi)$ then

$$-\mathbb{L}^2 Y_n^m(\theta,\varphi) = \left[\frac{1}{\sin(\theta)}\frac{d}{d\theta}\left(\sin(\theta)\frac{d}{d\theta}\right) + \frac{1}{\sin^2(\theta)}\frac{d^2}{d\varphi^2}\right]Y_n^m(\theta,\varphi)$$

$$= -n(n+1)Y_n^m(\theta,\varphi)$$

Where $Y_n^m(\theta,\varphi)$ called *Spherical Harmonics* and $\mathbb{L}$ the angular momentum operator.

Associated Legendre Functions

Spherical Harmonics (part2)

- The spherical harmonics are product of two functions, a function depend on the azimuthal angle φ:

$$\Phi(\varphi) = \frac{1}{\sqrt{2\pi}} e^{im\varphi}$$

and another function depends on polar angle θ:

$$\mathcal{P}_n^m(\cos(\theta)) = \sqrt{\frac{2n+1}{2}\frac{(n-m)!}{(n+m)!}} P_n^m(\cos(\theta))$$

where the spherical harmonics defined as

$$Y_n^m(\theta,\varphi) = (-1)^m \sqrt{\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}} P_n^m(\cos(\theta)) e^{im\varphi}$$

Where the factor $(-1)^m$

.

Associated Legendre Functions

Spherical Harmonics (Orthogonality)

- The functions $\Phi(\varphi)$ are orthogonal with respect to the azimuthal angle φ.

- The functions $\mathcal{P}_n^m(\cos(\theta))$ are orthogonal with respect to the polar angle θ.

- The spherical harmonic functions of two angles are orthogonal such that

$$\int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi} Y_{n_1}^{m_1 *}(\theta,\varphi) Y_{n_2}^{m_2}(\theta,\varphi)\sin(\theta)\,d\theta\, d\varphi = \delta_{n_1,n_2}\delta_{m_1,m_2}$$

Associated Legendre Functions

Spherical Harmonics (Laplace Series)

- Spherical Harmonics form a complete set. Means that any function $f(\theta,\varphi)$ evaluated over the sphere can be expanded in a uniformly convergent double series of spherical harmonics:

$$f(\theta,\varphi) = \sum_{m,n} a_{mn} Y_n^m(\theta,\varphi)$$