Home Assignment # 2 (II semester 1440/1441)

Max. Marks: 25

Note: All Questions Carry equal marks.

Q#1) Evaluate the Integral $\int_0^1 \int_x^{\sqrt[3]{x}} e^{\frac{1}{4}(2-y^2)}\, dy\, dx$.

Soln. Given $x \le y \le x^{\frac{1}{3}}$, $0 \le x \le 1$ } Vertical strip

we change it to $y^3 \le x \le y$, $0 \le y \le 1$ } Horizontal strip

Hence, we have $\int_0^1 \int_{y^3}^{y} e^{\frac{1}{4}y^2(2-y^2)}\, dx\, dy$ $\left\{ y = x,\ y = x^3 \ \therefore x = y^3 \right.$

$$= \int_0^1 (y-y^3)\, e^{\frac{1}{4}y^2(2-y^2)}\, dy$$

Put $t = \frac{1}{4}y^2(2-y^2)$

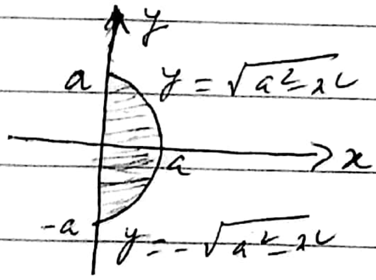
$\therefore dt = (y-y^3)dy$

$$\therefore \int_0^{\frac{1}{4}} e^t\, dt = \left[e^t\right]_0^{\frac{1}{4}} = \left(e^{\frac{1}{4}} - 1\right)$$

Q#2) Sketch the region of Integration and use polar Coordinates to find the value of the integral:

$$\int_0^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{x^2+y^2}{1+(x^2+y^2)^2}\, dy\, dx.$$

Soln. we have $\int_0^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{x^2+y^2}{1+(x^2+y^2)^2}\, dy\, dx$

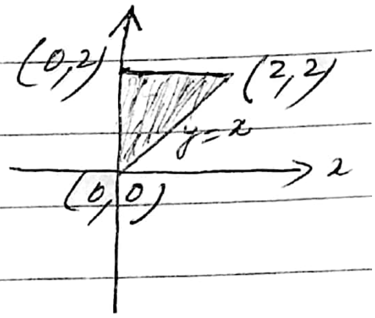


$$= \int_{-\pi/2}^{\pi/2} \int_0^a \frac{r^2}{1+r^4}\, r\, dr\, d\theta$$

$$= \frac{1}{4}\pi \ln(1+a^4).$$

Q#3) Find the surface area of the surface $z = y^2$ over the triangle in the xy-plane with vertices $(0,0), (0,2), (2,2)$.

Soln. we have $z = y^2 = g(x,y)$ $\therefore g_x = 0,\ g_y = 2y$



$$\therefore S\cdot A = \iint_R \sqrt{1+f_x+f_y}\, dA$$

$$= \int_0^2\int_0^y \sqrt{1+4y^2}\, dx\, dy$$

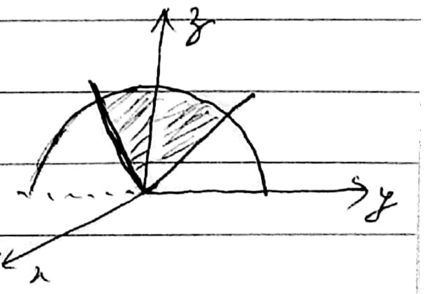
$$= \int_0^2 y\sqrt{1+4y^2}\, dy \qquad \text{Put } 1+4y^2 = t \qquad 8y\,dy = dt$$

$$= \frac{1\times 4}{4\,8\times 3}\left[(1+4y^2)^{3/2}\right]_0^2 \qquad \frac{1}{8}\int\sqrt{t}\,dt = \frac{1}{8}\times\frac{t^{3/2}}{3/2}$$

$$= \frac{1}{12}\left[(17)^{3/2}-1\right]$$

Q#4) Evaluate the triple integral $\iiint_Q \sqrt{x^2+y^2+z^2}\, dV$, where Q is the solid bounded by the graphs of the equations $z = \sqrt{4-x^2-y^2}$ and $z = \sqrt{x^2+y^2}$.

Soln. we have $\iiint_Q \sqrt{x^2+y^2+z^2}\, dV$



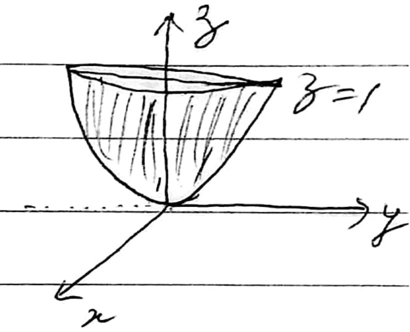
$$= \int_0^{2\pi}\int_0^{\pi/4}\int_0^2 \rho\cdot\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4}\left[\frac{\rho^4}{4}\right]_0^2 \sin\varphi\, d\varphi\, d\theta$$

$$= 4\int_0^{2\pi}\left[-\cos\varphi\right]_0^{\pi/4} d\theta = 4\times 2\pi\left(1-\frac{1}{\sqrt{2}}\right)$$

$$= 8\pi\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)$$

Q#5) Find the moment of Inertia about the z-axis of the solid having the shape of the region Q bounded by the graphs of the equations $z = x^2+y^2$, $z=1$ and density $\delta = \sqrt{x^2+y^2}$.



Soln: $I_z = \iiint_Q \sqrt{x^2+y^2}\,(x^2+y^2)\,dV$

$$= \int_0^{2\pi}\int_0^1\int_{r^2}^1 r\cdot r^2\cdot r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 (1-r^2)\,r^4\,dr\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{r^5}{5}-\frac{r^7}{7}\right]_0^1 d\theta = 2\pi\left(\frac{1}{5}-\frac{1}{7}\right)$$

$$= 2\pi\left(\frac{7-5}{35}\right) = \frac{4\pi}{35}$$