King Saud University
Department of Mathematics
M-203
(Differential & Integral Calculus)
First Mid-Term Examination
(II-Semester 1428/29)

Max. Marks:20 **Time:90 minutes**

Q.No:1 (a) Determine whether or not the sequence $\left\{\sqrt{n^2+n}-n\right\}_{n=1}^{\infty}$ converges, and if it Converges fid its limit.

Solution:

$$a_n=\sqrt{n^2+n}-n\times\frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n}=\frac{n^2+n-^2}{\sqrt{n^2+n}+n}=\frac{n}{\sqrt{n^2+n}+n}$$

$$=\frac{1}{\sqrt{1+1/n}+1}\Rightarrow \underset{n\to\infty}{Lim}(a_m)=\underset{n\to\infty}{Lim}\left[\frac{1}{\sqrt{1+1/n}+1}\right]=\frac{1}{2}.$$

(b) Use partial sums to determine the convergence or divergence of the series:

$$\ln\left(\frac{1}{2}\right)+\ln\left(\frac{2}{3}\right)+\ln\left(\frac{3}{4}\right)+\ldots\ldots\ldots\ldots\ldots\ldots+\ln\left(\frac{n}{n+1}\right)\ldots\ldots\ldots\ldots\ldots$$

Solution: Given series can be re-written as follows:

$$[\ln(1)-\ln(2)]+[\ln(2)-\ln(3)]+[\ln(3)-\ln(4)]+\ldots\ldots+[\ln(n)-\ln(n+1)]+\ldots\ldots$$

Therefore First partial sum $S_1=\ln(1)-\ln(2)=-\ln(2)$

Second Partial sum $S_2=[\ln(1)-\ln(2)]+[\ln(2)-\ln(3)]=-\ln(3)$

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nth partial sum $S_n=-\ln(n+1)\Rightarrow \underset{n\to\infty}{Lim}\,S_n=-\infty.$

Hence the series is divergent.

Q.No:2 (a)Determine whether the following infinite series is convergent or divergent

$$\sum_{n=2}^{\infty}\frac{1}{n(\ln(n))^3}.$$

Solution: **Here we use the integral test with the following function**

$$f(x)=\frac{1}{n(\ln(x))^3}$$

i) $f'(x)\le 0$ **for** $x\ge 2\Rightarrow$ **it is decreasing;**

ii) $$\int_2^{\infty}\frac{1}{x(\ln(x))^2}dx=\underset{t\to\infty}{Lim}\int_2^{t}\frac{1}{x(\ln(x))^2}dx=\underset{t\to\infty}{Lim}\left[-\frac{1}{2(\ln(x))^2}\right]_2^t$$

$$= \lim_{t\to\infty}\left[-\frac{1}{2(\ln(t))^2}+\frac{1}{2(\ln(2))^2}\right]=\frac{1}{2(\ln(2))^2}.$$

Hence Convergent.

(b) Test the convergence of the series $\sum_{n=1}^{\infty}\frac{Sin^2\left(\frac{1}{n}\right)}{n^2}$.

Solution: **<u>Try the Sandwich theorem:</u>**

$$0 \le Sin^2\left(\frac{1}{n}\right) \le 1$$

$$\frac{0}{n^2} \le \frac{Sin^2\left(\frac{1}{n}\right)}{n^2} \le \frac{1}{n^2} \quad \text{for all n.}$$

$\Rightarrow$ **since** $\sum_{n=1}^{\infty}\frac{1}{n^2}$ **is convergent,** $\sum_{n=1}^{\infty}\frac{Sin^2\left(\frac{1}{n}\right)}{n^2}$ **is also convergent**

by Basic Comparison test.

(C) Determine whether the series $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n\sqrt{n+1}}$ **converges absolutely, converges Conditionally, or diverges.**

Solution: **First check absolute convergence**

$$\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\left|(-1)^n\frac{1}{n\sqrt{n+1}}\right| = \sum_{n=1}^{\infty}\left|\frac{1}{n\sqrt{n+1}}\right| = \sum a_n$$

Now compare with the series $\sum_{n=1}^{\infty}\frac{1}{n^{3/2}} = \sum_{n=1}^{\infty}b_n$ **, which is Convergent.**

$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n^{3/2}}{n^{3/2}\sqrt{1+1/n}} = 1$. **Hence both series converge or Diverge together. So the given series is convergent.**

Q.No: 3 (a) Find the interval of convergence and the radius of convergence of the power Series $\sum_{n=1}^{\infty}(-1)^n\frac{2}{n}x^n$.

Solution: $\lim_{n\to\infty}\left|\frac{U_{n+1}}{U_n}\right| = \lim_{n\to\infty}\left(\frac{2}{n+1}\times\frac{n}{2}|x|\right) = |x|$

$\Rightarrow$ **given series is absolutely convergent if** $|x|<1 \Rightarrow -1<x<1$.,

Now check convergence at $x=-1$ **and** $x=1$.

At $x=-1$ $\quad \sum_{n=1}^{\infty}(-1)^n \frac{2}{n}(-1)^n = \sum_{n=1}^{\infty}\frac{2}{n}$ **it is clearly divergent**

At $x=1$ $\quad \sum_{n=1}^{\infty}(-1)^n \frac{2}{n}(1)^n = \sum_{n=1}^{\infty}(-1)^n \frac{2}{n}$ **convergent AS.**

Interval of convergence $-1<x\leq 1$ **and radius of convergence** $=\frac{1-(-!)}{2}=1$.

(b)Find the Maclaurin's series for the function $f(x)=e^x$ **and use it to approximate**

The integral $\int_0^1 x^4 e^x dx.$

Solution:

$$f(x)=e^x \Rightarrow f(0)=1, f^{/}(0)=1, f^{//}(0)=1,\ldots\ldots\ldots\ldots, f^{(n)}(o)=1,\ldots\ldots.$$

$$f(0)+xf^{/}(0)+\frac{x^2}{2!}f^{/}(0)+\ldots\ldots\ldots\ldots\ldots\ldots\ldots+\frac{x^n}{n!}f^{(n)}(0)+\ldots\ldots..$$

$$e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots+\frac{x^n}{n!}\ldots\ldots\ldots..$$

Now $\int_0^1 x^4\ [1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots+\frac{x^n}{n!}\ldots\ldots\ldots..]dx$

$$=\int_0^1[x^4+x^5+\frac{x^6}{2!}+\frac{x^7}{3!}+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots+\frac{x^{n+4}}{n!}\ldots\ldots\ldots..]dx$$

$$=[\frac{x^5}{5}+x+\frac{x^6}{2!\times 6}+\frac{x^7}{3!\times 7}+\ldots\ldots\ldots\ldots..]_0^1=0.45$$