King Saud University
Department Of Mathematics
M-203
(Differential and Integral Calculus)

First Mid-Term Examination
(I-Semester 1432/33)

Max. Marks: 20 Time: 90 Minutes

Q. No: 1 Determine whether or not the sequence $\left\{\left(1-\frac{1}{n}\right)^n\right\}_{n=1}^{\infty}$ converges, and if it converges find its limit……………………………………………………..[3]

Solution: Let $y=\left(1-\frac{1}{n}\right)^n$ it is $\left(1^{\infty}\right)-form$

$$\Rightarrow \ln(y)=n\ln(1-\frac{1}{n})$$

$$\Rightarrow \underset{n\to\infty}{Lim}\ln(y)=\underset{n\to\infty}{Lim}\, n\ln\left(1-\frac{1}{n}\right) \quad \text{it is} \quad (0.\infty)-form$$

$$=\lim_{n\to\infty}\frac{\ln\left(1-\frac{1}{n}\right)}{\frac{1}{n}} \quad \text{it is now} \quad \frac{0}{0}-form$$

$$=\lim_{n\to\infty}\frac{\frac{1}{1-1/n}\left(\frac{1}{n^2}\right)}{\left(-\frac{1}{n^2}\right)} = \lim_{n\to\infty}\left(-\frac{1}{1-\frac{1}{n}}\right)=-1$$

$$\Rightarrow \lim_{n\to\infty}\ln(y)=-1\Rightarrow \lim_{n\to\infty}y=e^{-1}=\frac{1}{e}\Rightarrow Convergent.$$

Q. No: 2 Determine whether the following infinite series converges or diverges. If it converges, find its sum

$$\sum_{n=1}^{\infty}\left[\frac{2^n+3^n}{6^n}\right]$$ ……………………………………………….[3]

Solution: $\sum_{n=1}^{\infty}\left[\frac{2^n+3^n}{6^n}\right]=\sum_{n=1}^{\infty}\frac{2^n}{6^n}+\sum_{n=1}^{\infty}\frac{3^n}{6^n}=\sum_{n=1}^{\infty}\left(\frac{2}{6}\right)^n+\sum_{n=1}^{\infty}\left(\frac{3}{6}\right)^n$ both are convergent geometric series.

Sum$=\frac{1/3}{1-1/3}+\frac{1/2}{1-1/2}=\frac{1}{3}.\frac{3}{2}+\frac{1}{2}.2=\frac{3}{2}.$

Q. No: 3 Determine whether the following series converges or diverges

$$\sum_{n=1}^{\infty} ne^{-n^2} \qquad \text{..[4]}$$

Solution: We use the integral test $f(x) = xe^{-x^2}$

$$f'(x) = e^{-x^2}(1-2x^2) < 0 \Rightarrow f \text{ is decreasing on } [1,\infty)$$

f is clearly continuous and positive on $[1,\infty)$.

Now, applying the integral test to $\int_1^{\infty} xe^{-x^2}\,dx$

$$\int_1^{\infty} xe^{-x^2}\,dx = \lim_{t\to\infty}\int_1^{t} xe^{-x^2}dx = \lim_{t\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_1^t = \lim_{t\to\infty}\left[-\frac{1}{2}e^{-x^2}+\frac{1}{2}\right] = \frac{1}{2}$$

Convergent.

Q, No: 4 Use the first two non-zero terms of the power series representation of the function $f(x) = \dfrac{1}{1+x^3}$, $|x|<1$ to approximate the value of $f(0.1)$.

.... [3]

Solution: We know $\dfrac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \text{..........................}$

Here we replace x by x^3 and get

$$f(x) = \frac{1}{1+(x^3)} = 1-(x^3)+(x^3)^2-(x^3)^3+(x^3)^4-\text{..........................}$$

Therefore $f(0.1) = 1-(0.1)^3 \approx 1-0.01 \approx 0.999$

Q. No: 5 Find the interval of convergence and radius of convergence of the power series $\sum_{n=1}^{\infty}(-1)^n \frac{(x-1)^n}{n2^n}$ .... [4]

Solution: By absolute ration test:

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-1)^{n+1}}{(n+1)2^{n+1}}\times\frac{n2^n}{(-1)^n(x-1)^n}\right| = \left|\frac{(x-1)}{2}\right|$$

For absolute convergence

$$\left|\frac{x-1}{2}\right| < 1 \Leftrightarrow |x-1| < 2 \Leftrightarrow -2 < x-1 < 2 \Leftrightarrow -1 < x < 3 \text{ .}$$

Now at $x=-1$, we have

$\sum_{n=1}^{\infty}(-1)^n \frac{(-2)^n}{n2^n} = \sum_{n=1}^{\infty} (-1)^2 \frac{2^{2n}}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{n}$. This is harmonic Series which is divergent.

Now at $x=3$, we have

$\sum_{n=1}^{\infty}(-1)^n \frac{(2)^n}{n2^n} = \sum_{n=1}^{\infty} (-1)^n \frac{2^2}{n2^n} = \sum_{n=1}^{\infty}(-1)^n \frac{1}{n}$. This is an AS which is convergent.

Hence interval of convergence is (-1, 3]

and the radius of convergence is $\rho = \frac{3-(-1)}{2} = 2.$

Q. No: 6 Find Maclaurin series for $f(x) = \cos x + \sin x$ ....[3]

Solution:

$$f(x) = \cos x + \sin x \Rightarrow f(0) = 1$$

$$f^{/}(x) = -\sin x + \cos x \Rightarrow f^{/}(0) = 1$$

$$f^{//}(x) = -\cos x - \sin x \Rightarrow f^{//}(0) = -1$$

$$f^{///}(x) = \sin x - \cos x \Rightarrow f^{///}(0) = -1$$

Hence the required Maclaurin Series for $f(x) = \cos x + \sin x$ is

$$= 1 + \frac{x}{1!} - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} - \text{........................}$$