Introduction

• Legendre functions or Legendre polynomials are the solutions of Legendre’s differential equation that appear when we separate the variables of Helmholtz’ equation, Laplace equation or Schrodinger equation using spherical coordinate.

• They are solutions of the following differential equation:

\[
\frac{d}{dx} \left[ (1 - x^2) \frac{dP_n(x)}{dx} \right] + n(n + 1)P_n(x) = 0
\]

• The constant \( n \) is an integer (\( n=0,1,2,\ldots \)).

• \( P_n(x) \) converge only when \(|x| < 1\).
Legendre Functions

* Legendre functions (or polynomials) is a power series solution of Legendre differential equation about the origin 
  \((x = 0)\).

  • The series solution should converge to be terminated in order to meet the physical requirements. The series solution converges when \(|x| < 1\) satisfies.

  • The Legendre polynomials can be expressed as Rodrigues’ Formula:

  \[
P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n
  \]

  The degree of Legendre polynomial depends on the value of \(n\), such as
Legendre Polynomial plots

\[ P_0(x) = 1 \]
\[ P_1(x) = x \]
\[ P_2(x) = \frac{1}{2}(3x^2 - 1) \]
\[ P_3(x) = \frac{1}{2}(5x^3 - 3x) \]
\[ P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3) \]
\[ P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x) \]
Legendre Polynomial

The generating function

The generating function of Legendre polynomials:

\[ g(t, x) = (1 - 2xt + t^2)^{-1/2} \]

Which has an important application in electric multipole expansions. If we expand this function as a binomial series if \(|t| < 1\) we obtain

\[ g(t, x) = \sum_{n=0}^{\infty} P_n(x) t^n \]
Legendre Polynomial

The generating function (example)

- Consider an electric charge $q$ placed on the $z$-axis at $z = 0$. The electrostatic potential of the charge at point $P$ is
  \[ \phi = k \frac{q}{|\vec{R}|} \]

- If we express the electrostatic potential in terms of spherical polar coordinates.

\[ \frac{1}{|\vec{R}|} = (r^2 - 2ar \cos(\theta) + a^2)^{-1/2} = \frac{1}{r} \left(1 - 2 \frac{a}{r} \cos(\theta) + \left(\frac{a}{r}\right)^2\right)^{-1/2} \]

In case of $\frac{a}{r} < 1$ we can express the electrostatic potential as a series of Legendre polynomial

\[ \phi = kq \sum_{n=0}^{\infty} P_n(\cos(\theta)) \left(\frac{a}{r}\right)^n \]
Legendre Polynomials

The recurrence function

the Legendre polynomials obey the three term recurrence relations:

\[(2n + 1)xP_n(x) = (n + 1)P_{n+1}(x) + nP_{n-1}(x)\]

\[P'_{n+1}(x) - P'_{n-1}(x) = (2n + 1)P_n(x)\]

\[(1 - x^2)P'_n(x) = nP_{n-1}(x) - nxP_n(x)\]

But these relations are valid for \(n = 1, 2, 3, \ldots\)
Legendre Polynomials

Special Properties

• Some special values

\[ P_n(1) = 1 \]
\[ P_n(-1) = (-1)^n \]

and

\[ P_{2n}(0) = (-1)^n \frac{(2n-1)!!}{(2n)!!} \]
\[ P_{2n+1}(0) = 0 \]

for \( n = 0, 1, 2, \ldots \)

Where \( n!! = \begin{cases} n(n-2)(n-4) \ldots 1 & \text{if } n \text{ is odd} \\ n(n-2)(n-4) \ldots 2 & \text{if } n \text{ is even} \\ 1 & \text{if } n = 0 \end{cases} \) (called double factorial)
Legendre Polynomials

Special Properties

• The Parity property: (with respect to $x = 0$, $\theta = \pi/2$)

$$P_n(-x) = (-1)^nP_n(x)$$
$$P_n(\cos(\pi - \theta)) = (-1)^nP_n(\cos(\theta))$$

If $n$ is odd the parity of the polynomial is odd, but if it is even the parity of the polynomial is even.

• Upper and lower Bounds for $P_n(\cos(\theta))$

$$|P_n(\cos(\theta))| \leq P_n(1) = 1$$
Legendre Polynomials

Orthogonality

* Legendre’s equation is self-adjoint. Which satisfies Sturm-Liouville theory where the solutions are expected to be orthogonal to satisfying certain boundary conditions. Legendre polynomials are a set of orthogonal functions on (-1,1).

\[
\int_{-1}^{1} P_n(x)P_m(x)dx = \frac{2}{2n + 1}\delta_{n,m}
\]

where

\[
\delta_{n,m} = \begin{cases} 
0 & \text{if } n \neq m \\
1 & \text{if } n = m 
\end{cases}
\]
Legendre Polynomials

**Legendre Series**

- According to Sturm-Liouville theory that Legendre polynomial form a complete set.

\[
f(x) = \sum_{n=0}^{\infty} a_n P_n(x)
\]

- The coefficients \(a_n\) are obtained by multiplying the series by \(P_m(x)\) and integrating in the interval \([-1,1]\)

\[
a_n = \frac{2m + 1}{2} \int_{-1}^{1} f(x)P_m(x)dx = 0
\]
Legendre Polynomials

Potential Applications

• Electrostatic potential, and gravitational potential involving spherical symmetry.

\[ U(\vec{r}_1 - \vec{r}_2) \propto \frac{1}{|\vec{r}_1 - \vec{r}_2|} = \frac{1}{r_>} \sum_{n=0}^{\infty} \left( \frac{r_<}{r_>} \right)^n P_n(\cos(\theta)) \]

where

\[ r_> = |r_1| \] for \( |r_1| > |r_2| \) and \( r_< = |r_2| \) for \( |r_2| > |r_1| \)
Legendre Polynomials

Alternate Definitions of Legendre Polynomials

• Schaefli Integral:

Schaefli integral is obtained by using Cauchy’s integral

\[ f(z) = \frac{1}{2\pi i} \oint \frac{f(t)}{t - z} dt \quad \text{with} \quad f(z) = (z^2 - 1)^n \]

by differentiating \( f(z) \) \( n \) times with respect to \( z \) and multiplying by \( 1/2^n n! \), then we can get

\[ P_n(z) = \frac{2^{-n}}{2\pi i} \oint \frac{(t^2 - 1)^n}{(t - z)^{n+1}} dt \]

with the contour enclosing the point \( t = z \). Schaefli integral is useful to define \( P_v(z) \) for nonintegral \( v \).
Associated Legendre Functions

If Laplace’s or Helmholtz’s equations are separated in spherical polar coordinates we get associated Legendre equation (with $x = \cos\theta$):

$$\frac{d}{dx} \left[ (1 - x^2) \frac{d P_n^m(x)}{dx} \right] + \left[ n(n + 1) - \frac{m^2}{1 - x^2} \right] P_n^m(x) = 0$$

The new integer $m$ associated with another polar angle $\phi$. The solutions of this equation are called associated Legendre functions

$$P_n^m(x) = (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_n(x)$$

If the separation only has the azimuthal part then $m = 0$, the equation become Legendre equation.
Associated Legendre Functions

**Properties**

- Since the highest power of $x$ in $P_n(x)$ is $x^n$ then $m \leq n$ (or the $m$-fold differentiation will drive our function to zero).
- The integer $m$ can be negative or positive and take the values $-n \leq m \leq n$.
- The relation between $P_n^m(x)$ and $P_n^{-m}(x)$ is given by
  \[
  P_n^{-m}(x) = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m(x)
  \]
- The generating function for the associated Legendre function
  \[
  \frac{(2m)! (1-x^2)^{m/2}}{2^m m! (1-2tx+t^2)^{m+1/2}} = \sum_{s=0}^{\infty} P_{s+m}^m(x) t^s
  \]
Associated Legendre Functions

Recurrence Relations

Because of the existence of two indices, we have wide variety of recurrence relation:

$$p_{n}^{m+1} - \frac{2mx}{(1-x^2)^{1/2}} p_{n}^{m} + [n(n+1) - m(m-1)]p_{n}^{m+1} = 0$$

$$(2n+1)xP_{n}^{m} = (n+m)P_{n-1}^{m} + (n-m+1)P_{n+1}^{m}$$

$$(2n+1)(1-x^2)^{1/2}P_{n}^{m} = P_{n+1}^{m+1} - P_{n-1}^{m+1}$$

$$= (n+m)(n+m-1)P_{n-1}^{m-1} + (n-m+1)(n-m+2)P_{n+1}^{m-1}$$

$$(1-x^2)^{1/2}P_{n}^{m'} = \frac{1}{2}P_{n}^{m+1} - \frac{1}{2}(n+m)(n-m+1)P_{n}^{m-1}$$
Associated Legendre Functions

Parity

• Associated Legendre function satisfies the parity relation as $x \to -x$ about the origin. Because $P_n^m$ $m$-fold differentiation yields a factor of $(-1)^m$:

$$P_n^m(-x) = (-1)^{n+m}P_n^m(x)$$

• From the parity property and the definition of $P_n^m(x)$ we can deduce that:

$$P_n^m(\pm 1) = 0 \quad \text{for } m \neq 0$$
Associated Legendre Functions

Orthogonality

• The orthogonality of associated Legendre function depends on the values of two indices $n$ and $m$:

• If $m$ is the same for both functions:

$$\int_{-1}^{1} P_p^m(x) P_q^m(x) \, dx = \frac{2}{2q + 1} \cdot \frac{(q + m)!}{(q - m)!} \delta_{p,q}$$

• If $n$ is the same for both functions:

$$\int_{-1}^{1} P_n^m(x) P_n^k(x)(1 - x^2)^{-1} \, dx = \frac{(n + m)!}{m(n - m)!} \delta_{m,k}$$

where $(1 - x^2)^{-1}$ is the weighting factor.
Associated Legendre Functions

**Spherical Harmonics**

- The Schrödinger equation for central force
  \[ \nabla^2 \psi + k^2 f(x) \psi = 0 \]

if use the spherical polar coordinates

\[
\left[ \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{\mathbf{l}^2}{r^2} + k^2 \right] \psi(r, \theta, \phi) = 0
\]

If the wavefunction expressed as \( \psi(r, \theta, \phi) = R(r)Y_n^m(\theta, \phi) \) then

\[
-\mathbf{l}^2 Y_n^m(\theta, \phi) = \left[ \frac{1}{\sin(\theta)} \frac{d}{d\theta} \left( \sin(\theta) \frac{d}{d\theta} \right) + \frac{1}{\sin^2(\theta)} \frac{d^2}{d\phi^2} \right] Y_n^m(\theta, \phi)
\]

\[ = -n(n + 1)Y_n^m(\theta, \phi) \]

Where \( Y_n^m(\theta, \phi) \) called *Spherical Harmonics* and \( \mathbf{l} \) the angular momentum operator.
The spherical harmonics are product of two functions, a function depend on the azimuthal angle $\phi$:

$$\Phi(\phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}$$

and another function depends on polar angle $\theta$:

$$\mathcal{P}_n^m(\cos(\theta)) = \sqrt{\frac{2n + 1 (n - m)!}{2 (n + m)!}} P_n^m(\cos(\theta))$$

where the spherical harmonics defined as

$$Y_n^m(\theta, \phi) = (-1)^m \sqrt{\frac{2n + 1 (n - m)!}{4\pi (n + m)!}} P_n^m(\cos(\theta)) e^{im\phi}$$

Where the factor $(-1)^m$. 
Associated Legendre Functions

Spherical Harmonics (Orthogonality)

- The functions $\Phi(\varphi)$ are orthogonal with respect to the azimuthal angle $\varphi$.

- The functions $\varphi_n^m(\cos(\theta))$ are orthogonal with respect to the polar angle $\theta$.

- The spherical harmonic functions of two angles are orthogonal such that

$$\int_{\varphi=0}^{2\pi} \int_{\theta=0}^{\pi} Y_{n_1}^{m_1*}(\theta, \varphi) Y_{n_2}^{m_2}(\theta, \varphi) \sin(\theta) d\theta d\varphi = \delta_{n_1, n_2} \delta_{m_1, m_2}$$
Associated Legendre Functions

*Spherical Harmonics (Laplace Series)*

- Spherical Harmonics form a complete set. Means that any function $f(\theta, \varphi)$ evaluated over the sphere can be expanded in a uniformly convergent double series of spherical harmonics:

\[
f(\theta, \varphi) = \sum_{m,n} a_{mn} Y_n^m(\theta, \varphi)
\]