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LONG-TIME DECAY OF LERAY SOLUTION OF 3D-NSE WITH EXPONENTIAL DAMPING

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Abstract

We study the uniqueness, the continuity in L^2 and the large time decay for the Leray solutions of the 3D incompressible Navier–Stokes equations with nonlinear exponential damping term $a(e^{b|u|^4} - 1)u$, $(a, b > 0)$.

Keywords: Navier–Stokes Equations; Friedrich Method; Global Weak Solution.

1. INTRODUCTION

In this paper, we investigate the questions of the existence, uniqueness and asymptotic study of global weak solution to the modified incompressible Navier–Stokes equations in $\mathbb{R}^3$

$$(S)\begin{cases}\partial_t u - \nu\Delta u + u.\nabla u & \text{in } \mathbb{R}^+ \times \mathbb{R}^3,\\ \quad + a(e^{b|u|^4} - 1)u & \\ \quad = -\nabla p & \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}^3,\\ u(0,x) = u^0(x) & \text{in } \mathbb{R}^3,\\ a, b > 0 & \end{cases}$$

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where $u = u(t,x) = (u_1, u_2, u_3)$, $p = p(t,x)$ denote, respectively, the unknown velocity and the unknown pressure of the fluid at the point $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^3$, the viscosity of fluid $\nu > 0$ and $u^0 = (u_1^0(x), u_2^0(x), u_3^0(x))$ is the initial given velocity. The damping is from the resistance to the motion of the flow. It describes various physical situations such as porous media flow, drag or friction effects, and some dissipative mechanisms (see Refs. 1–4 and references therein). The fact that $\operatorname{div} u = 0$, allows to write the term $(u.\nabla u) := u_1\partial_1 u + u_2\partial_2 u + u_3\partial_3 u$ in the following form $\operatorname{div}(u \otimes u) := (\operatorname{div}(u_1 u), \operatorname{div}(u_2 u), \operatorname{div}(u_3 u))$. If the initial velocity u^0 is quite regular, the divergence-free condition determines the pressure p.

Without loss of generality and in order to simplify the proofs of our results, we consider the viscosity unitary ($\nu = 1$).

The global existence of weak solution of initial value problem of classical incompressible Navier–Stokes were proved by Leray and Hopf (see Refs. 5 and 6) long before. Uniqueness remains an open problem for the dimensions $d \geq 3$.

The polynomial damping $\alpha|u|^{\beta-1}u$ is studied in Ref. 7 by Cai and Jiu, where they proved the global existence of weak solution in

$$L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3)) \cap L^{\beta+1}(\mathbb{R}^+, L^{\beta+1}(\mathbb{R}^3)).$$

The exponential damping $a(e^{b|u|^2} - 1)u$ is studied in Ref. 8 by Benameur, where he proved the global existence of weak solution in

$$L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3)) \cap \mathcal{E}_b,$$

where $\mathcal{E}_b = \{f \in \mathcal{M}(\mathbb{R}^+ \times \mathbb{R}^3) : (e^{b|f|^2} - 1)|f|^2 \in L^1(\mathbb{R}^+ \times \mathbb{R}^3)\}$ and $\mathcal{M}(\mathbb{R}^+ \times \mathbb{R}^3)$ the space of measurable functions on $\mathbb{R}^+ \times \mathbb{R}^3$.

The purpose of this paper is to study the well-posedness and the asymptotic study of the incompressible Navier–Stokes equations with exponential damping $a(e^{b|u|^4} - 1)u$. We will show that the Cauchy problem (S) has a global weak solutions for any $a, b \in (0, \infty)$. We apply the Friedrich method to construct the approximate solutions and make more delicate estimates to proceed to compactness arguments. In particular, we obtain new more *a priori* estimates

$$\|u(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(s)\|_{L^2}^2 ds + 2a\int_0^t \|(e^{b|u(s)|^4} - 1)|u(s)|^2\|_{L^1} ds \leq \|u^0\|_{L^2}^2,$$

comparing with the Navier–Stokes equations, to guarantee that the solution u belongs to

$$L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3)) \cap \mathcal{F}_b,$$

where $\mathcal{F}_b = \{f \in \mathcal{M}(\mathbb{R}^+ \times \mathbb{R}^3) : (e^{b|f|^4} - 1)|f|^2 \in L^1(\mathbb{R}^+ \times \mathbb{R}^3)\}$.

To prove the uniqueness we use an energy method and the approximates systems. The proof of the asymptotic study is based on a decomposition of the solution in high and low frequencies and the uniqueness of such solution in a well chosen time t_0.

In our case of exponential damping, we are trying to find more regularity of Leray solution in $\cap_p L^p(\mathbb{R}^+, L^p(\mathbb{R}^3))$. In particular, we give a new energy estimate. Our main result is the following:

Theorem 1. *Let $u^0 \in L^2(\mathbb{R}^3)$ be a divergence-free vector fields, then there is a unique global solution of the system (S) : $u \in C_b(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3)) \cap \mathcal{E}_b$. Moreover, for all $t \geq 0$*

$$\|u(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(s)\|_{L^2}^2 ds + 2a\int_0^t \|(e^{b|u(s)|^4} - 1)|u(s)|^2\|_{L^1} ds \leq \|u^0\|_{L^2}^2.$$

Moreover, we have

$$\limsup_{t\to\infty} \|u(t)\|_{L^2} = 0. \tag{1}$$

Remark 2. (1) The new results in this theorem is the uniqueness of the global solution, the continuity of the solution in the $L^2(\mathbb{R}^3)$ space and the asymptotic behavior at infinity.

(2) Generally, for $\mathbf{r} \geq 1$ the following problem is as follows:

$$(P_\mathbf{r}) \begin{cases} \partial_t u - \nu\Delta u + u.\nabla u + a(e^{b|u|^\mathbf{r}} - 1)u = -\nabla p & \text{in } \mathbb{R}^+ \times \mathbb{R}^3, \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}^3, \\ u(0,x) = u^0(x) & \text{in } \mathbb{R}^3, \\ a, b > 0 \end{cases}$$

and by adapting the same proof of result of Ref. 8, we show the global existence of such a solution in $C_b(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3)) \cap \mathcal{E}_b^\mathbf{r}$, where

$$\mathcal{E}_b^\mathbf{r} = \{f \in \mathcal{M}(\mathbb{R}^+ \times \mathbb{R}^3) : (e^{b|f|^\mathbf{r}} - 1)|f|^2$$

$\in L^1(\mathbb{R}^+ \times \mathbb{R}^3)\}$.

Moreover, we get

$$\|u(t)\|_{L^2}^2 + 2\int_0^t \|\nabla u(s)\|_{L^2}^2 ds + 2a\int_0^t \|(e^{b|u(s)|^{\mathbf{r}}} - 1)|u(s)|^2\|_{L^1} ds \leq \|u^0\|_{L^2}^2.$$

The asymptotic result (1) is true for all $\mathbf{r} \geq \frac{7}{3}$, and the index $\frac{7}{3}$ is a critical technical condition (See (13)–(14)).

(3) With the same arguments, we can study the following modified Navier–Stokes equations

$$(S_3)\ \begin{cases} \partial_t u + \nu(-\Delta)^\delta u + u.\nabla u & \text{in } \mathbb{R}^+ \times \mathbb{R}^3, \\ \quad + a(e^{b|u|^r} - 1)u = -\nabla p & \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^+ \times \mathbb{R}^3, \\ u(0,x) = u^0(x) & \text{in } \mathbb{R}^3, \\ a, b > 0 & \end{cases}$$

where r, δ well chosen.

(4) The system studied in this paper can still be treated in any dimension. In particular, in one dimension and taking inspiration from Ref. 9, we can study the following fractal case:

$$\begin{cases} \partial_t u + \dfrac{1}{2}\Gamma_{\mathcal{H}} u - \nu\Delta_1 u \\ \quad + a(e^{b|u|^r} - 1)u + \partial p = 0, \\ \partial^* u = 0. \end{cases}$$

2. NOTATIONS AND PRELIMINARY RESULTS

For a function $f\colon \mathbb{R}^3 \to \bar{\mathbb{R}}$ and $R > 0$, the Friedrich operator J_R is defined by $J_R(D)f = \mathcal{F}^{-1}(\chi_{B_R}\widehat{f})$, where B_R is the ball of center 0 and radius R. If $L^2_\sigma(\mathbb{R}^3)$ denotes the space of divergence-free vector fields in $L^2(\mathbb{R}^3)$, the Leray projector $\mathbb{P}\colon (L^2(\mathbb{R}^3))^3 \to (L^2_\sigma(\mathbb{R}^3))^3$ is defined by

$$\mathcal{F}(\mathbb{P}f) = \widehat{f}(\xi) - \left(\widehat{f}(\xi).\frac{\xi}{|\xi|}\right)\frac{\xi}{|\xi|} = M(\xi)\widehat{f}(\xi),$$

where $M(\xi)$ is the matrix $(\delta_{k,\ell} - \frac{\xi_k\xi_\ell}{|\xi|^2})_{1\leq k,\ell\leq 3}$. If $u \in \mathcal{S}(\mathbb{R}^3)^3$,

$$\mathbb{P}(u)_k(x) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{\mathbb{R}^3}\left(\delta_{kj} - \frac{\xi_k\xi_j}{|\xi|^2}\right)\widehat{u}_j(\xi)e^{i\xi\cdot x}\,d\xi,$$

where $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space. Define also the operator $A_R(D)$ on $L^2(\mathbb{R}^3)$ by

$$A_R(D)u = \mathbb{P}J_R(D)u = \mathcal{F}^{-1}(M(\xi)\chi_{B_R}(\xi)\widehat{u}).$$

To simplify the exposition of the main result, we first collect some preliminary results and we give some new technical lemmas.

Proposition 3 (Ref. 10). *Let H be a Hilbert space.*

(1) *The unit ball is weakly compact, that is, if (x_n) is a bounded sequence in H, then there is a subsequence $(x_{\varphi(n)})$ such that*

$$(x_{\varphi(n)}|y) \to (x|y), \quad \forall\, y \in H.$$

(2) *If $x \in H$ and (x_n) a bounded sequence in H such that $\lim_{n\to+\infty}(x_n|y) = (x|y)$, for all $y \in H$, then $\|x\| \leq \liminf_{n\to\infty}\|x_n\|$.*

(3) *If $x \in H$ and (x_n) is a bounded sequence in H such that $\lim_{n\to+\infty}(x_n|y) = (x|y)$, for all $y \in H$ and $\limsup_{n\to\infty}\|x_n\| \leq \|x\|$, then $\lim_{n\to\infty}\|x_n - x\| = 0$.*

We recall the following product law in the homogeneous Sobolev spaces:

Lemma 4 (Ref. 11). *Let s_1, s_2 be two real numbers and $d \in \mathbb{N}$.*

(1) *If $s_1 < \frac{d}{2}$ and $s_1+s_2 > 0$, there exists a constant $C_1 = C_1(d, s_1, s_2)$, such that: if $f, g \in \dot{H}^{s_1}(\mathbb{R}^d) \cap \dot{H}^{s_2}(\mathbb{R}^d)$, then $f.g \in \dot{H}^{s_1+s_2-\frac{d}{2}}(\mathbb{R}^d)$ and*

$$\|fg\|_{\dot{H}^{s_1+s_2-\frac{d}{2}}} \leq C_1(\|f\|_{\dot{H}^{s_1}}\|g\|_{\dot{H}^{s_2}} + \|f\|_{\dot{H}^{s_2}}\|g\|_{\dot{H}^{s_1}}).$$

(2) *If $s_1, s_2 < \frac{d}{2}$ and $s_1 + s_2 > 0$ there exists a constant $C_2 = C_2(d, s_1, s_2)$ such that if $f \in \dot{H}^{s_1}(\mathbb{R}^d)$ and $g \in \dot{H}^{s_2}(\mathbb{R}^d)$, then $f.g \in \dot{H}^{s_1+s_2-\frac{d}{2}}(\mathbb{R}^d)$ and*

$$\|fg\|_{\dot{H}^{s_1+s_2-\frac{d}{2}}} \leq C_2\|f\|_{\dot{H}^{s_1}}\|g\|_{\dot{H}^{s_2}}.$$

Lemma 5. *Let $\beta > 0$ and $d \in \mathbb{N}$. Then, for all $x, y \in \mathbb{R}^d$, we have*

$$\langle |x|^\beta x - |y|^\beta y, x - y\rangle \geq \frac{1}{2}(|x|^\beta + |y|^\beta)|x-y|^2, \qquad (2)$$

and, for $\mathbf{r} > 0$, we have

$$\langle (e^{b|x|^{\mathbf{r}}} - 1)x - (e^{b|y|^{\mathbf{r}}} - 1)y, x - y\rangle \geq \frac{1}{2}((e^{b|x|^{\mathbf{r}}} - 1) + (e^{b|y|^{\mathbf{r}}} - 1)x)|x-y|^2. \qquad (3)$$

Proof. Suppose that $|x| > |y| > 0$. For $u > v > 0$, we have

$$2\langle ux - vy, x - y\rangle - (u + v)|x - y|^2 = (u - v)(|x|^2 - |y|^2) \geq 0. \quad (4)$$

It suffices to take $u = |x|^\beta$ and $v = |y|^\beta$, we get the inequality (2).

Suppose that $|x| > |y| > 0$. In use of the inequality (4) with $u = (e^{b|x|^{\mathbf{r}}} - 1)$ and $v = (e^{b|y|^{\mathbf{r}}} - 1)$, we get

$$2\langle (e^{b|x|^{\mathbf{r}}} - 1)x - (e^{b|y|^{\mathbf{r}}} - 1)y, x - y\rangle - ((e^{b|x|^{\mathbf{r}}} - 1) + (e^{b|y|^{\mathbf{r}}} - 1))|x - y|^2 = (e^{b|x|^{\mathbf{r}}} - e^{b|y|^{\mathbf{r}}})|x - y|^2 \geq 0.$$

This proves the inequality (3).

The following result is a generalization of Proposition 3.1 in Ref. 8. □

Proposition 6. *Let $\nu_1, \nu_2, \nu_3 \in [0, \infty)$, $r_1, r_2, r_3 \in (0, \infty)$ and $f^0 \in L^2_\sigma(\mathbb{R}^3)$. For $n \in \mathbb{N}$, let $F_n : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}^3$ be a measurable function in $C^1(\mathbb{R}^+, L^2(\mathbb{R}^3))$ such that*

$$A_n(D)F_n = F_n, \quad F_n(0, x) = A_n(D)f^0(x)$$

and

(E1) $\partial_t F_n + \sum_{k=1}^3 \nu_k |D_k|^{2r_k} F_n + A_n(D)\mathrm{div}\,(F_n \otimes F_n) + A_n(D)h(|F_n|)F_n = 0,$

(E2)

$$\|F_n(t,.)\|^2_{L^2} + 2\sum_{k=1}^3 \nu_k \int_0^t \||D_k|^{r_k} F_n(s,.)\|^2_{L^2} ds + 2a\int_0^t \|h(|F_n(s,.)|)|F_n(s,.)|^2\|_{L^1} ds \leq \|f^0\|^2_{L^2},$$

where $h(z) = a(e^{bz^{\mathbf{r}}} - 1)$, with $\mathbf{r} \geq 1$ and $a, b > 0$. Then for every $\varepsilon > 0$ there is $\delta = \delta(\varepsilon, a, b, \nu_1, \nu_2, \nu_3, r_1, r_2, r_3, \|f^0\|_{L^2}) > 0$ such that for all $t_1, t_2 \in \mathbb{R}^+$, we have

$$(|t_2 - t_1| < \delta \Rightarrow \|F_n(t_2) - F_n(t_1)\|_{H^{-s_0}} < \varepsilon), \quad \forall\, n \in \mathbb{N}, \quad (5)$$

with $s_0 \geq \max(3, 2r_1, 2r_2, 2r_3)$.

Proof. Integrate $(E1)$ on the interval $[t_1, t_2] \subset \mathbb{R}^+$ and take the inner product in H^{-s_0}, we get

$$\|F_n(t_2) - F_n(t_1)\|_{H^{-s_0}} \leq \int_{t_1}^{t_2} \sum_{k=1}^3 \nu_k \||D_k|^{2r_k} F_n(t)\|_{H^{-s_0}} dt + \int_{t_1}^{t_2} \|A_n(D)\mathrm{div}\,(F_n \otimes F_n)(t)\|_{H^{-s_0}} dt + \int_{t_1}^{t_2} \|A_n(D)h(|F_n|)F_n(t)\|_{H^{-s_0}} dt.$$

Let

$$I_{1,n}(t_1, t_2) = \int_{t_1}^{t_2} \sum_{k=1}^3 \nu_k \||D_k|^{2r_k} F_n(t)\|_{H^{-s_0}} dt,$$

$$I_{2,n}(t_1, t_2) = \int_{t_1}^{t_2} \|A_n(D)\mathrm{div}\,(F_n \otimes F_n)(t)\|_{H^{-s_0}} dt,$$

and

$$I_{3,n}(t_1, t_2) = \int_{t_1}^{t_2} \|A_n(D)h(|F_n|)F_n(t)\|_{H^{-s_0}} dt.$$

We have

$$I_{1,n}(t_1, t_2) \leq \sum_{k=1}^3 \nu_k \int_{t_1}^{t_2} \|F_n(t)\|_{H^{2r_k - s_0}} dt \underset{2r_k - s_2 \leq 0}{\leq} \left(\sum_{k=1}^3 \nu_k\right) \int_{t_1}^{t_2} \|F_n(t)\|_{L^2} dt \leq \left(\sum_{k=1}^3 \nu_k\right) \|f^0\|_{L^2}(t_2 - t_1), \quad (6)$$

$$I_{2,n}(t_1, t_2) = \int_{t_1}^{t_2} \|A_n(D)\mathrm{div}\,(F_n \otimes F_n)(t)\|_{H^{-s_0}} dt \leq \int_{t_1}^{t_2} \|\mathrm{div}\,(F_n \otimes F_n)(s)\|_{H^{-s_2}} dt \leq \int_{t_1}^{t_2} \|(F_n \otimes F_n)(t)\|_{H^{-s_0+1}} dt \leq \int_{t_1}^{t_2} \|(F_n \otimes F_n)(t)\|_{H^{-2}} dt.$$

□

Recall that if $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ is an integrable function, then for all $s > \frac{3}{2}$,

$$\|f\|^2_{H^{-s}} = \int_{\mathbb{R}^3} (1 + |\xi|^2)^{-s} |\widehat{f}(\xi)|^2 d\xi \leq \left(\int_{\mathbb{R}^3} (1 + |\xi|^2)^{-s} d\xi\right) \|\widehat{f}\|^2_\infty$$

$$\le \left(\int_{\mathbb{R}^3}(1+|\xi|^2)^{-s}d\xi\right)\|f\|^2_{L^1}.$$

We deduce that

$$\|f\|^2_{H^{-s}} \le \left(\int_{\mathbb{R}^3}(1+|\xi|^2)^{-s}d\xi\right)\|f\|^2_{L^1} \quad (7)$$

and there exists $C>0$ such that

$$\begin{aligned} I_{2,n}(t_1,t_2) &\le C\int_{t_1}^{t_2}\|(F_n\otimes F_n)(t)\|_{L^1}dt \\ &\le C\int_{t_1}^{t_2}\|F_n(t)\|^2_{L^2}dt \\ &\le C(t_2-t_1)\|f^0\|^2_{L^2}. \end{aligned} \quad (8)$$

To estimate the integral $I_{3,n}(t_1,t_2)$, consider for $R>1$ the sub-level sets

$$X_n(R,t)=\{x\in\mathbb{R}^3 : |F_n(t,x)|\le R\}.$$

We remark that, for all $x\in X_n(R,t)$

$$(e^{b|F_n(t,x)|^{\mathbf{r}}}-1)|F_n(t,x)| \le \left(\frac{e^{bR^{\mathbf{r}}}-1}{R}\right)|F_n(t,x)|^2.$$

Let $M(R)=\frac{e^{bR^{\mathbf{r}}}-1}{R}$. From (7), there exists $C_1,C_2,C_3>0$ such that

$$\begin{aligned} I_{3,n}(t_1,t_2) &= \int_{t_1}^{t_2}\|A_n(D)h(|F_n|)F_n(t)\|_{H^{-s_0}}dt \\ &\le C_1\int_{t_1}^{t_2}\|h(|F_n|)F_n(t)\|_{L^1}dt \\ &\le C_1\int_{t_1}^{t_2}\int_{X_n(R,t)}h(|F_n|)|F_n|dxdt \\ &\quad + C_1\int_{t_1}^{t_2}\int_{X_n(R,t)^c}h(|F_n|)|F_n|dxdt \\ &\le C_2\int_{t_1}^{t_2}\int_{X_n(R,t)}|F_n|^2dxdt \\ &\quad + \frac{C_1}{R}\int_{t_1}^{t_2}\int_{X_n(R,t)^c}h(|F_n|)|F_n|^2dxdt \\ &\le C_2M(R)\int_{t_1}^{t_2}\|F_n(t)\|_2^2dt \\ &\quad + \frac{C_3}{R}\int_{t_1}^{t_2}\|h(|F_n|)|F_n|^2\|_{L^1}dt \\ &\le C_2M(R)\|f^0\|_2^2(t_2-t_1)+\frac{C_3}{R}\|f^0\|_2^2. \end{aligned}$$

Hence

$$I_{3,n}(t_1,t_2)\le C_2M(R)\|f^0\|_2^2(t_2-t_1)+\frac{C_3}{R}\|f^0\|_2^2. \quad (9)$$

Now using the inequalities (6), (8) and (9), for $\varepsilon>0$, consider R such that $\frac{C_3}{R}\|f^0\|_2^2<\frac{\varepsilon}{4}$ and

$$0<\delta<\min\left(\frac{\varepsilon}{4((\sum_{k=1}^3\nu_k)\|f^0\|_{L^2}+1)}, \frac{\varepsilon}{4(C\|f^0\|_2^2+1)}, \frac{\varepsilon}{4(C_2M(R)\|f^0\|_2^2+1)}.\right).$$

For such δ, we get (5).

3. PROOF OF THE MAIN THEOREM (1)

The proof is given in four steps:

3.1. Existence of Weak Solution

In this step, we build approximate solutions of the system (S) inspired by the method used in Refs. 8 and 11, hence we construct a global solution. For this, consider the approximate system with parameter $n\in\mathbb{N}$

$$(S_n)\begin{cases} \partial_t u-\Delta J_n u+J_n(J_n u.\nabla J_n u) & \text{in } \mathbb{R}^+\times\mathbb{R}^3, \\ \quad +aJ_n[(e^{b|J_n u|^4}-1)J_n u] & \\ \quad =-\nabla p_n & \\ p_n=(-\Delta)^{-1}(\mathrm{div}\, J_n & \text{in } \mathbb{R}^+\times\mathbb{R}^3, \\ \quad\times(J_n u.\nabla J_n u) & \\ \quad + a\mathrm{div}\, J_n[(e^{b|J_n u|^2}-1) & \\ \quad\times J_n u])\mathrm{div}\, u=0 & \\ u(0,x)=J_n u^0(x) & \text{in } \mathbb{R}^3. \end{cases}$$

J_n is the Friedrich operator defined in the second section.

- By Cauchy–Lipschitz theorem, we obtain a unique solution $u_n\in C^1(\mathbb{R}^+,L^2_\sigma(\mathbb{R}^3))$ of $(S_{2,n})$. Moreover, $J_n u_n=u_n$ such that

$$\|u_n(t)\|^2_{L^2}+2\int_0^t\|\nabla u_n\|^2_{L^2} + 2a\int_0^t\|(e^{b|u_n|^4}-1)|u_n|^2\|_{L^1} \le \|u^0\|^2_{L^2}. \quad (10)$$

- The sequence $(u_n)_n$ is bounded in $L^\infty(\mathbb{R}^+,L^2(\mathbb{R}^3))$ and on $L^2(\mathbb{R}^+,\dot{H}^1(\mathbb{R}^3))$. Using Proposition 6 and the interpolation method, we deduce that the sequence $(u_n)_n$ is equicontinuous on $H^{-1}(\mathbb{R}^3)$.

- Let $(T_q)_q$ be a strictly increasing sequence such that $\lim_{q\to+\infty} T_q = \infty$. Consider a sequence of functions $(\theta_q)_q$ in $C_0^\infty(\mathbb{R}^3)$ such that
$$\begin{cases}\theta_q(x)=1 & \text{for } |x|\le q+\dfrac{5}{4},\\ \theta_q(x)=0 & \text{for } |x|\ge q+2,\\ 0\le\theta_q\le 1.\end{cases}$$
Using (10), the equicontinuity of the sequence $(u_n)_n$ on $H^{-1}(\mathbb{R}^3)$ and classical argument by combining Ascoli's theorem and the Cantor diagonal process, there exists a subsequence $(u_{\varphi(n)})_n$ and $u\in L^\infty(\mathbb{R}^+,L^2(\mathbb{R}^3))\cap C(\mathbb{R}^+,H^{-3}(\mathbb{R}^3))$ such that for all $q\in\mathbb{N}$,
$$\lim_{n\to\infty}\|\theta_q(u_{\varphi(n)}-u)\|_{L^\infty([0,T_q],H^{-4})}=0. \quad (11)$$
In particular, the sequence $(u_{\varphi(n)}(t))_n$ converges weakly in $L^2(\mathbb{R}^3)$ to $u(t)$ for all $t\ge 0$.
- Combining the above inequalities, we obtain
$$\|u(t)\|_{L^2}^2+2\int_0^t\|\nabla u(s)\|_{L^2}^2ds + 2a\int_0^t\|(e^{b|u(s)|^4}-1)|u(s)|^2\|_{L^1}ds \le \|u^0\|_{L^2}^2,$$
for all $t\ge 0$.
- u is a solution of the system (S).

3.2. Continuity of the Solution in L^2

In this section, we give a simple proof of the continuity of the solution u of the system (S) and we prove also that $u\in C(\mathbb{R}^+,L^2(\mathbb{R}^3))$. The construction of the solution is based on the Friedrich approximation method. We point out that we can use this method to show the same results as in Ref. 4.

- By inequality (12), we get
$$\limsup_{t\to 0}\|u(t)\|_{L^2}\le\|u^0\|_{L^2}.$$

Then, Proposition 3-(3) implies that
$$\limsup_{t\to 0}\|u(t)-u^0\|_{L^2}=0,$$
which ensures the continuity of u at 0.

- Consider the functions
$$v_{n,\varepsilon}(t,.)=u_{\varphi(n)}(t+\varepsilon,.),$$
$$p_{n,\varepsilon}(t,.)=p_{\varphi(n)}(t+\varepsilon,.),$$

for $n\in\mathbb{N}$ and $\varepsilon>0$. We have
$$\partial_t u_{\varphi(n)}-\Delta u_{\varphi(n)}+J_{\varphi(n)}(u_{\varphi(n)}.\nabla u_{\varphi(n)}) + aJ_{\varphi(n)}(e^{b|u_{\varphi(n)}|^4}-1)u_{\varphi(n)} = -\nabla p_{\varphi(n)},$$
$$\partial_t v_{n,\varepsilon}-\Delta v_{n,\varepsilon}+J_{\varphi(n)}(v_{n,\varepsilon}.\nabla v_{n,\varepsilon}) + aJ_{\varphi(n)}(e^{b|v_{n,\varepsilon}|^4}-1)v_{n,\varepsilon} = -\nabla p_{n,\varepsilon}.$$

The function $w_{n,\varepsilon}=u_{\varphi(n)}-v_{n,\varepsilon}$ fulfills the following:
$$\begin{aligned}\partial_t w_{n,\varepsilon}-\Delta w_{n,\varepsilon}&+aJ_{\varphi(n)}(e^{b|u_{\varphi(n)}|^4}-1)u_{\varphi(n)} - aJ_{\varphi(n)}(e^{b|v_{n,\varepsilon}|^4}-1)v_{n,\varepsilon}\\ &= -\nabla(p_{\varphi(n)}-p_{n,\varepsilon})+J_{\varphi(n)}(w_{n,\varepsilon}.\nabla w_{n,\varepsilon})\\ &\quad - J_{\varphi(n)}(w_{n,\varepsilon}.\nabla u_{\varphi(n)})-J_{\varphi(n)}(u_{\varphi(n)}.\nabla w_{n,\varepsilon}).\end{aligned}$$

Taking the scalar product in $L^2(\mathbb{R}^3)$ with $w_{n,\varepsilon}$ and using the properties div $w_{n,\varepsilon}=0$ and $\langle w_{n,\varepsilon}.\nabla w_{n,\varepsilon},w_{n,\varepsilon}\rangle=0$, we get
$$\begin{aligned}\frac12\frac{d}{dt}\|w_{n,\varepsilon}\|_{L^2}^2+\|\nabla w_{n,\varepsilon}\|_{L^2}^2 &+ a\langle J_{\varphi(n)}((e^{b|u_{\varphi(n)}|^4}-1)u_{\varphi(n)} - (e^{b|v_{n,\varepsilon}|^4}-1)v_{n,\varepsilon});w_{n,\varepsilon}\rangle_{L^2}\\ &= -\langle J_{\varphi(n)}(w_{n,\varepsilon}.\nabla u_{\varphi(n)});w_{n,\varepsilon}\rangle_{L^2}. \quad (12)\end{aligned}$$

Using inequality (3), we get
$$\begin{aligned}&\langle J_{\varphi(n)}((e^{b|u_{\varphi(n)}|^2}-1)u_{\varphi(n)} - (e^{b|v_{n,\varepsilon}|^4}-1)v_{n,\varepsilon});w_{n,\varepsilon}\rangle_{L^2}\\ &= \langle(e^{b|u_{\varphi(n)}|^4}-1)u_{\varphi(n)} - (e^{b|v_{n,\varepsilon}|^4}-1)v_{n,\varepsilon};J_{\varphi(n)}w_{n,\varepsilon}\rangle_{L^2}\\ &= \langle(e^{b|u_{\varphi(n)}|^4}-1)u_{\varphi(n)} - (e^{b|v_{n,\varepsilon}|^4}-1)v_{n,\varepsilon};w_{n,\varepsilon}\rangle_{L^2}\\ &\ge \frac12\int_{\mathbb{R}^3}((e^{b|u_{\varphi(n)}|^4}-1) + (e^{b|v_{n,\varepsilon}|^4}-1))|w_{n,\varepsilon}|^2\\ &\ge \frac12\int_{\mathbb{R}^3}(e^{b|u_{\varphi(n)}|^4}-1)|w_{n,\varepsilon}|^2\\ &\ge \frac{b}{2}\int_{\mathbb{R}^3}|u_{\varphi(n)}|^4|w_{n,\varepsilon}|^2,\end{aligned}$$
$$|\langle J_{\varphi(n)}(w_{n,\varepsilon}.\nabla u_{\varphi(n)});w_{n,\varepsilon}\rangle_{L^2}| \le \int_{\mathbb{R}^3}|w_{n,\varepsilon}|.|u_{\varphi(n)}|.|\nabla w_{n,\varepsilon}|$$

$$\leq \frac{1}{2}\int_{\mathbb{R}^3} |w_{n,\varepsilon}|^2|u_{\varphi(n)}|^2 + \frac{1}{2}\|\nabla w_{n,\varepsilon}\|_{L^2}^2.$$

Again by using the elementary inequality $xy \leq \frac{ab}{8}x^2 + \frac{2}{ab}y^2$, for $x, y \geq 0$, we get

$$|\langle J_{\varphi(n)}(w_{n,\varepsilon}.\nabla u_{\varphi(n)}); w_{n,\varepsilon}\rangle_{L^2}| \leq \frac{ab}{8}\int_{\mathbb{R}^3} |u_{\varphi(n)}|^4|w_{n,\varepsilon}|^2 + \frac{2}{ab}\|w_{n,\varepsilon}\|_{L^2}^2 + \frac{1}{2}\|\nabla w_{n,\varepsilon}\|_{L^2}^2.$$

Combining the identity (3) and the inequality (12), we get

$$\frac{1}{2}\frac{d}{dt}\|w_{n,\varepsilon}\|_{L^2}^2 + \frac{1}{2}\|\nabla w_{n,\varepsilon}\|_{L^2}^2 \leq \frac{2}{ab}\|w_{n,\varepsilon}\|_{L^2}^2.$$

By Gronwall's Lemma, we get

$$\|w_{n,\varepsilon}(t)\|_{L^2}^2 \leq \|w_{n,\varepsilon}(0)\|_{L^2}^2 e^{\frac{4t}{ab}}.$$

But

$$\|u_{\varphi(n)}(t+\varepsilon) - u_{\varphi(n)}(t)\|_{L^2}^2 \leq \|u_{\varphi(n)}(\varepsilon) - u_{\varphi(n)}(0)\|_{L^2}^2 e^{\frac{4t}{ab}}.$$

For $t_0 > 0$ and $\varepsilon \in (0, t_0)$, we have

$$\|u_{\varphi(n)}(t_0+\varepsilon) - u_{\varphi(n)}(t_0)\|_{L^2}^2 \leq \|u_{\varphi(n)}(\varepsilon) - u_{\varphi(n)}(0)\|_{L^2}^2 \exp\left(\frac{4t_0}{ab}\right),$$

$$\|u_{\varphi(n)}(t_0-\varepsilon) - u_{\varphi(n)}(t_0)\|_{L^2}^2 \leq \|u_{\varphi(n)}(\varepsilon) - u_{\varphi(n)}(0)\|_{L^2}^2 \exp\left(\frac{4t_0}{ab}\right).$$

So

$$\begin{aligned}\|u_{\varphi(n)}(\varepsilon) - u_{\varphi(n)}(0)\|_{L^2}^2 &= \|J_{\varphi(n)}u_{\varphi(n)}(\varepsilon) - J_{\varphi(n)}u_{\varphi(n)}(0)\|_{L^2}^2 \\ &= \|\chi_{\varphi(n)}(\widehat{u_{\varphi(n)}} - \widehat{u^0})\|_{\varphi(n)}^2 \\ &\leq \|u_{\varphi(n)}(\varepsilon) - u^0\|_{L^2}^2 \\ &\leq 2\|u^0\|_{L^2}^2 - 2\mathrm{Re}\langle u_{\varphi(n)}(\varepsilon), u^0\rangle.\end{aligned}$$

But $\lim_{n\to+\infty}\langle u_{\varphi(n)}(\varepsilon), u^0\rangle = \langle u(\varepsilon), u^0\rangle$. Hence

$$\liminf_{n\to\infty} \|u_{\varphi(n)}(\varepsilon) - u_{\varphi(n)}(0)\|_{L^2}^2 \leq 2\|u^0\|_{L^2}^2 - 2\mathrm{Re}\langle u(\varepsilon); u^0\rangle_{L^2}.$$

Moreover, for all $q, N \in \mathbb{N}$

$$\begin{aligned}\|J_N(\theta_q.(u_{\varphi(n)}(t_0 \pm \varepsilon) - u_{\varphi(n)}(t_0)))\|_{L^2}^2 &\leq \|\theta_q.(u_{\varphi(n)}(t_0 \pm \varepsilon) - u_{\varphi(n)}(t_0))\|_{L^2}^2 \\ &\leq \|u_{\varphi(n)}(t_0 \pm \varepsilon) - u_{\varphi(n)}(t_0)\|_{L^2}^2.\end{aligned}$$

Using (11) we get, for q big enough

$$\|J_N(\theta_q.(u(t_0 \pm \varepsilon) - u(t_0)))\|_{L^2}^2 \leq \liminf_{n\to\infty} \|u_{\varphi(n)}(t_0 \pm \varepsilon) - u_{\varphi(n)}(t_0)\|_{L^2}^2.$$

Then

$$\|J_N(\theta_q.(u(t_0 \pm \varepsilon) - u(t_0)))\|_{L^2}^2 \leq 2(\|u^0\|_{L^2}^2 - \mathrm{Re}\langle u(\varepsilon); u^0\rangle_{L^2}) \exp\left(\frac{4t_0}{ab}\right).$$

By applying the monotone convergence theorem in the order $N \to \infty$ and $q \to \infty$, we get

$$\|u(t_0 \pm \varepsilon, .) - u(t_0, .))\|_{L^2}^2 \leq 2(\|u^0\|_{L^2}^2 - \mathrm{Re}\langle u(\varepsilon); u^0\rangle_{L^2}) \exp\left(\frac{4t_0}{ab}\right).$$

Using the continuity at 0 and make $\varepsilon \to 0$, we get the continuity at t_0.

3.3. Uniqueness of the Solution

Let u, v be two solutions of (S) in the space

$$C_b(\mathbb{R}^+, L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}^+, \dot{H}^1(\mathbb{R}^3)) \cap \mathcal{F}_\beta.$$

The function $w = u - v$ satisfies the following:

$$\partial_t w - \Delta w + a((e^{b|u|^4} - 1)u - (e^{b|v|^4} - 1)v) = -\nabla(p - \tilde{p}) + w.\nabla w - w.\nabla u - u.\nabla w.$$

Taking the scalar product in L^2 with w, we get

$$\frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2 + \|\nabla w\|_{L^2}^2 + a\langle((e^{b|u|^4} - 1)u - (e^{b|v|^2} - 1)v); w\rangle_{L^2} = -\langle w.\nabla u; w\rangle_{L^2}.$$

The idea is to lower the term $\langle((e^{b|u|^4}-1)u-(e^{b|v|^2}-1)v); w\rangle_{L^2}$ with the help of Lemma 5 and then divide the term find into two equal pieces, one to absorb the nonlinear term and the other is used in the last inequality.

By using inequality (3), we get

$$\begin{aligned}\langle((e^{b|u|^4} - 1)u - (e^{b|v|^4} - 1)v); w\rangle_{L^2} &\geq \frac{1}{2}\int_{\mathbb{R}^3} ((e^{b|u|^4} - 1) + (e^{b|v|^4} - 1))|w|^2 \\ &\geq \frac{b}{2}\int_{\mathbb{R}^3} |u|^4|w|^2.\end{aligned}$$

Moreover, we have

$$|\langle w.\nabla u; w\rangle_{L^2}|$$

$$= |\langle \mathrm{div}\,(w \otimes u); w\rangle_{L^2}|$$
$$= |\langle w \otimes u; \nabla w\rangle_{L^2}|$$
$$\leq \int_{\mathbb{R}^3} |w|.|u|.|\nabla w|$$
$$\leq \frac{1}{2}\int_{\mathbb{R}^3} |w|^2|u|^2 + \frac{1}{2}\|\nabla w\|_{L^2}^2$$
$$\leq \frac{ab}{8}\int_{\mathbb{R}^3} |u|^4|w|^2 + \frac{1}{2ab}\|w\|_{L^2}^2 + \frac{1}{2}\|\nabla w\|_{L^2}^2.$$

Combining the above inequalities, we get

$$\frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2 + \frac{1}{2}\|\nabla w\|_{L^2}^2 + \frac{a}{4}\int_{\mathbb{R}^3}((e^{b|u|^2}-1)+(e^{b|v|^2}-1))|w|^2 \leq \frac{1}{2ab}\|w\|_{L^2}^2$$

and, Gronwall's Lemma gives

$$\|w\|_{L^2}^2 + \int_0^t \|\nabla w\|_{L^2}^2 + \frac{a}{2}\int_0^t\int_{\mathbb{R}^3}((e^{b|u|^4}-1)+(e^{b|v|^4}-1))|w|^2 \leq \|w^0\|_{L^2}^2 e^{\frac{t}{ab}}.$$

As $w^0 = 0$, then $w = 0$ and $u = v$ which implies the uniqueness.

3.4. Asymptotic Study of the Global Solution

In this section, we prove the asymptotic behavior (1). For this we prove some preliminaries lemmas:

Lemma 7. *If u is a global solution of* (1), *then* $(e^{b|u|^4} - 1)u \in L^1(\mathbb{R}^+ \times \mathbb{R}^3)$.

Proof. If $X_1 = \{(t,x) : \ |u(t,x)| \leq 1\}$ and $X_2 = \{(t,x) : \ |u(t,x)| > 1\}$, we have

$$\int_0^\infty \|(e^{b|u(s,.)|^4}-1)u)(s,.)\|_{L^1}ds = K_1 + K_2,$$

where $K_1 = \int_{X_1}(e^{b|u(s,x)|^4}-1)|u(s,x)|dxds$ and $K_2 = \int_{X_2}(e^{b|u(s,x)|^4}-1)|u(s,x)|dxds$,

$$K_1 = \int_{X_1}(e^{b|u(s,x)|^4}-1)|u(s,x)|dxds$$
$$= \int_{X_1} b|u(s,x)|^{\frac{5}{3}}\frac{(e^{b|u(s,x)|^4}-1)}{b|u(s,x)|^4}|u(s,x)|^{\frac{10}{3}}dxds$$
$$\leq b(e^b-1)\int_0^\infty\int_{\mathbb{R}^3}|u(s,x)|^{\frac{10}{3}}dxds$$
$$\leq be^b\int_0^\infty \|u(s,.)\|_{L^{\frac{10}{3}}}^{\frac{10}{3}}ds.$$

By using the Sobolev injection $\dot{H}^{\frac{3}{5}}(\mathbb{R}^3) \hookrightarrow L^{\frac{10}{3}}(\mathbb{R}^3)$, we get

$$K_1 \leq C\int_0^\infty \|u(s,.)\|_{\dot{H}^{\frac{3}{5}}}^{\frac{10}{3}}ds. \tag{13}$$

By interpolation inequality

$$\|u(s)\|_{\dot{H}^{\frac{3}{5}}} \leq \|u(s)\|_{\dot{H}^0}^{\frac{2}{5}}\|u(s)\|_{\dot{H}^1}^{\frac{3}{5}},$$

we obtain

$$K_1 \leq C\int_0^\infty \|u(s,.)\|_{L^2}^{\frac{4}{3}}\|\nabla u(s)\|_{L^2}^2$$
$$\leq C\|u^0\|_{L^2}^{\frac{4}{3}}\int_0^\infty \|\nabla u(s)\|_{L^2}^2. \tag{14}$$

For the term K_2, we have

$$K_2 = \int_{X_2}(e^{b|u(s,x)|^4}-1)|u(s,x)|dxds$$
$$\leq \int_0^\infty\int_{\mathbb{R}^3}(e^{b|u(s,x)|^4}-1)|u(s,x)|^2dxds.$$

Hence

$$\|(e^{b|u|^4}-1)u\|_{L^1(\mathbb{R}^+\times\mathbb{R}^3)} \leq C\|u^0\|_{L^2}^{\frac{4}{3}}\int_0^\infty \|\nabla u(s,.)\|_{L^2}^2ds + \int_0^\infty\int_{\mathbb{R}^3}(e^{b|u(s,x)|^4}-1)|u(s,x)|^2dxds.$$

Therefore $(e^{b|u|^4}-1)u \in L^1(\mathbb{R}^+\times\mathbb{R}^3)$. □

Lemma 8. *If u is a global solution of* (1), *then* $\lim_{t\to\infty}\|u(t)\|_{H^{-2}} = 0$.

Proof. Let $\varepsilon > 0$. By the energy inequality (1) and Lemma 7, there exists $t_0 \geq 0$ such that

$$\|\nabla u\|_{L^2([t_0,\infty)\times\mathbb{R}^3)} < \frac{\varepsilon}{4}, \tag{15}$$

$$\|(e^{b|u|^4}-1)u\|_{L^1([t_0,\infty)\times\mathbb{R}^3)} < \frac{\varepsilon}{4}. \tag{16}$$

Now, consider the following system:

$$(S')\begin{cases} \partial_t v - \nu\Delta v + v.\nabla v & \text{in } \mathbb{R}^+\times\mathbb{R}^3,\\ \quad + a(e^{b|v|^4}-1)v = -\nabla q & \\ \mathrm{div}\, v = 0 & \text{in } \mathbb{R}^+\times\mathbb{R}^3,\\ v(0,x) = u(t_0,x) & \text{in } \mathbb{R}^3. \end{cases}$$

By the existence and uniqueness part, the system (S') has a unique global solution v such that $v(t_0) =$

$u(t_0,x)$ and $q(t)=p(t_0+t)$. We recall the following energy estimate for this system:

$$\begin{aligned}\|v(t)\|_{L^2}^2 &+ 2\int_0^t \|\nabla v(s)\|_{L^2}^2 ds\\ &+ 2a\int_0^t \|(e^{b|v(s)|^2}-1)|v(s)|^2\|_{L^1}\\ &\le \|u(t_0)\|_{L^2}^2\\ &\le \|u^0\|_{L^2}^2.\end{aligned}$$

By the Duhamel's formula $v(t,x) = e^{t\Delta}v^0(x) + f(t,x)+g(t,x)$, where

$$f(t,x) = -\int_0^t e^{(t-s)\Delta}\mathbb{P}\mathrm{div}\,(v\otimes v)(s,x)ds$$

and

$$g(t,x) = -\int_0^t e^{(t-s)\Delta}\mathbb{P}\mathrm{div}\,(e^{b|v(s,x)|^2}-1)v(s,x)ds.$$

By the Dominated Convergence Theorem, we have $\lim_{t\to\infty}\|e^{t\Delta}v^0\|_{L^2} = 0$ and hence $\lim_{t\to\infty}\|e^{t\Delta}v^0\|_{H^{-2}} = 0$. Moreover, we have

$$\begin{aligned}&\|f(t)\|_{H^{-2}}^2\\ &\quad\le \|f(t)\|_{H^{-\frac12}}^2 \le \|f(t)\|_{\dot H^{-\frac12}}^2\\ &\quad\le \int_{\mathbb{R}^3}\frac{1}{|\xi|}\\ &\qquad\times\left(\int_0^t e^{-(t-s)|\xi|^2}|\mathcal{F}\mathrm{div}(v\otimes v)(s,\xi)|ds\right)^2 d\xi\\ &\quad\le \int_{\mathbb{R}^3}|\xi|\left(\int_0^t e^{-(t-s)|\xi|^2}|\mathcal{F}(v\otimes v)(s,\xi)|ds\right)^2 d\xi.\end{aligned}$$

As

$$\begin{aligned}&\left(\int_0^t e^{-(t-s)|\xi|^2}|\mathcal{F}(v\otimes v)(s,\xi)|ds\right)^2\\ &\quad\le \left(\int_0^t e^{-2(t-s)|\xi|^2}ds\right)\int_0^t|\mathcal{F}(v\otimes v)(s,\xi)|^2ds\\ &\quad\le |\xi|^{-2}\int_0^t|\mathcal{F}(v\otimes v)(s,\xi)|^2ds,\end{aligned}$$

we obtain

$$\begin{aligned}\|f(t)\|_{H^{-2}}^2dt &\le \int_{\mathbb{R}^3}|\xi|^{-1}\int_0^t|\mathcal{F}(v\otimes v)(s,\xi)|^2dsd\xi\\ &\le \int_0^t\left(\int_{\mathbb{R}^3}|\xi|^{-1}|(v\otimes v)(s,\xi)|^2d\xi\right)ds\\ &= \int_0^t\|v\otimes v)(s)\|_{\dot H^{-\frac12}}^2 ds.\end{aligned}$$

Using the product law in homogeneous Sobolev spaces, with $s_1=0, s_2=1$, we get

$$\|f(t)\|_{H^{-2}}^2dt \le C\int_0^t\|v(s)\|_{L^2}^2\|\nabla v(s)\|_{L^2}^2ds.$$

Using inequalities (15) and (16), we get

$$\begin{aligned}\|f(t)\|_{H^{-2}}^2dt &\le C\|u^0\|_{L^2}^2\int_0^t\|\nabla u(t_0+s)\|_{L^2}^2ds\\ &\le C\|u^0\|_{L^2}^2\int_0^\infty\|\nabla u(t_0+s)\|_{L^2}^2ds\\ &\le C\|u^0\|_{L^2}^2\int_{t_0}^\infty\|\nabla u(s)\|_{L^2}^2ds\\ &\le C\|u^0\|_{L^2}^2\frac{\varepsilon^2}{9(C\|u^0\|_{L^2}^2+1)},\end{aligned}$$

which implies

$$\|f(t)\|_{H^{-2}} < \frac{\varepsilon}{3}, \quad \forall\, t\ge 0.$$

For an estimation of $\|g(t)\|_{H^{-2}}$ and by using (7) with $s=2$, we get

$$\begin{aligned}&\|g(t)\|_{H^{-2}}^2dt\\ &\quad\le \int_{\mathbb{R}^3}\frac{1}{(1+|\xi|^2)^2}\\ &\qquad\times\left(\int_0^t e^{-(t-s)|\xi|^2}|\mathcal{F}((e^{b|v|^4}-1)v)(s,\xi)|ds\right)^2 d\xi\\ &\quad\le C\left(\int_0^t\|(e^{b|v(s,.)|^4}-1)v(s)\|_{L^1(\mathbb{R}^3)}ds\right)^2\\ &\quad\le C\|(e^{b|v|^4}-1)v\|_{L^1(\mathbb{R}^+\times\mathbb{R}^3)}^2,\end{aligned}$$

where $C=\int_{\mathbb{R}^3}(1+|\xi|^2)^{-2}d\xi$. Also by using inequality (16), we get

$$\begin{aligned}&\|g(t)\|_{H^{-2}}^2dt\\ &\quad\le C\|(e^{b|u(t_0+.)|^4}-1)u(t_0+.)\|_{L^1(\mathbb{R}^+\times\mathbb{R}^3)}^2\\ &\quad\le C\|(e^{b|u|^4}-1)u\|_{L^1([t_0,\infty)\times\mathbb{R}^3)}^2\\ &\quad\le C\frac{\varepsilon^2}{9C}.\end{aligned}$$

which implies that $\|g(t)\|_{H^{-2}} < \frac{\varepsilon}{3}$, $\forall t\ge 0$.

Combining the above inequalities, we obtain

$$\lim_{t\to\infty}\|u(t)\|_{H^{-2}} = 0. \qquad \square$$

Lemma 9. *If u is a global solution of* (1), *then* $\lim_{t\to\infty}\|u(t)\|_{L^2}=0$.

Proof. We have $u = w_1 + w_2$, where

$$w_1 = \mathbf{1}_{|D|<1} u = \mathcal{F}^{-1}(\mathbf{1}_{|\xi|<1}\widehat{u}),$$
$$w_2 = \mathbf{1}_{|D|\geq 1} u = \mathcal{F}^{-1}(\mathbf{1}_{|\xi|\geq 1}\widehat{u}).$$

By the second step, we get

$$\begin{aligned}\|w_1(t)\|_{L^2} &= c_0\|w_1(t)\|_{H^0}\\ &\leq 2c_0\||w_1(t)\|_{H^{-2}}\\ &\leq 2\||u(t)\|_{H^{-2}},\end{aligned}$$

which implies

$$\lim_{t\to\infty} \|w_1(t)\|_{L^2} = 0.$$

Let $\varepsilon > 0$. There is a time $t_1 > 0$ such that

$$\|w_1(t)\|_{L^2} < \frac{\varepsilon}{2}, \quad \forall\, t \geq t_1.$$

We have

$$\begin{aligned}\int_{t_1}^{\infty} \|w_2(t)\|_{L^2}^2 dt &\leq \int_{t_1}^{\infty} \|\nabla w_2(t)\|_{L^2}^2 dt\\ &\leq \int_{t_1}^{\infty} \|\nabla u(t)\|_{L^2}^2 dt < \infty.\end{aligned}$$

As $t \to \|w_2(t)\|_{L^2}$ is continuous, then there is a time $t_2 \geq t_1$ such that

$$\|w_2(t_2)\|_{L^2} < \frac{\varepsilon}{2}.$$

Particularly

$$\|u(t_2)\|_{L^2}^2 = \|w_1(t_2)\|_{L^2}^2 + \|w_2(t_2)\|_{L^2}^2 < \frac{\varepsilon^2}{2}.$$

By using the following energy estimate:

$$\begin{aligned}&\|u(t)\|_{L^2}^2 + 2\int_{t_2}^{t} \|\nabla u(s)\|_{L^2}^2 ds\\ &\quad + 2a\int_{t_2}^{t} \|(e^{b|u(s)|^4} - 1)|u(s)|^2\|_{L^1} ds\\ &\leq \|u(t_2)\|_{L^2}^2, \quad \forall\, t \geq t_2,\end{aligned}$$

we get

$$\|u(t)\|_{L^2} < \varepsilon, \quad \forall\, t \geq t_2,$$

and the proof is completed. □

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