M 104 - GENERAL MATHEMATICS -2-

Dr. Tariq A. AlFadhel

Solution of the Mid-Term Exam

First semester 1443 H

Q.1 Find the elements of the conic section of equation $y^2 - 2y + 4x = 3$, then sketch it.

Solution :

$y^2 - 2y + 4x = 3$

$y^2 - 2y = -4x + 3$

By completing the square.

$y^2 - 2y + 1 = -4x + 3 + 1$

$y^2 - 2y + 1 = -4x + 4$

$(y-1)^2 = -4(x-1)$

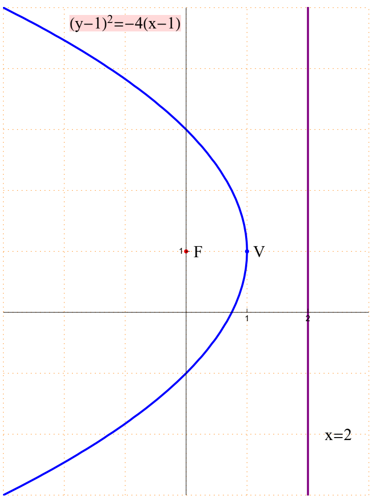
The conic section is a Parabola, and it opens to the left.

The vertex is $V(1,1)$

$4a = 4 \implies a = 1$

The Focus is $F(0,1)$

The equation of the directrix is $x = 2$



Q.2 Find the standard equation of the ellipse with vertices at $(-4,2)$, $(6,2)$ and one of its two foci at $(5,2)$, then sketch it.

Solution :

The two vertices and one focus are located on a line parallel to the x-axis.

The standard equation of the ellipse is $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$, where $a > b$.

$P(h,k) = \left(\dfrac{-4+6}{2}, \dfrac{2+2}{2}\right) = (1,2)$, hence $h = 1$ and $k = 2$

c is the distance between the focus $(5,2)$ and P , hence $c = 4$

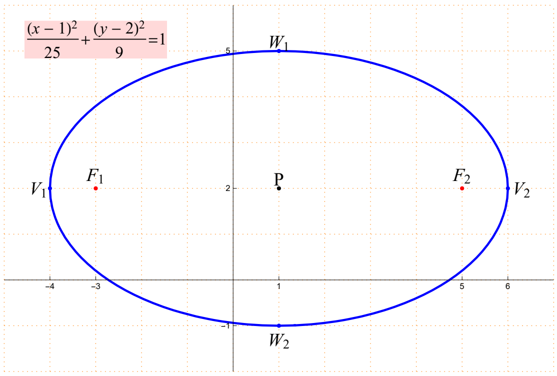
a is the distance between one of the vertices and P , hence $a = 5$

$c^2 = a^2 - b^2 \implies 16 = 25 - b^2 \implies b^2 = 25 - 16 = 9 \implies b = 3$

The standard equation of the ellipse is $\dfrac{(x-1)^2}{25} + \dfrac{(y-2)^2}{9} = 1$

The other focus is $(-3,2)$

The end-points of the minor axis are $(1,-1)$ and $(1,5)$.



Q.3 Calculate, whenever it is possible, the products of $2\mathbf{AB}$ and $\mathbf{BA}$ of matrices

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix} , \mathbf{B} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 1 \end{pmatrix} .$$

Solution :

$$2\mathbf{AB} = 2\,(\mathbf{AB}) = 2\left[\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 1 \end{pmatrix}\right]$$

$$= 2\begin{pmatrix} 1+0+2 & -1-1+1 \\ 1+0+0 & -1+1+0 \end{pmatrix} = 2\begin{pmatrix} 3 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 6 & -2 \\ 2 & 0 \end{pmatrix}$$

$$\mathbf{BA} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1-1 & -1-1 & 1+0 \\ 0+1 & 0+1 & 0+0 \\ 2+1 & -2+1 & 2+0 \end{pmatrix} = \begin{pmatrix} 0 & -2 & 1 \\ 1 & 1 & 0 \\ 3 & -1 & 2 \end{pmatrix}$$

Q.4 Consider the system of the linear equations:

$$\begin{cases} x & - & 2y & + & z & = & 4 \\ -x & + & 2y & + & z & = & -2 \\ 2x & - & 3y & - & z & = & 3 \end{cases}$$

(a) Solve this system using Cramer's rule.

(b) Solve this system using Gauss elimination method.

Solution :

(a) Using Cramer's rule :

$$\mathbf{A} = \begin{pmatrix} 1 & -2 & 1 \\ -1 & 2 & 1 \\ 2 & -3 & -1 \end{pmatrix}$$

$$\begin{matrix} 1 & -2 & 1 & 1 & -2 \\ -1 & 2 & 1 & -1 & 2 \\ 2 & -3 & -1 & 2 & -3 \end{matrix}$$

$$|\mathbf{A}| = (-2-4+3) - (4-3-2) = -3 - (-1) = -2 \neq 0$$

$$\mathbf{A}_x = \begin{pmatrix} 4 & -2 & 1 \\ -2 & 2 & 1 \\ 3 & -3 & -1 \end{pmatrix}$$

$$\begin{matrix} 4 & -2 & 1 & 4 & -2 \\ -2 & 2 & 1 & -2 & 2 \\ 3 & -3 & -1 & 3 & -3 \end{matrix}$$

$$|\mathbf{A}_x| = (-8-6+6) - (6-12-4) = -8 - (-10) = -8 + 10 = 2$$

$$x = \frac{|\mathbf{A}_x|}{|\mathbf{A}|} = \frac{2}{-2} = -1$$

$$\mathbf{A}_y = \begin{pmatrix} 1 & 4 & 1 \\ -1 & -2 & 1 \\ 2 & 3 & -1 \end{pmatrix}$$

$$\begin{matrix} 1 & 4 & 1 & 1 & 4 \\ -1 & -2 & 1 & -1 & -2 \\ 2 & 3 & -1 & 2 & 3 \end{matrix}$$

$$|\mathbf{A}_y| = (2+8-3) - (-4+3+4) = 7 - 3 = 4$$

$$y = \frac{|\mathbf{A}_y|}{|\mathbf{A}|} = \frac{4}{-2} = -2$$

$$\mathbf{A}_z = \begin{pmatrix} 1 & -2 & 4 \\ -1 & 2 & -2 \\ 2 & -3 & 3 \end{pmatrix}$$

$$\begin{matrix} 1 & -2 & 4 & 1 & -2 \\ -1 & 2 & -2 & -1 & 2 \\ 2 & -3 & 3 & 2 & -3 \end{matrix}$$

$|\mathbf{A}_z| = (6+8+12) - (16+6+6) = 26 - 28 = -2$

$z = \dfrac{|\mathbf{A}_z|}{|\mathbf{A}|} = \dfrac{-2}{-2} = 1$

(b) Using Gauss elimination method: The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ -1 & 2 & 1 & -2 \\ 2 & -3 & -1 & 3 \end{array}\right) \xrightarrow{R_1+R_2} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 0 & 2 & 2 \\ 2 & -3 & -1 & 3 \end{array}\right)$$

$$\xrightarrow{R_2 \longleftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 2 & -3 & -1 & 3 \\ 0 & 0 & 2 & 2 \end{array}\right) \xrightarrow{-2R_1+R_2} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 4 \\ 0 & 1 & -3 & -5 \\ 0 & 0 & 2 & 2 \end{array}\right)$$

$2z = 2 \implies z = 1$

$y - 3z = -5 \implies y - 3 = -5 \implies y = -2$

$x - 2y + z = 4 \implies x - 2(-2) + 1 = 4 \implies x + 5 = 4 \implies x = -1$

The solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$

Q.5 Evaluate the integrals:

(a) $\displaystyle\int (3x-1)\sqrt{3x^2-2x+1}\; dx$

(b) $\displaystyle\int (5x+4)^5\; dx$

(c) $\displaystyle\int x^3\; \ln x\; dx$

(d) $\displaystyle\int \frac{3\cos(3x) + 2\sin(2x)}{\sin(3x) - \cos(2x)}\; dx$

Solution :

(a) $\displaystyle\int (3x-1)\sqrt{3x^2-2x+1}\; dx = \int \left(3x^2-2x+1\right)^{\frac{1}{2}} (3x-1)\; dx$

$$= \frac{1}{2}\int \left(3x^2-2x+1\right)^{\frac{1}{2}} (6x-2)\; dx = \frac{1}{2}\, \frac{\left(3x^2-2x+1\right)^{\frac{3}{2}}}{\frac{3}{2}} + c$$

$$= \frac{\left(3x^2 - 2x + 1\right)^{\frac{3}{2}}}{3} + c$$

Using the formula $\displaystyle\int [f(x)]^n \;\; f'(x) \; dx = \frac{[f(x)]^{n+1}}{n+1} + c$, where $n \neq 1$

(b) $\displaystyle\int (5x + 4)^5 \; dx = \frac{1}{5} \int (5x + 4)^5 \; (5) \; dx = \frac{1}{5} \; \frac{(5x + 4)^6}{6} + c$

Using the formula $\displaystyle\int [f(x)]^n \;\; f'(x) \; dx = \frac{[f(x)]^{n+1}}{n+1} + c$, where $n \neq 1$

(c) $\displaystyle\int x^3 \;\; \ln x \; dx$

Using integration by parts:

$$\begin{array}{ll} u = \ln x & dv = x^3 \; dx \\ du = \dfrac{1}{x} \; dx & v = \dfrac{x^3}{3} \end{array}$$

$$\int x^3 \;\; \ln x \; dx = \frac{x^3}{3} \ln x - \int \frac{x^3}{3} \; \frac{1}{x} \; dx = \frac{x^3}{3} \ln x - \frac{1}{3} \int x^2 \; dx$$

$$= \frac{x^3}{3} \ln x - \frac{1}{3} \frac{x^3}{3} + c = \frac{x^3}{3} \ln x - \frac{x^3}{9} + c$$

(d) $\displaystyle\int \frac{3\cos(3x) + 2\sin(2x)}{\sin(3x) - \cos(2x)} \; dx = \ln|\sin(3x) - \cos(2x)| + c$

Using the formula $\displaystyle\int \frac{f'(x)}{f(x)} \; dx = \ln|f(x)| + c$

M 104 - GENERAL MATHEMATICS -2-

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Solution of the Final Exam

First semester 1443 H

Part 1: Multiple Choice Questions:

1. The center of the conic section of equation $4x^2 + 8x - y^2 + 2y - 1 = 0$ is

(a). $(-1, 1)$ (b). $(1, -1)$ (c). $(4, 1)$ (d). $(4, -1)$

Solution :

$4x^2 + 8x - y^2 + 2y - 1 = 0$

$4x^2 + 8x - y^2 + 2y = 1$

$4(x^2 + 2x) - (y^2 - 2y) = 1$

By completing the square:

$4(x^2 + 2x + 1) - (y^2 - 2y + 1) = 1 + 4 - 1$

$4(x + 1)^2 - (y - 1)^2 = 4$

$\dfrac{4(x+1)^2}{4} - \dfrac{(y-1)^2}{4} = \dfrac{4}{4}$

$(x + 1)^2 - \dfrac{(y-1)^2}{4} = 1$

The center of the conic section is $P(-1, 1)$.

The right answer is (a).

2. The equation of the ellipse of foci $(-3, 6)$;$(-3, 2)$ and length of major axis 14 is given by:

(a). $\dfrac{(x-6)^2}{14} + \dfrac{(y-2)^2}{3} = 1$ (b). $\dfrac{(x+3)^2}{45} + \dfrac{(y-4)^2}{49} = 1$

(c). $\dfrac{(x-3)^2}{9} + \dfrac{(y+6)^2}{4} = 1$ (d). $\dfrac{(x+3)^2}{9} + \dfrac{(y-2)^2}{36} = 1$

Solution :

The two foci are located in a line parallel to the y-axis, hence the equation of the ellipse is: $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$, where $b > a$.

The center P is the middle point of the two foci, hence:

$P(h, k) = \left(\dfrac{-3 + (-3)}{2}, \dfrac{6+2}{2} \right) = (-3, 4)$.

c is the distance between P and one of the foci, hence $c = 2$.

The length of the major axis is 14 means that $2b = 14 \implies b = 7$.

$c^2 = b^2 - a^2 \implies 4 = 49 - a^2 \implies a^2 = 49 - 4 = 45 \implies a = \sqrt{45}$.

The equation of the ellipse is $\dfrac{(x+3)^2}{45} + \dfrac{(y-4)^2}{49} = 1$.

The right answer is (b).

3. If $\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & -3 & 2 \\ 1 & 2 & -1 \end{pmatrix}$, then $\mathbf{A}\left(\mathbf{B}^T\right)$ equals

(a). $\begin{pmatrix} 0 & 3 & 4 \\ 0 & 2 & 2 \end{pmatrix}$ (b). $\begin{pmatrix} 0 & 3 \\ 0 & 2 \end{pmatrix}$ (c). $\begin{pmatrix} 7 & -3 \\ -7 & 4 \end{pmatrix}$ (d). undefined

Solution :

$$\mathbf{A}\left(\mathbf{B}^T\right) = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -3 & 2 \\ 2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0+3+4 & 1-2-2 \\ 0-3-4 & 0+2+2 \end{pmatrix} = \begin{pmatrix} 7 & -3 \\ -7 & 4 \end{pmatrix}$$

The right answer is (c).

4. The determinant $\begin{vmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 2 \\ 2 & 2 & 3 & 4 \end{vmatrix}$ is equal to

(a). -2 (b). 0 (c). 6 (d). 8

Solution :

Note that $C_4 = 2C_1$, hence $\begin{vmatrix} 1 & 1 & 1 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 2 \\ 2 & 2 & 3 & 4 \end{vmatrix} = 0$.

The right answer is (b).

5. The integral $\displaystyle\int x^3 \left(2 + x^4\right)^5 \, dx$ is equal to

(a). $\dfrac{\left(2+x^4\right)^6}{24} + c$ (b). $\dfrac{x^4\left(2+x^4\right)^6}{24} + c$

(c). $\dfrac{\left(2+x^4\right)^6}{6} + c$ (d). $\dfrac{x^4\left(2+x^4\right)^5}{4} + c$

Solution :

$$\int x^3 \left(2+x^4\right)^5 \; dx = \frac{1}{4} \int \left(2+x^4\right)^5 (4x^3) \; dx$$

$$= \frac{1}{4} \; \frac{\left(2+x^4\right)^6}{6} + c = \frac{\left(2+x^4\right)^6}{24} + c$$

The right answer is (a).

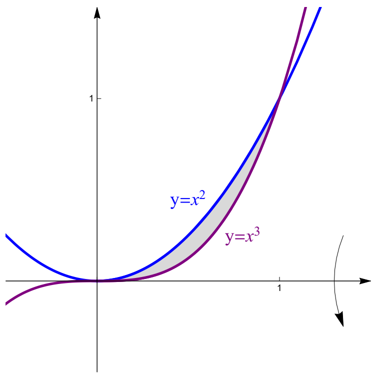
6. The volume of the solid, obtained by revolving the region bounded by the curves $y = x^2$, $y = x^3$ about the x-axis, is equal to

$(a).\ \frac{\pi}{7}$ $\quad (b).\ \frac{\pi}{12}$ $\quad (c).\ \frac{2\pi}{15}$ $\quad (d).\ \frac{2\pi}{35}$

Solution :

The points of intersection of $y = x^2$ and $y = x^3$:

$x^3 = x^2 \implies x^3 - x^2 = 0 \implies x^2(x-1) = 0 \implies x = 0, x = 1.$



Using Washer method:

$$\mathbf{V} = \pi \int_0^1 \left[(x^2)^2 - (x^3)^2\right] \; dx = \pi \int_0^1 \left(x^4 - x^6\right) \; dx = \pi \left[\frac{x^5}{5} - \frac{x^7}{7}\right]_0^1$$

$$= \pi \left[\left(\frac{1}{5} - \frac{1}{7}\right) - (0-0)\right] = \pi \left(\frac{7-5}{35}\right) = \frac{2\pi}{35}.$$

The right answer is (d).

7. The point with rectangular coordinates $(-1, \sqrt{3})$, has polar coordinates:

$(a).\ \left(2, \frac{\pi}{3}\right)$ $\quad (b).\ \left(2, \frac{2\pi}{3}\right)$ $\quad (c).\ \left(\sqrt{3}, \frac{\pi}{2}\right)$ $\quad (d).\ \left(\sqrt{3}, \frac{\pi}{4}\right)$

Solution :

$(x, y) = \left(-1, \sqrt{3}\right) \implies x = -1 \; , \; y = \sqrt{3}.$

$r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2} = \sqrt{1+3} = \sqrt{4} = 2.$

$\tan\theta = \dfrac{y}{x} = \dfrac{\sqrt{3}}{-1} = -\sqrt{3} \implies \theta = \dfrac{2\pi}{3}.$

$(r, \theta) = \left(2, \dfrac{2\pi}{3}\right).$

The right answer is (b).

8. Let $f(x, y) = x^3y^2 + y\sin\left(\dfrac{x}{y}\right)$. The partial derivative $\dfrac{\partial f}{\partial x}$ is equal to

$(a).\ 3x^2y^2 + \cos x$ $\quad(b).\ 6x^2y + \cos x$
$(c).\ 3x^2y^2 + \cos\left(\dfrac{x}{y}\right)$ $\quad(d).\ 6x^2y + \cos\left(\dfrac{x}{y}\right)$

Solution :

$\dfrac{\partial f}{\partial x} = (3x^2)y^2 + y\cos\left(\dfrac{x}{y}\right)\dfrac{1}{y} = 3x^2y^2 + \cos\left(\dfrac{x}{y}\right).$

The right answer is (c).

9. If $y = y(x)$ is defined implicitly by $e^{xy} = xy + 1$, for $x, y > 0$, then $\dfrac{dy}{dx}$ is equal to

$(a).\ xe^{xy} - y \quad (b).\ -\dfrac{e^{xy}}{y} \quad (c).\ -\dfrac{e^{xy}}{x} \quad (d).\ -\dfrac{y}{x}$

Solution :

$e^{xy} = xy + 1 \implies e^{xy} - xy - 1 = 0.$

Let $F(x, y) = e^{xy} - xy - 1$, then $F(x, y) = 0$.

$\dfrac{dy}{dx} = -\dfrac{F_x}{F_y} = -\dfrac{e^{xy}y - y}{e^{xy}x - x} = -\dfrac{y\left(e^{xy} - 1\right)}{x\left(e^{xy} - 1\right)} = -\dfrac{y}{x}.$

The right answer is (d).

10. The general solution of the differential equation $y' - \dfrac{3x^2}{2y} = 0$ is

$(a).\ 2y = 3x^2 + c \quad (b).\ y - x^3\ln|2y| = c \quad (c).\ y^2 = x^3 + c \quad (d).\ y - \dfrac{x^3}{y^3} = c$

Solution :

$\dfrac{dy}{dx} = \dfrac{3x^2}{2y}$

$2y\ dy = 3x^2\ dx$ (Separable differential equation)

$y^2 = x^3 + c$

The right answer is (c).

Part 2: Essay Questions

11. Find the elements of the conic section $4x^2 - 9y^2 - 8x - 36y - 68 = 0$ and then sketch it.

Solution :

$4x^2 - 9y^2 - 8x - 36y - 68 = 0$

$4x^2 - 8x - 9y^2 - 36y = 68$

$4(x^2 - 2x) - 9(y^2 + 4y) = 68$

By completing the square :

$4(x^2 - 2x + 1) - 9(y^2 + 4y + 4) = 68 + 4 - 36$

$4(x-1)^2 - 9(y+2)^2 = 36$

$\dfrac{4(x-1)^2}{36} - \dfrac{9(y+2)^2}{36} = \dfrac{36}{36}$

$\dfrac{(x-1)^2}{9} - \dfrac{(y+2)^2}{4} = 1$

The conic section is a hyperbola.

The center is $P(h,k) = (1,-2)$.

$a^2 = 9 \implies a = 3$ and $b^2 = 4 \implies b = 2$.

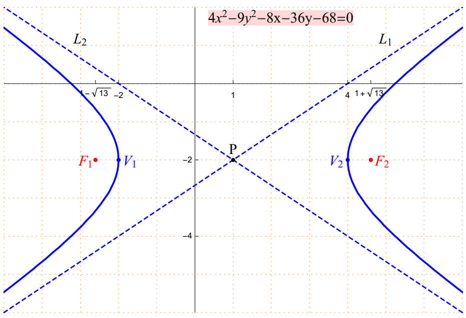
$c^2 = a^2 + b^2 = 9 + 4 = 13 \implies c = \sqrt{13}$.

The vertices are $V_1(-2,-2)$ and $V_2(4,-2)$.

The foci are $F_1\left(1-\sqrt{13},-2\right)$ and $F_2\left(1+\sqrt{13},-2\right)$.

The equations of the asymptotes are :

$L_1 :\ y + 2 = \dfrac{2}{3}(x-1)$ and $L_2 :\ y + 2 = -\dfrac{2}{3}(x-1)$



12. Solve by using Gauss Elimination Method the system

$$\begin{cases} x & + & y & + & z & = & 2 \\ x & - & y & + & 2z & = & 0 \\ 2x & & & + & z & = & 2 \end{cases}$$

Solution :

The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & -1 & 2 & 0 \\ 2 & 0 & 1 & 2 \end{array}\right) \xrightarrow{-R_1+R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 1 & -2 \\ 2 & 0 & 1 & 2 \end{array}\right)$$

$$\xrightarrow{-2R_1+R_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 1 & -2 \\ 0 & -2 & -1 & -2 \end{array}\right) \xrightarrow{-R_2+R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -2 & 1 & -2 \\ 0 & 0 & -2 & 0 \end{array}\right)$$

$-2z = 0 \implies z = 0$

$-2y + z = -2 \implies -2y = -2 \implies y = 1$

$x + y + z = 2 \implies x + 1 = 2 \implies x = 1$

The solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

13. Compute the integral $\displaystyle\int \frac{2x^2 - 2x - 2}{x(x+1)(x-1)}\, dx$

Solution : Using partial fractions method.

$$\frac{2x^2 - 2x - 2}{x(x-1)(x+1)} = \frac{A_1}{x} + \frac{A_2}{x-1} + \frac{A_3}{x+1}$$

$2x^2 - 2x - 2 = A_1(x-1)(x+1) + A_2 x(x+1) + A_3 x(x-1)$

$2x^2 - 2x - 2 = A_1(x^2 - 1) + A_2(x^2 + x) + A_3(x^2 - x)$

$2x^2 - 2x - 2 = A_1 x^2 - A_1 + A_2 x^2 + A_2 x + A_3 x^2 - A_3 x$

$2x^2 - 2x - 2 = (A_1 + A_2 + A_3)\, x^2 + (A_2 - A_3)\, x - A_1$

$$\begin{array}{ll} A_1 + A_2 + A_3 = 2 & \longrightarrow \ (1) \\ A_2 - A_3 = -2 & \longrightarrow \ (2) \\ -A_1 = -2 & \longrightarrow \ (3) \end{array}$$

From equation (1) : $A_1 = 2$.

Equation (1) becomes : $A_2 + A_3 = 0 \longrightarrow$ (4).

Equation (4) + Equation (2) : $2A_2 = -2 \implies A_2 = -1$.

From equation (4) : $A_3 = 1$.

$$\int \frac{2x^2-2x-2}{x(x+1)(x-1)}\ dx = \int \left(\frac{2}{x} + \frac{-1}{x-1} + \frac{1}{x+1}\right)\ dx$$

$$= 2\int \frac{1}{x}\ dx - \int \frac{1}{x-1}\ dx + \int \frac{1}{x+1}\ dx$$

$$= 2\ln|x| - \ln|x-1| + \ln|x+1| + c$$

14. If $w = x^2+xy+3y^2$, $x = u^2+v$ and $y = v^2$, use the chain rule to compute $\dfrac{\partial w}{\partial u}$.

Solution :

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial w}{\partial x} = 2x+y \text{ and } \frac{\partial x}{\partial u} = 2u,$$

$$\frac{\partial w}{\partial y} = x+6y \text{ and } \frac{\partial y}{\partial u} = 0.$$

$$\frac{\partial w}{\partial u} = (2x+y)(2u) + (x+6y)(0) = (2x+y)(2u) = \left[2(u^2+v)+v^2\right](2u)$$

$$= 2u\left[2u^2+2v+v^2\right] = 4u^2+4uv+2uv^2.$$

15. Find the general solution of the linear differential equation $xy'+2y = 5x^3$.

Solution :

$$xy' + 2y = 5x^3$$

$$y' + \left(\frac{2}{x}\right) y = 5x^2$$

It is a First-order differential equation .

$P(x) = \dfrac{2}{x}$ and $Q(x) = 5x^2$

The integrating factor is :

$$u(x) = e^{\int P(x)\ dx} = e^{\int \frac{2}{x}\ dx} = e^{2\int \frac{1}{x}\ dx} = e^{2\ln|x|} = e^{\ln(x^2)} = x^2.$$

The general solution of the differential equation is :

$$y = \frac{1}{u(x)}\int u(x)Q(x)\ dx = \frac{1}{x^2}\int x^2\ (5x^2)\ dx = \frac{1}{x^2}\int 5x^4\ dx$$

$$= \frac{1}{x^2}\left(x^5+c\right) = x^3 + \frac{c}{x^2}$$

M 104 - GENERAL MATHEMATICS -2-

Dr. Tariq A. AlFadhel

Solution of the Mid-Term Exam

Second semester 1443 H

Q.1 Find the elements of the conic section of equation

$4x^2 + 9y^2 - 8x - 36y + 4 = 0$, then sketch it.

Solution :

$4x^2 + 9y^2 - 8x - 36y + 4 = 0$

$4x^2 - 8x + 9y^2 - 36y = -4$

$4(x^2 - 2x) + 9(y^2 - 4y) = -4$

By completing the square.

$4(x^2 - 2x + 1) + 9(y^2 - 4y + 4) = -4 + 4 + 36$

$4(x - 1)^2 + 9(y - 2)^2 = 36$

$\dfrac{4(x-1)^2}{36} + \dfrac{9(y-2)^2}{36} = \dfrac{36}{36}$

$\dfrac{(x-1)^2}{9} + \dfrac{(y-2)^2}{4} = 1$

The conic section is an ellipse.

The center is $P(1, 2)$

$a^2 = 9 \implies a = 3$

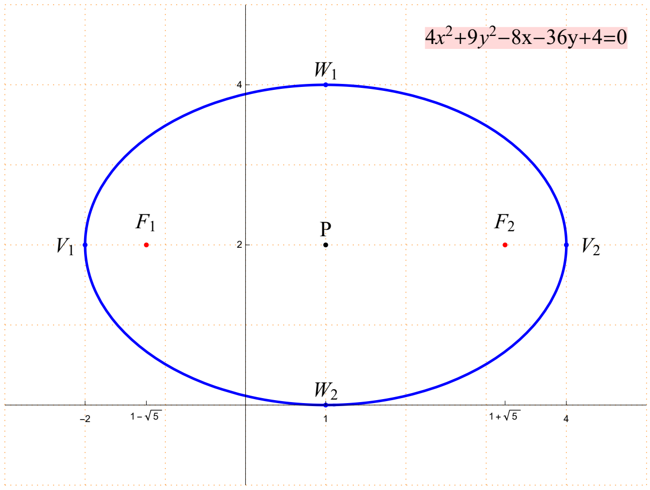
$b^2 = 4 \implies b = 2$

$c^2 = a^2 - b^2 = 9 - 4 = 5 \implies c = \sqrt{5}$

The vertices are $V_1(-2, 2)$ and $V_2(4, 2)$

The foci are $F_1\left(1 - \sqrt{5}, 2\right)$ and $F_2\left(1 + \sqrt{5}, 2\right)$

The end-points of the minor axis are $W_1(1, 4)$ and $W_2(1, 0)$



Q.2 Find the standard equation of the hyperbola with foci $(2,3)$,(-6,3) and the the distance between its two vertices equals to 6, then sketch it.

Solution :

The two foci are located on a line parallel to the x-axis.

The standard equation of the hyperbola is $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1.$

The center is the middle point between the two foci.

$P(h,k) = \left(\dfrac{2+(-6)}{2}, \dfrac{3+3}{2}\right) = (-2,3)$, hence $h=-2$ and $k=3$.

c is the distance between one of the foci and P , hence $c=4$.

The distance between its two vertices equals 6 means $2a=6 \implies a=3$.

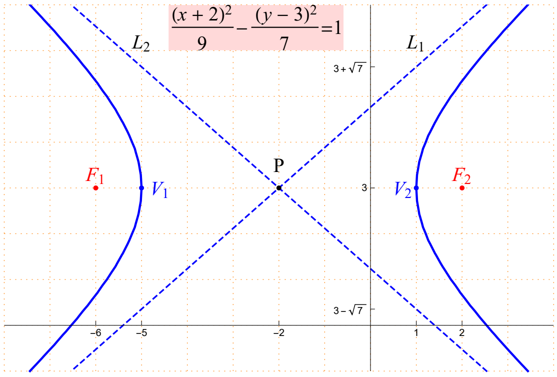
$c^2 = a^2+b^2 \implies 16 = 9+b^2 \implies b^2 = 16-9 = 7 \implies b = \sqrt{7}$.

The standard equation of the hyperbola is $\dfrac{(x+2)^2}{9} - \dfrac{(y-3)^2}{7} = 1.$

The vertices are $V_1(-5,3)$ and $V_2(1,3)$.

The equations of the asymptotes are :

$L_1 \ : \ y-3 = \dfrac{\sqrt{7}}{3}\,(x+2)$ and $L_2 \ : \ y-3 = -\dfrac{\sqrt{7}}{3}\,(x+2)$.



Q.3 Calculate, whenever it is possible, $2\mathbf{A} - \mathbf{B}^T$ and $\mathbf{AB}$ for matrices

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 3 & -2 \end{pmatrix},\ \mathbf{B} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ -1 & 3 \end{pmatrix}.$$

Solution :

$$2\mathbf{A} - \mathbf{B}^T = \begin{pmatrix} 4 & 2 & -2 \\ 2 & 6 & -4 \end{pmatrix} - \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 2 & -1 \\ 0 & 5 & -7 \end{pmatrix}$$

$$\mathbf{AB} = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 3 & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ -1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 2+0+1 & 4+1-3 \\ 1+0+2 & 2+3-6 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 3 & -1 \end{pmatrix}$$

Q.4 Consider the system of the linear equations:

$$\begin{cases} 2x + y + z = 1 \\ x - y = 0 \\ y - z = 3 \end{cases}$$

(a) Solve this system using Cramer's rule.

(b) Solve this system using Gauss elimination method.

Solution :

(a) Using Cramer's rule :

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

$$\begin{array}{ccccc} 2 & 1 & 1 & 2 & 1 \\ 1 & -1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 & 1 \end{array}$$

$|\mathbf{A}| = (2+0+1) - (0+0-1) = 3 - (-1) = 3 + 1 = 4 \neq 0$

$\mathbf{A}_x = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 3 & 1 & -1 \end{pmatrix}$

$$\begin{array}{ccccc} 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 0 & -1 \\ 3 & 1 & -1 & 3 & 1 \end{array}$$

$|\mathbf{A}_x| = (1+0+0) - (-3+0+0) = 1 - (-3) = 1 + 3 = 4$

$x = \dfrac{|\mathbf{A}_x|}{|\mathbf{A}|} = \dfrac{4}{4} = 1$

$\mathbf{A}_y = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 3 & -1 \end{pmatrix}$

$$\begin{array}{ccccc} 2 & 1 & 1 & 2 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 3 & -1 & 0 & 3 \end{array}$$

$|\mathbf{A}_y| = (0+0+3) - (0+0-1) = 3 - (-1) = 3 + 1 = 4$

$y = \dfrac{|\mathbf{A}_y|}{|\mathbf{A}|} = \dfrac{4}{4} = 1$

$\mathbf{A}_z = \begin{pmatrix} 2 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 3 \end{pmatrix}$

$$\begin{array}{ccccc} 2 & 1 & 1 & 2 & 1 \\ 1 & -1 & 0 & 1 & -1 \\ 0 & 1 & 3 & 0 & 1 \end{array}$$

$|\mathbf{A}_z| = (-6+0+1) - (0+0+3) = -5 - 3 = -8$

$z = \dfrac{|\mathbf{A}_z|}{|\mathbf{A}|} = \dfrac{-8}{4} = -2$

(b) Using Gauss elimination method: The augmented matrix is

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 3 \end{array}\right) \xrightarrow{R_1 \longleftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 2 & 1 & 1 & 1 \\ 0 & 1 & -1 & 3 \end{array}\right)$$

$$\xrightarrow{-2R_1+R_2} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 3 & 1 & 1 \\ 0 & 1 & -1 & 3 \end{array}\right) \xrightarrow{R_2 \longleftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 3 \\ 0 & 3 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{-3R_2+R_3} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 4 & -8 \end{array}\right)$$

$4z = -8 \implies z = \dfrac{-8}{4} = -2$

$y - z = 3 \implies y - (-2) = 3 \implies y + 2 = 3 \implies y = 3 - 2 = 1$

$x - y = 0 \implies x - 1 = 0 \implies x = 1$

The solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

Q.5 Evaluate the integrals:

(a) $\displaystyle\int \frac{4x^3+1}{\sqrt{x^4+x+1}}\ dx$

(b) $\displaystyle\int (x+1)\ e^{x^2+2x}\ dx$

(c) $\displaystyle\int (2x+1)\ \cos x\ dx$

(d) $\displaystyle\int (2x+1)\ \ln x\ dx$

Solution :

(a) $\displaystyle\int \frac{4x^3+1}{\sqrt{x^4+x+1}}\ dx = \int \left(x^4+x+1\right)^{-\frac{1}{2}} (4x^3+1)\ dx$

$= \dfrac{\left(x^4+x+1\right)^{\frac{1}{2}}}{\frac{1}{2}} + c = 2\ \left(x^4+x+1\right)^{\frac{1}{2}} + c$

Using the formula $\displaystyle\int [f(x)]^n\ \ f'(x)\ dx = \frac{[f(x)]^{n+1}}{n+1} + c$, where $n \neq 1$

(b) $\displaystyle\int (x+1)\ e^{x^2+2x}\ dx = \frac{1}{2}\int e^{x^2+2x}\ \ [2(x+1)]\ \ dx$

$= \dfrac{1}{2}\displaystyle\int e^{x^2+2x}\ (2x+2)\ dx = \frac{1}{2}\ e^{x^2+2x} + c$

Using the formula $\displaystyle\int e^{f(x)}\ f'(x)\ dx = e^{f(x)} + c$

(c) $\int (2x+1)\ \cos x\ dx$

Using integration by parts:

$$\begin{array}{ll} u = 2x+1 & dv = \cos x\ dx \\ du = 2\ dx & v = \sin x \end{array}$$

$$\int (2x+1)\ \cos x\ dx = (2x+1)\sin x - \int 2\sin x\ dx$$

$$= (2x+1)\sin x - 2\int \sin x\ dx = (2x+1)\sin x - 2(-\cos x) + c$$

$$= (2x+1)\sin x + 2\cos x + c$$

(d) $\int (2x+1)\ \ln x\ dx$

Using integration by parts:

$$\begin{array}{ll} u = \ln x & dv = (2x+1)\ dx \\ du = \dfrac{1}{x}\ dx & v = x^2 + x \end{array}$$

$$\int (2x+1)\ \ln x\ dx = (x^2+x)\ln x - \int (x^2+x)\ \frac{1}{x}\ dx$$

$$= (x^2+x)\ln x - \int (x+1)\ dx = (x^2+x)\ln x - \left(\frac{x^2}{2} + x\right) + c$$

$$= (x^2+x)\ln x - \frac{x^2}{2} - x + c$$

M 104 - GENERAL MATHEMATICS -2-

Dr. Tariq A. AlFadhel

Solution of the Final Exam

Second semester 1443 H

Q.1 **(a)** Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$. Compute (if possible) $\mathbf{AB}$ and $\mathbf{BA}$.

(b) Compute the determinant $\begin{vmatrix} -1 & 6 & 2 \\ 0 & 0 & 5 \\ 0 & 3 & 4 \end{vmatrix}$.

(c) Solve by using Gauss Elimination Method the linear system

$$\begin{cases} x + y + 3z = 7 \\ -2x - y - z = -4 \\ 3x + 2y - 2z = -1 \end{cases}$$

Solution :

(a) $\mathbf{AB} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 0+0+0 & 1+0+0 \\ 0+1+0 & 1+1+0 \\ 0+1+1 & 1+1+0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \\ 2 & 2 \end{pmatrix}$

$\mathbf{BA}$ can not be computed.

(b) **Solution (1):** Using Sarrus Method

$$\begin{matrix} -1 & 6 & 2 & -1 & 6 \\ 0 & 0 & 5 & 0 & 0 \\ 0 & 3 & 4 & 0 & 3 \end{matrix}$$

$$\begin{vmatrix} -1 & 6 & 2 \\ 0 & 0 & 5 \\ 0 & 3 & 4 \end{vmatrix} = (0+0+0) - (0+(-15)+0) = 0 - (-15) = 15$$

Solution (2) : Using the properties of determinants:

$$\begin{vmatrix} -1 & 6 & 2 \\ 0 & 0 & 5 \\ 0 & 3 & 4 \end{vmatrix} \xrightarrow{R_2 \longleftrightarrow R_3} (-1) \times \begin{vmatrix} -1 & 6 & 2 \\ 0 & 3 & 4 \\ 0 & 0 & 5 \end{vmatrix} = (-1) \times (-1 \times 3 \times 5) = 15$$

(c) Using Gauss Elimination Method :

$$\left(\begin{array}{ccc|c} 1 & 1 & 3 & 7 \\ -2 & -1 & -1 & -4 \\ 3 & 2 & -2 & -1 \end{array}\right) \xrightarrow{2R_1+R_2} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 7 \\ 0 & 1 & 5 & 10 \\ 3 & 2 & -2 & -1 \end{array}\right)$$

$$\xrightarrow{-3R_1+R_3} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 7 \\ 0 & 1 & 5 & 10 \\ 0 & -1 & -11 & -22 \end{array}\right) \xrightarrow{R_2+R_3} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 7 \\ 0 & 1 & 5 & 10 \\ 0 & 0 & -6 & -12 \end{array}\right)$$

$$-6z = -12 \implies z = \frac{-12}{-6} = 2$$

$$y + 5z = 10 \implies y + 5(2) = 10 \implies y + 10 = 10 \implies y = 0$$

$$x + y + 3z = 7 \implies x + 0 + 3(2) = 7 \implies x + 6 = 7 \implies x = 1$$

The solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

Q.2 **(a)** Find the standard equation of the ellipse with foci $(3, 6)$ and $(3, -2)$, and vertex $(3, -3)$ and then sketch it.

(b) Find the elements of the conic section $9x^2 - 4y^2 - 18x - 24y + 9 = 0$ and then sketch it.

Solution :

(a) The two foci and the vertex are located on a line parallel to the y-axis.

The standard equation of the ellipse is $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$, where $b > a$.

$P(h, k) = \left(\dfrac{3+3}{2}, \dfrac{6+(-2)}{2}\right) = (3, 2)$, hence $h = 3$ and $k = 2$

b is the distance between the vertex $(3, -3)$ and P , hence $b = 5$

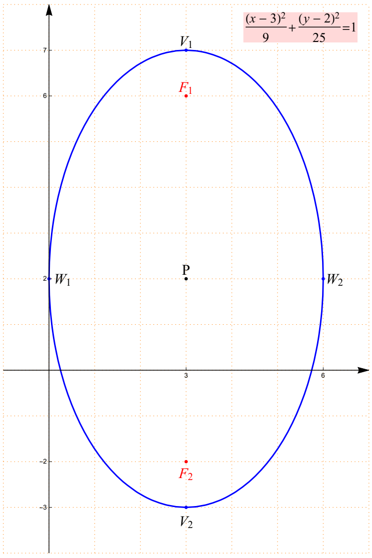
c is the distance between one of the foci and P , hence $c = 3$

$c^2 = b^2 - a^2 \implies 16 = 25 - a^2 \implies a^2 = 25 - 16 = 9 \implies a = 3$

The standard equation of the ellipse is $\dfrac{(x-3)^2}{9} - \dfrac{(y-2)^2}{25} = 1$

The other vertex is $(3, 7)$

The end-points of the minor axis are $(0, 2)$ and $(6, 2)$.



(b) $9x^2 - 4y^2 - 18x - 24y + 9 = 0$

$9x^2 - 18x - 4y^2 - 24y = -9$

$9(x^2 - 2x) - 4(y^2 + 6y) = -9$

By completing the square

$9(x^2 - 2x + 1) - 4(y^2 + 6y + 9) = -9 + 9 - 36$

$9(x-1)^2 - 4(y+3)^2 = -36$

$$\frac{9(x-1)^2}{-36} - \frac{4(y+3)^2}{-36} = 1$$

$$-\frac{(x-1)^2}{4} + \frac{(y+3)^2}{9} = 1 \implies \frac{(y+3)^2}{9} - \frac{(x-1)^2}{4} = 1$$

The conic section is a hyperbola.

The center is $P(1, -3)$.

$a^2 = 4 \implies a = 2.$

$b^2 = 9 \implies b = 3.$

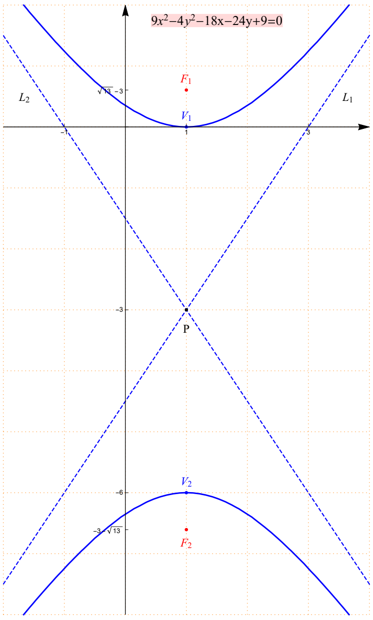
$c^2 = a^2 + b^2 = 4 + 9 = 13 \implies c = \sqrt{13}.$

The vertices are $V_1(1, 0)$ and $V_2(1, -6)$

The foci are $F_1\left(1, -3 + \sqrt{13}\right)$ and $F_2\left(1, -3 - \sqrt{13}\right)$.

The equations of the asymptotes are :

$$L_1 \ : \ y + 3 = \frac{3}{2}\,(x - 1) \text{ and } L_2 \ : \ y + 3 = -\frac{3}{2}\,(x - 1)$$



Q.3 **(a)** Compute the integrals :

(i) $\int x\,\sqrt{x^2+4}\,dx$ (ii) $\int \tan^{-1} x\,dx$ (iii) $\int \dfrac{x+3}{(3-x)(x-2)}\,dx$

(b) Sketch and find the area of the region bounded by the curves :

$y = 4 - x^2$ and $y = x3$.

(c) The region bounded by the curves curves $y = \sqrt{x}$, $y = 1$, $y = 2$ and $x = 0$ is rotated about the y-axis to form a solid S . Find the volume of S.

Solution :

(a) (i) $\int x\,\sqrt{x^2+4}\,dx = \int x\ \left(x^2+4\right)^{\frac{1}{2}}\ dx\frac{1}{2}\int \left(x^2+4\right)^{\frac{1}{2}}(2x)\ dx$

$$= \frac{1}{2}\frac{\left(x^2+4\right)^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{1}{3}\ \left(x^2+4\right)^{\frac{3}{2}} + c$$

(ii) $\int \tan^{-1} x\ dx$

Using integration by parts

$u = \tan^{-1} xx$ $\qquad dv = dx$

$du = \dfrac{1}{1+x^2}\ dx$ $\qquad v = x$

$$\int \tan^{-1} x\ dx = x\ \tan^{-1} x - \int x\ \frac{1}{1+x^2}\ dx$$

$= x\ \tan^{-1} x - \frac{1}{2}\int \frac{2x}{1+x^2}\ dx = x\ \tan^{-1} x - \frac{1}{2}\ln|1+x^2| + c$

(iii) $\int \frac{x+3}{(3-x)(x-2)}\ dx$

Using the method of partial fractions

$\frac{x+3}{(3-x)(x-2)} = \frac{A_1}{3-x} + \frac{A_2}{x-2}$

$x+3 = A_1(x-2) + A_2(3-x)$

Put $x = 3$ then $3+3 = A_1(3-2) \implies A_1 = 6$

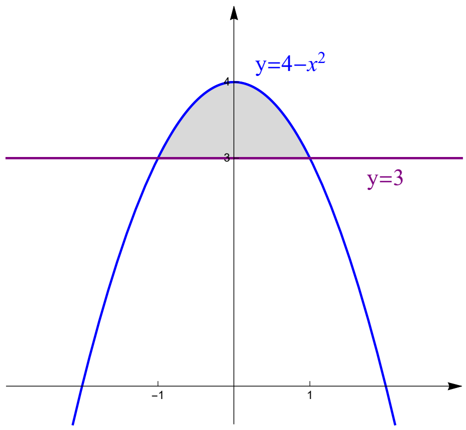
Put $x = 2$ then $2+3 = A_2(3-2) \implies A_2 = 5$

$\int \frac{x+3}{(3-x)(x-2)}\ dx = \int \left(\frac{6}{3-x} + \frac{5}{x-2}\right)\ dx$

$= -6\int \frac{-1}{3-x}\ dx + 5\int \frac{1}{x-2}\ dx = -6\ln|3-x| + 5\ln|x-2| + c$

(b) $y = 4 - x^2$ is a parabola opens downwards with vertex $(0, 4)$.

$y = 3$ is a straight line parallel to the x-axis and passing through $(0, 3)$.



Points of intersection of $y = 4 - x^2$ and $y = 3$:

$3 = 4 - x^2 \implies x^2 - 1 = 0 \implies (x-1)(x+1) = 0 \implies x = -1\ ,\ x = 1$

Area $= \int_{-1}^{1} \left[(4-x^2) - 3\right]\ dx = \int_{-1}^{1} (1-x^2)\ dx = \left[x - \frac{x^3}{3}\right]_{-1}^{1}$

$= \left(1 - \frac{1^3}{3}\right) - \left(-1 - \frac{(-1)^3}{3}\right) = 1 - \frac{1}{3} - \left(-1 + \frac{1}{3}\right)$

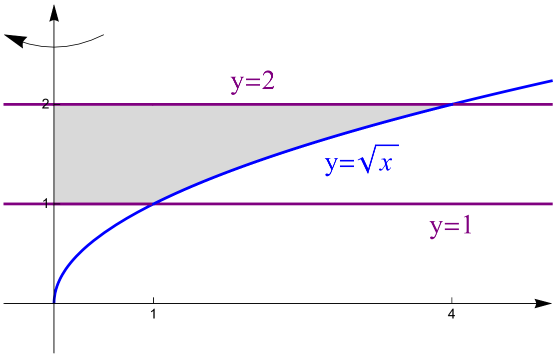
$= 1 - \frac{1}{3} + 1 - \frac{1}{3} = 2 - \frac{2}{3} = \frac{4}{3}$

(c) $x = 0$ is the y-axis .

$y = 1$ is a straight line parallel to the x-axis and passes through $(0, 1)$.

$y = 2$ is a straight line parallel to the x-axis and passes through $(0, 2)$.

$y = \sqrt{x}$ is the upper-half of the parabola $x = y^2$ which opens to the right with vertex $(0, 0)$,



$y = \sqrt{x} \implies x = y^2$

Using Disk Method :

$$\text{Volume} = \pi \int_1^2 (y^2)^2 \, dy = \pi \int_1^2 y^4 \, dy = \pi \left[\frac{y^5}{5} \right]_1^2$$

$$= \pi \left[\frac{2^5}{5} - \frac{1^5}{5} \right] = \pi \left(\frac{32 - 1}{5} \right) = \frac{31\pi}{5}$$

Q.4 **(a)** We define $z(x, y)$ implicitly by the equation $x^2y + \sin(xyz) = 1$. Compute the partial derivative $\dfrac{\partial z}{\partial y}$.

(b) Solve the differential equation : $xy^2 + y' \, e^{-x} = 0$.

Solution :

(a) $x^2y + \sin(xyz) = 1 \implies x^2y + \sin(xyz) - 1 = 0.$

Put $F(x, y, z) = x^2y + \sin(xyz) - 1$, then $F(x, y, z) = 0$.

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{x^2 + \cos(xyz) \; (xz) - 0}{0 + \cos(xyz) \; (xy) - 0} = -\frac{x^2 + xz\cos(xyz)}{xy\cos(xyz)}$$

(b) $xy^2 + y' \, e^{-x} = 0$

$y' \, e^{-x} = -xy^2$

$\dfrac{dy}{dx} \, e^{-x} = -xy^2$

$$\frac{1}{y^2}\ dy = -x\ e^x\ dx$$

It is a separable differential equation.

$$\int \frac{1}{y^2}\ dy = -\int x\ e^x\ dx$$

$$\frac{y^{-1}}{-1} = -\left(x\ e^x - e^x + c\right)$$

$$\frac{1}{y} = x\ e^x - e^x + c$$

$$y = \frac{1}{x\ e^x - e^x + c}.$$

Note that $\int x\ e^x\ dx$ can be solved by parts :

$$\begin{array}{ll} u = x & dv = e^x\ dx \\ du = dx & v = e^x \end{array}$$

$$\int x\ e^x\ dx = x\ e^x - \int e^x\ dx = x\ e^x - e^x + c$$