Department of Mathematics, College of Science,
King Saud University
M-203, Final Examination, Semester-I, 1442/1443H

Time: 3 hours | Max. Marks-40

Q.1 (a) Determine whether the sequence $\left\{\frac{\ln(n^2+n+1)}{\ln(n+3)}\right\}$ converges or diverges, and if it converges, find its limit. [3]

(b) Test the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{\tan^{-1}(\frac{1}{n})}{n+1}$. [4]

(c) Find the interval of convergence and the radius of convergence of the power series [5]

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(x-2)^{3n}}{8^n}$$

Q. 2 (a) Evaluate the integral

$$\iint_R 2(x-y)dA,$$

where R is the region bounded by the graphs of the equations $y = x^2$, $x+y = 6$ and $y = 0$. [4]

(b) Find the surface area of the part of the paraboloid $z = 5 - x^2 - y^2$ that lies above the plane $z = 4$. [4]

(c) Find the centroid of the solid bounded by the graphs of the equations $z = \sqrt{x^2+y^2}$ and $z = 1$. [4]

Q.3 (a) Evaluate

$$\int_{(1,0)}^{(1,1)} [(y^2+2xy)dx + (x^2+2xy)dy]$$

by showing that the line integral is independent of path. [3]

(b) Use Green's theorem to find the area enclosed by the ellipse $x^2 + y^2 = 36$. [4]

(c) For the force $\vec{F} = zx\,\vec{i} - y\,\vec{j} + z\,\vec{k}$ find its flux through the surface of the sphere $x^2+y^2+z^2 = 4$. [4]

(d) For the force $\vec{F} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$ and the surface S of the paraboloid $z = x^2+y^2$ that lies inside the cylinder $x^2+y^2 = 1$, verify the Stokes's theorem. [5]

M-203

Final Exam. (I Semester 1442/1443)

Solutions

Q#1) (a) Determine whether the sequence $\left\{\frac{\ln(n^2+n+1)}{\ln(n+3)}\right\}$ converges or diverges and if it converges, find its limit. [3]

Soln. we have $a_n = \frac{\ln(n^2+n+1)}{\ln(n+3)}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln(n^2+n+1)}{\ln(n+3)}$$

$$= \lim_{n\to\infty} \frac{\ln\left[n^2\left(1+\frac{1}{n}+\frac{1}{n^2}\right)\right]}{\ln\left[n\left(1+\frac{3}{n}\right)\right]} \quad (1)$$

$$= \lim_{n\to\infty} \frac{2\ln(n)+\ln\left(1+\frac{1}{n}+\frac{1}{n^2}\right)}{\ln(n)+\ln\left(1+\frac{3}{n}\right)}$$

$$\underset{(1)}{=} 2; \text{ conv.} \quad (1)$$

Q#1 (b) Test the convergence or divergence of the Series $\sum_{n=1}^{\infty} \frac{\tan^{-1}\left(\frac{1}{n}\right)}{n+1}$. [4]

Soln. Take $\sum \frac{1}{n^2} = \sum b_n$ which is conv. Apply LCT

$$\therefore \lim_{n\to\infty} \frac{\tan^{-1}\left(\frac{1}{n}\right)}{(n+1)} \times n^2 = \lim_{n\to\infty} n\tan^{-1}\left(\frac{1}{n}\right)\underbrace{\left(\frac{n}{n+1}\right)}^{=1}$$

$$= \lim_{n\to\infty} \frac{\tan^{-1}\left(\frac{1}{n}\right)}{\frac{1}{n}} = 1 > 0$$

Hence by LCT, $\sum_{n=1}^{\infty} \frac{\tan^{-1}\left(\frac{1}{n}\right)}{n+1}$ is conv. (2)

(2)

Q#1(c) Find the Interval of converges and radius of conv. of the power Series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-2)^{3n}}{8^n}$. [5]

Soln. $\lim_{n\to\infty} \left| \frac{(-1)^{n+2}(x-2)^{3n+3}}{8^{n+1}} \times \frac{8^n}{(-1)^{n+1}(x-2)^{3n}} \right|$

$= \left| \frac{(x-2)^3}{2^3} \right|$ and it converges (abs) $\Leftrightarrow \left| \frac{(x-2)^3}{2^3} \right| < 1$

$\Leftrightarrow |x-2| < 2 \Leftrightarrow -2 < (x-2) < 2$

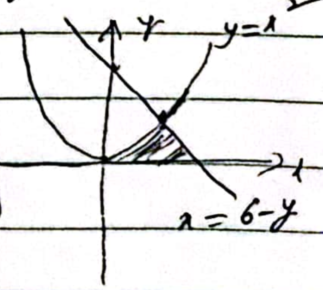
$\Leftrightarrow 0 < x < 4$ (2)

At $x=0$, we have $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(-2)^{3n}}{2^{3n}}$ which is div (1)

At $x=4$, we have $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^{3n}}{2^{3n}}$ which is also div. (1)

$\therefore$ Interval of convergence: $(0, 4)$ and radius of convergence $r = 2$. (1)

Q#2(a) Evaluate the integral $\iint 2(x-y)\,dA$ where R is the region bounded by the graphs of the equations $y = x^2$, $x+y=6$ and $y=0$. [4]



Soln. $R = \{(x,y) \mid 0 \le y \le 4, \sqrt{y} \le x \le 6-y\}$

$\int_0^2\int_0^{x^2} \cdots + \int_2^6\int_0^{6-x} \cdots$

$\iint_R 2(x-y)\,dA = \int_0^4 \int_{\sqrt{y}}^{6-y} (2x-2y)\,dx\,dy$ (2)

$= \int_0^4 \left[x^2 - 2yx\right]_{\sqrt{y}}^{6-y} dy = \int_0^4 \left[3y^2 - 25y + 2y^{3/2} + 36\right] dy$

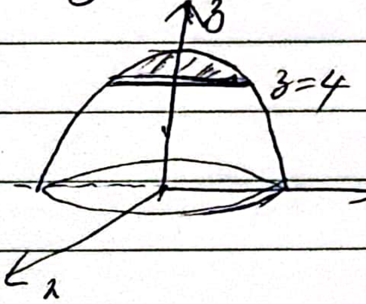
$= \left[y^3 - 25\frac{y^2}{2} + 2\cdot\frac{2}{5} y^{5/2} + 36y \right]_0^4 = \frac{168}{5}$ (1)

(1)

$\simeq 33.6$

Q#2(b) Find the Surface area of the part of the paraboloid $z = 5 - x^2 - y^2$ that lies above the plane $z = 4$ [4]

Sol: we have $z = 5 - x^2 - y^2 = g(x,y)$



$\therefore g_x = -2x$ and $g_y = -2y$ (1)

$\therefore S.A.A = \iint_R \sqrt{1+4x^2+4y^2}\, dA$

$= \int_0^{2\pi}\int_0^1 \sqrt{1+4r^2}\, r\, dr\, d\theta$ (2)

Put $1+4r^2 = u$, $8r\,dr = du$

$\frac{1}{8}\int \sqrt{u}\, du$

$= \frac{1}{48} u^{3/2} \times \frac{2}{3}$

$= \frac{1}{12}\left[(1+4r^2)^{3/2}\right]_0^1$

$= \frac{1}{12}\left[5^{3/2} - 1\right]$

$= 2\pi \times \frac{1}{6 \cancel{12}}(5^{3/2} - 1)$

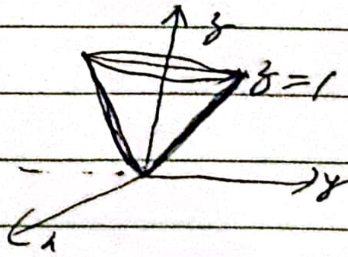
$= \frac{\pi}{6}(5^{3/2} - 1)$. (1)

$\approx 1.6\cancel{97}$...

Q#2(c) Find the centroid of the Solid bounded by the graphs of the equations $z = \sqrt{x^2+y^2}$ and $z = 1$. [4]

Soln. Mass: $m = \iiint_Q \delta(x,y,z)\, dV$. we take $\delta(x,y,z) = 1$.

$\therefore m = \int_0^{2\pi}\int_0^1\int_r^1 dz\, r\, dr\, d\theta$ (3)



$= \int_0^{2\pi}\int_0^1 (1-r)\, r\, dr\, d\theta$

$= \int_0^{2\pi}\left[\frac{r^2}{2} - \frac{r^3}{3}\right]_0^1 = 2\pi\left(\frac{1}{2} - \frac{1}{3}\right) = 2\pi\left(\frac{1}{\cancel{3}6}\right)$

$= \frac{\pi}{3}$ (2)

By Symmetry of Figure: $\bar{x} = \bar{y} = 0$. we find $\bar{z}$. ≈ 1.046

$\bar{z} = \frac{M_{xy}}{m}$; $M_{xy} = \int_0^{2\pi}\int_0^1\int_r^1 z\, dz\, r\, dr\, d\theta = \frac{\pi}{4}$ $\approx 0.7\cancel{8}$ (1)

$\therefore \bar{z} = \frac{M_{xy}}{m} = \frac{\pi/4}{\pi/3} = \frac{3}{4}$ $\therefore (\bar{x}, \bar{y}, \bar{z}) = (0, 0, 3/4)$.

Q # 3(C) For the force $\vec{F} = zx\vec{i} - y\vec{j} + z\vec{k}$, find its flux through the surface of the sphere $x^2+y^2+z^2 = 4$.

[illegible] [4]

Soln. $\iint_S \vec{F}\cdot\vec{n}\, ds = \int_0^{2\pi}\int_0^{\pi}\int_0^{2} \rho\cos\varphi\, \rho^2 \sin\varphi\, d\rho\, d\varphi\, d\theta$ 3

$$= \int_0^{2\pi}\int_0^{\pi} \left[\frac{\rho^4}{4}\right]_0^2 \cos\varphi \sin\varphi\, d\varphi\, d\theta$$

$$= 4\int_0^{2\pi}\int_0^{\pi} \frac{\sin 2\varphi}{2}\, d\varphi\, d\theta$$

$$= 4\int_0^{2\pi} \left[-\frac{\cos 2\varphi}{4}\right]_0^{\pi} d\theta$$

$$= \underline{\underline{0}}$$ 1

Q # 3(d) For the force $\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$ and the Surface S of the paraboloid $z = x^2+y^2$ that lies inside the cylinder $x^2+y^2=1$, verify the Stokes's theorem. [5]

Soln. We have Stokes's theorem $\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\text{curl}\,\vec{F})\cdot\vec{n}\, ds$ (1) —(1)

First, we prove L.H.S. of (1): $\oint_C \vec{F}\cdot d\vec{r}$

Boundary of S is given by

$C: x^2+y^2=1,\ z=1;\ x=\cos t,\ y=\sin t\ z=1$
$0 \le t \le 2\pi$

$$\therefore \oint \vec{F}\cdot d\vec{r} = \int_0^{2\pi} x\,dx + y\,dy + z\,dz \quad (1)$$

$$= \int_0^{2\pi} \cancel{\cos t\cdot(-\sin t)\,dt} + \cancel{\sin t(\cos t)\,dt}$$

$$= \int_0^{2\pi} 0\cdot dt = 0. \quad (1)$$

Now, we prove R.H.S. of (1). That is: $\iint_S (\text{curl}\,\vec{F})\cdot\vec{n}\,ds$

$$= \iint \quad \text{curl}\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \vec{0}$$

Hence $\iint_S (\text{curl}\,\vec{F})\cdot\vec{n}\,ds = 0$ (1)

Therefore L.H.S of (1) = R.H.S. of (1).

That is $\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\text{curl}\,\vec{F})\cdot\vec{n}\,ds$. (1)