

King Saud University
Department of Mathematics
M-203
(Differential & Integral Calculus)
First Mid-Term Examination
(II-Semester 1428/29)

Max. Marks:20
Time:90 minutes

$$\left\{ \sqrt{n^2 + n} - n \right\}_{n=1}^{\infty}$$

Q.No:1 (a) Determine whether or not the sequence converges, and if it converges find its limit.

$$a_n = \sqrt{n^2 + n} - n \times \frac{\sqrt{n^2 + n} + n}{\sqrt{n^2 + n} + n} = \frac{n^2 + n - n^2}{\sqrt{n^2 + n} + n} = \frac{n}{\sqrt{n^2 + n} + n}$$

Solution:

$$= \frac{1}{\sqrt{1 + \frac{1}{n}} + 1} \Rightarrow \lim_{n \rightarrow \infty} (a_n) = \lim_{n \rightarrow \infty} \left[\frac{1}{\sqrt{1 + \frac{1}{n}} + 1} \right] = \frac{1}{2}$$

(b) Use partial sums to determine the convergence or divergence of the series:

$$\ln\left(\frac{1}{2}\right) + \ln\left(\frac{2}{3}\right) + \ln\left(\frac{3}{4}\right) + \dots + \ln\left(\frac{n}{n+1}\right) + \dots$$

Solution: Given series can be re-written as follows:

$$[\ln(1) - \ln(2)] + [\ln(2) - \ln(3)] + [\ln(3) - \ln(4)] + \dots + [\ln(n) - \ln(n+1)] + \dots$$

$$S_1 = \ln(1) - \ln(2) = -\ln(2)$$

Therefore First partial sum
Second Partial sum

$$S_2 = [\ln(1) - \ln(2)] + [\ln(2) - \ln(3)] = -\ln(3)$$

⋮

⋮

$$S_n = -\ln(n+1) \Rightarrow \lim_{n \rightarrow \infty} S_n = -\infty$$

nth partial sum

Hence the series is divergent.

Q.No:2 (a) Determine whether the following infinite series is convergent or divergent

$$\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^3}$$

Solution: Here we use the integral test with the following function

$$f(x) = \frac{1}{n(\ln(x))^3}$$

i) $f'(x) \leq 0$ for $x \geq 2 \Rightarrow$ it is decreasing;

$$\int_2^{\infty} \frac{1}{x(\ln(x))^2} dx = \lim_{t \rightarrow \infty} \int_2^t \frac{1}{x(\ln(x))^2} dx = \lim_{t \rightarrow \infty} \left[-\frac{1}{2(\ln(x))^2} \right]_2^t$$

ii)

$$= \lim_{t \rightarrow \infty} \left[-\frac{1}{2(\ln(t))^2} + \frac{1}{2(\ln(2))^2} \right] = \frac{1}{2(\ln(2))^2}$$

Hence Convergent.

$$\sum_{n=1}^{\infty} \frac{\sin^2\left(\frac{1}{n}\right)}{n^2}$$

(b) Test the convergence of the series

Solution: Try the Sandwich theorem:

$$0 \leq \sin^2\left(\frac{1}{n}\right) \leq 1$$

$$\frac{0}{n^2} \leq \frac{\sin^2\left(\frac{1}{n}\right)}{n^2} \leq \frac{1}{n^2}$$

for all n.

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ is convergent, } \sum_{n=1}^{\infty} \frac{\sin^2\left(\frac{1}{n}\right)}{n^2} \text{ is also convergent}$$

by Basic Comparison test.

$$\sum_{n=1}^{\infty} (-1)^n \frac{1}{n\sqrt{n+1}}$$

(C) Determine whether the series absolutely, converges Conditionally, or diverges.

Solution: First check absolute convergence

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \left| (-1)^n \frac{1}{n\sqrt{n+1}} \right| = \sum_{n=1}^{\infty} \left| \frac{1}{n\sqrt{n+1}} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} = \sum_{n=1}^{\infty} b_n$$

Now compare with the series

which is

Convergent.

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n^{3/2}}{n^{3/2} \sqrt{1 + \frac{1}{n}}} = 1$$

Hence both series

converge or

Diverge together. So the given series is

convergent.

Q.No: 3 (a) Find the interval of convergence and the radius of convergence of the power

$$\sum_{n=1}^{\infty} (-1)^n \frac{2}{n} x^n$$

Series

$$\lim_{n \rightarrow \infty} \left| \frac{U_{n+1}}{U_n} \right| = \lim_{n \rightarrow \infty} \left(\frac{2}{n+1} \times \frac{n}{2} |x| \right) = |x|$$

Solution:

\Rightarrow

given series is absolutely convergent if

$$|x| < 1 \Rightarrow -1 < x < 1$$

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Now check convergence at $x = -1$ and $x = 1$.

$$\sum_{n=1}^{\infty} (-1)^n \frac{2}{n} (-1)^n = \sum_{n=1}^{\infty} \frac{2}{n}$$

At

it is clearly

divergent

$$\sum_{n=1}^{\infty} (-1)^n \frac{2}{n} (1)^n = \sum_{n=1}^{\infty} (-1)^n \frac{2}{n}$$

At

convergent AS.

$$-1 < x \leq 1$$

Interval of convergence

and radius of

$$= \frac{1 - (-1)}{2} = 1$$

convergence

$$f(x) = e^x$$

(b) Find the Maclaurin's series for the function and use it to approximate

$$\int_0^1 x^4 e^x dx.$$

The integral

Solution:

$$f(x) = e^x \Rightarrow f(0) = 1, f'(0) = 1, f''(0) = 1, \dots, f^{(n)}(0) = 1, \dots$$

$$f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^n}{n!} f^{(n)}(0) + \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

$$\int_0^1 x^4 \left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots \right] dx$$

Now

$$\int_0^1 \left[x^4 + x^5 + \frac{x^6}{2!} + \frac{x^7}{3!} + \dots + \frac{x^{n+4}}{n!} + \dots \right] dx$$

=

$$\left[\frac{x^5}{5} + x + \frac{x^6}{2 \times 6} + \frac{x^7}{3 \times 7} + \dots \right]_0^1 = 0.45$$

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