M 104 - GENERAL MATHEMATICS -2-
Dr. Tariq A. AlFadhel
Solution of the Mid-Term Exam
First Term 1444 H

Q.1 Find the elements of the conic section of equation

$4y^2 = -9x^2 + 18x + 27$, then sketch it. [4]

Solution :

$4y^2 = -9x^2 + 18x + 27$

$9x^2 - 18x + 4y^2 = 27$

$9(x^2 - 2x) + 4y^2 = 27$

By completing the square.

$9(x^2 - 2x + 1) + 4y^2 = 27 + 9$

$9(x-1)^2 + 4y^2 = 36$

$\dfrac{9(x-1)^2}{36} + \dfrac{4y^2}{36} = \dfrac{36}{36}$

$\dfrac{(x-1)^2}{4} + \dfrac{y^2}{9} = 1$

The conic section is an ellipse.

The center is $P(1,0)$

$a^2 = 4 \implies a = 2$

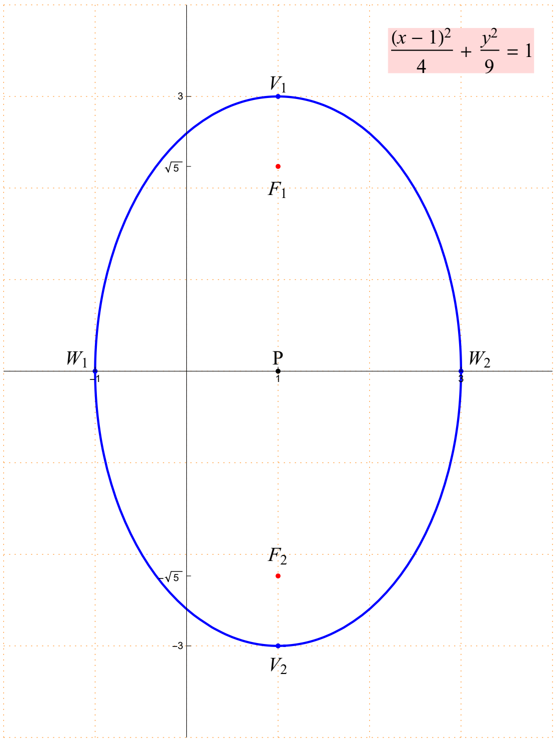
$b^2 = 9 \implies b = 3$

$c^2 = b^2 - a^2 = 9 - 4 = 5 \implies c = \sqrt{5}$

The vertices are $V_1(1,3)$ and $V_2(1,-3)$

The foci are $F_1\left(1,\sqrt{5}\right)$ and $F_2\left(1,-\sqrt{5}\right)$

The end-points of the minor axis are $W_1(-1,0)$ and $W_2(3,0)$



Q.2 Find the standard equation of the parabola with vertex $(2,3)$ and Focus $(2,1)$, then sketch it. [4]

Solution :

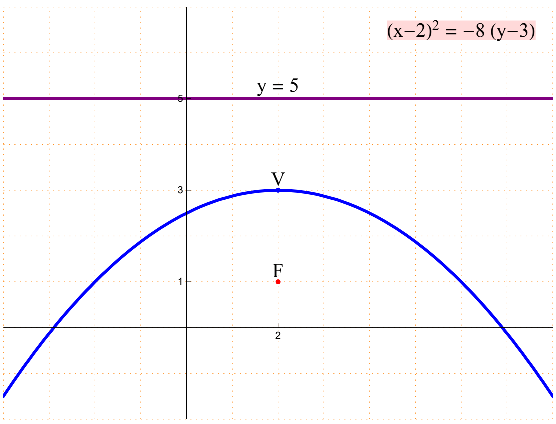
The parabola opens downwards.

The standard equation of the parabola is $(x-2)^2 = -4a(y-3)$.

a is the distance between $V(2,3)$ and $F(2,1)$, hence $a = 2$.

The standard equation of the parabola is $(x-2)^2 = -8(y-3)$.

The equation of the directrix is $y = 5$.



Q.3 Calculate, whenever it is possible, $\mathbf{A}+\mathbf{B}^T$ and $\mathbf{AB}$ for matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & -2 \end{pmatrix}.$$ [4]

Solution :

$$\mathbf{A}+\mathbf{B}^T = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} + \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 3 \\ 0 & 3 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{AB} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 1 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 1-1+2 & 0+1+2 & 0+0-4 \\ 0-2+1 & 0+2+1 & 0+0-2 \\ 0+0+2 & 0+0+2 & 0+0-4 \end{pmatrix} = \begin{pmatrix} 2 & 3 & -4 \\ -1 & 3 & -2 \\ 2 & 2 & -4 \end{pmatrix}$$

Q.4 Consider the system of the linear equations:

$$\begin{cases} 2x - 2y + z = 2 \\ x - y + z = 2 \\ 2x + 2y - z = 2 \end{cases}$$

(a) Solve this system using Cramer's rule. [4]

(b) Solve this system using Gauss-Jordan elimination method. [4]

Solution :

(a) Using Cramer's rule :

$$\mathbf{A} = \begin{pmatrix} 2 & -2 & 1 \\ 1 & -1 & 1 \\ 2 & 2 & -1 \end{pmatrix}$$

$$\begin{matrix} 2 & -2 & 1 & 2 & -2 \\ 1 & -1 & 1 & 1 & -1 \\ 2 & 2 & -1 & 2 & 2 \end{matrix}$$

$$|\mathbf{A}| = (2-4+2) - (-2+4+2) = 0 - 4 = -4 \neq 0$$

$$\mathbf{A}_x = \begin{pmatrix} 2 & -2 & 1 \\ 2 & -1 & 1 \\ 2 & 2 & -1 \end{pmatrix}$$

$$\begin{matrix} 2 & -2 & 1 & 2 & -2 \\ 2 & -1 & 1 & 2 & -1 \\ 2 & 2 & -1 & 2 & 2 \end{matrix}$$

$|\mathbf{A}_x| = (2-4+4) - (-2+4+4) = 2-6 = -4$

$x = \dfrac{|\mathbf{A}_x|}{|\mathbf{A}|} = \dfrac{-4}{-4} = 1$

$\mathbf{A}_y = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 2 & 1 \\ 2 & 2 & -1 \end{pmatrix}$

$$\begin{matrix} 2 & 2 & 1 & 2 & 2 \\ 1 & 2 & 1 & 1 & 2 \\ 2 & 2 & -1 & 2 & 2 \end{matrix}$$

$|\mathbf{A}_y| = (-4+4+2) - (4+4-2) = 2-6 = -4$

$y = \dfrac{|\mathbf{A}_y|}{|\mathbf{A}|} = \dfrac{-4}{-4} = 1$

$\mathbf{A}_z = \begin{pmatrix} 2 & -2 & 2 \\ 1 & -1 & 2 \\ 2 & 2 & 2 \end{pmatrix}$

$$\begin{matrix} 2 & -2 & 2 & 2 & -2 \\ 1 & -1 & 2 & 1 & -1 \\ 2 & 2 & 2 & 2 & 2 \end{matrix}$$

$|\mathbf{A}_z| = (-4-8+4) - (-4+8-4) = -8-0 = -8$

$z = \dfrac{|\mathbf{A}_z|}{|\mathbf{A}|} = \dfrac{-8}{-4} = 2$

(b) Using Gauss-Jordan elimination method: The augmented matrix is

$$\left(\begin{array}{ccc|c} 2 & -2 & 1 & 2 \\ 1 & -1 & 1 & 2 \\ 2 & 2 & -1 & 2 \end{array}\right) \xrightarrow{R_1 \longleftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 2 & -2 & 1 & 2 \\ 2 & 2 & -1 & 2 \end{array}\right)$$

$$\xrightarrow{-2R_1+R_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 0 & -1 & -2 \\ 2 & 2 & -1 & 2 \end{array}\right) \xrightarrow{R_2 \longleftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 2 & 2 & -1 & 2 \\ 0 & 0 & -1 & -2 \end{array}\right)$$

$$\xrightarrow{-2R_1+R_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 4 & -3 & -2 \\ 0 & 0 & -1 & -2 \end{array}\right) \xrightarrow{-R_3} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 4 & -3 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$$\xrightarrow{3R_3+R_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 4 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right) \xrightarrow{-R_3+R_1} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 4 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$$\xrightarrow{\frac{1}{4}\ R_2} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right) \xrightarrow{R_2+R_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

The solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

Q.5 Evaluate the integrals:

(a) $\int \left(4x^3 - \frac{2}{x^3} + e^x\right) \; dx$ [2]

(b) $\int 20x^3 \left(x^4+2\right)^4 \; dx$ [2]

(c) $\int \sec^2 x \; \ln|\sin x| \; dx$ [3]

(d) $\int \frac{x+1}{(x-2)(x-1)} \; dx$ [3]

Solution :

(a) $\int \left(4x^3 - \frac{2}{x^3} + e^x\right) \; dx = \int 4x^3 \; dx - \int \frac{2}{x^3} \; dx + \int e^x \; dx$

$= \int 4x^3 \; dx - 2\int x^{-3} \; dx + \int e^x \; dx = x^4 - 2\left(\frac{x^{-2}}{-2}\right) + e^x + c$

$= x^4 + x^{-2} + e^x + c$

(b) $\int 20x^3 \left(x^4+2\right)^4 \; dx = 5\int \left(x^4+2\right)^4 \; (4x^3) \; dx$

$= 5\left[\frac{\left(x^4+2\right)^5}{5}\right] + c = \left(x^4+2\right)^5 + c$

Using the formula $\int [f(x)]^n \; f'(x) \; dx = \frac{[f(x)]^{n+1}}{n+1} + c$, where $n \neq 1$

(c) $\int \sec^2 x \; \ln|\sin x| \; dx$

Using integration by parts:

$u = \ln|\sin x| \qquad dv = \sec^2 x \; dx$

$du = \frac{\cos x}{\sin x} \; dx = \cot x \; dx \qquad v = \tan x$

$\int \sec^2 x \; \ln|\sin x| \; dx = \tan x \; \ln|\sin x| - \int \tan x \; \cot x \; dx$

$= \tan x \; \ln|\sin x| - \int 1 \; dx = \tan x \; \ln|\sin x| - x + c$

(d) $\int \frac{x+1}{(x-2)(x-1)}\, dx$

Using the method of partial fractions :

$$\frac{x+1}{(x-2)(x-1)} = \frac{A_1}{x-2} + \frac{A_2}{x-1}$$

$$\frac{x+1}{(x-2)(x-1)} = \frac{A_1(x-1) + A_2(x-2)}{(x-2)(x-1)}$$

$$x+1 = A_1(x-1) + A_2(x-2)$$

Put $x = 2$ then $2+1 = A_1(2-1) + A_2(2-2) \implies A_1 = 3$

Put $x = 1$ then $1+1 = A_1(1-1) + A_2(1-2) \implies 2 = -A_2 \implies A_2 = -2$

$$\int \frac{x+1}{(x-2)(x-1)}\, dx = \int \left(\frac{3}{x-2} + \frac{-2}{x-1} \right)\, dx$$

$$= \int \frac{3}{x-2}\, dx + \int \frac{-2}{x-1}\, dx = 3\int \frac{1}{x-2}\, dx - 2\int \frac{1}{x-1}\, dx$$

$$= 3\ln|x-2| - 2\ln|x-1| + c$$

M 104 - GENERAL MATHEMATICS -2-

Dr. Tariq A. AlFadhel

Solution of the Final Exam

First Term 1444 H

Q.1 **(a)** Let $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$. Compute (if possible) **AB** and **BA**. [3]

(b) Compute the determinant $\begin{vmatrix} 1 & 1 & -1 \\ 0 & -2 & -5 \\ 1 & 2 & 1 \end{vmatrix}$. [2]

(c) Solve by using Gauss-Jordan Elimination Method the linear system

$$\begin{cases} x + y - z = -3 \\ \quad - 2y + 5z = 1 \\ x + 2y + z = 1 \end{cases} \quad [4]$$

Solution :

(a) **AB** can not be computed.

$$\mathbf{BA} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-1 & 0-1 \\ 0+1 & 0+1 \\ 1+0 & 0+0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$$

(b) Using Sarrus Method

$$\begin{matrix} 1 & 1 & -1 & 1 & 1 \\ 0 & -2 & -5 & 0 & -2 \\ 1 & 2 & 1 & 1 & 2 \end{matrix}$$

$$\begin{vmatrix} 1 & 1 & -1 \\ 0 & -2 & -5 \\ 1 & 2 & 1 \end{vmatrix} = (-2-5+0) - (2-10+0) = -7-(-8) = -7+8 = 1$$

(c) Using Gauss-Jordan Elimination Method :

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & -3 \\ 0 & -2 & 5 & 1 \\ 1 & 2 & 1 & 1 \end{array}\right) \xrightarrow{-R_1+R_3} \left(\begin{array}{ccc|c} 1 & 1 & -1 & -3 \\ 0 & -2 & 5 & 1 \\ 0 & 1 & 2 & 4 \end{array}\right)$$

$$\xrightarrow{R_2 \longleftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 1 & -1 & -3 \\ 0 & 1 & 2 & 4 \\ 0 & -2 & 5 & 1 \end{array}\right) \xrightarrow{2R_2+R_3} \left(\begin{array}{ccc|c} 1 & 1 & -1 & -3 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 9 & 9 \end{array}\right)$$

$$\xrightarrow{\frac{1}{9}R_3} \left(\begin{array}{ccc|c} 1 & 1 & -1 & -3 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow{-2R_3+R_2} \left(\begin{array}{ccc|c} 1 & 1 & -1 & -3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{R_3+R_1} \left(\begin{array}{ccc|c} 1 & 1 & 0 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow{-R_2+R_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

The solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \\ 1 \end{pmatrix}$

Q.2 **(a)** Find the standard equation of the ellipse with end points of minor axis are $(1,4)$ and $(1,-2)$, and the distance between its foci is 8, and then sketch its graph. [4]

(b) Find the elements of the conic section $y = 4x - x^2$ and then sketch it. [3]

Solution :

(a) The end points of minor axis are located on a line parallel to the y-axis.

The standard equation of the ellipse is $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$, where $a > b$.

$P(h,k) = \left(\dfrac{1+1}{2}, \dfrac{4+(-2)}{2}\right) = (1,1)$, hence $h = 1$ and $k = 1$

$2b$ is the distance between the end points of minor axis,
hence $2b = 6 \implies b = 3$

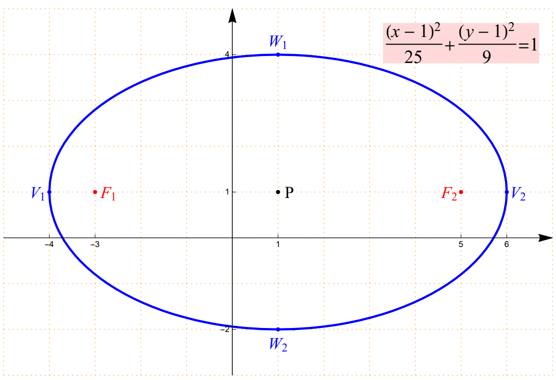
$2c$ is the distance between its two foci , hence $2c = 8 \implies c = 4$

$c^2 = a^2 - b^2 \implies 16 = a^2 - 9 \implies a^2 = 16 + 9 = 25 \implies a = 5$

The standard equation of the ellipse is $\dfrac{(x-1)^2}{25} + \dfrac{(y-1)^2}{9} = 1$

The vertices are $V_1(-4,1)$ and $V_2(6,1)$.

The foci are $F_1(-3,1)$ and $F_2(5,1)$.



(b) $y = 4x - x^2$

$y = -x^2 + 4x$

$y = -(x^2 - 4x)$

By completing the square

$y = -(x^2 - 4x + 4) + 4$

$(y - 4) = -(x - 2)^2$

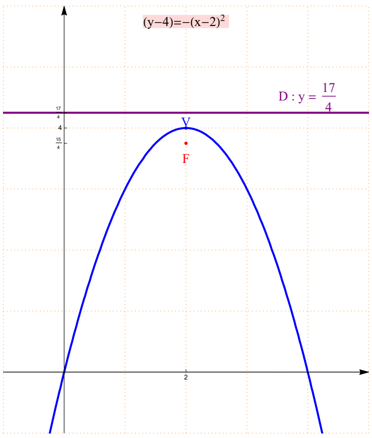
The conic section is a parabola opens downwards.

The vertex is $V(2, 4)$.

$-4a = -1 \implies a = \frac{1}{4}$.

The Focus is $V\left(2, 4 - \frac{1}{4}\right) = \left(2, \frac{15}{4}\right)$

The equation of the directrix is : $y = 4 + \frac{1}{4} = \frac{17}{4}$



Q.3 **(a)** Compute the integrals : [2,3,3]

(i) $\int 8x(x^2 + 24)^3 \, dx$ (ii) $\int (\ln x)^2 \, dx$ (iii) $\int \frac{3x}{x^2 - 2x - 8} \, dx$

(b) Sketch the region bounded by the curves $y = x^2$, $y = 2x + 3$, $x = 1$ and $x = 2$ and compute its area. [3]

(c) The region bounded by the curves curves $y = 4x - x^2$ and $y = x$ is rotated about the y-axis to form a solid S . Use the method of cylindrical shells to find the volume of S. [4]

(d) Give the Cartesian coordinates of the points in polar coordinates :

$M\left(\sqrt{2}, \frac{\pi}{4}\right)$ and $N(2, \pi)$ [2]

Solution :

(a) (i) $\int 8x(x^2+24)^3 \; dx = 4 \int (x^2+24)^3 \; (2x) \; dx$

$= 4 \; \frac{(x^2+24)^4}{4} + c = (x^2+24)^4 + c$

(ii) $\int (\ln x)^2 \; dx$

Using integration by parts

$u = (\ln x)^2 \qquad dv = dx$

$du = 2 \ln x \left(\frac{1}{x}\right) \; dx \qquad v = x$

$$\int (\ln x)^2 \; dx = x (\ln x)^2 - \int 2 \ln x \left(\frac{1}{x}\right) \; x \; dx = x (\ln x)^2 - 2 \int \ln x \; dx$$

Using integration by parts again

$u = \ln x \qquad dv = dx$

$du = \frac{1}{x} \; dx \qquad v = x$

$$\int (\ln x)^2 \; dx = x (\ln x)^2 - 2 \left(x \; \ln x - \int x \; \frac{1}{x} \; dx\right)$$

$$= x (\ln x)^2 - 2x \; \ln x + 2 \int \; dx = x (\ln x)^2 - 2x \; \ln x + 2x + c$$

(iii) $\int \frac{3x}{x^2 - 2x - 8} \; dx$

Using the method of partial fractions

$$\frac{3x}{x^2-2x-8} = \frac{3x}{(x+2)(x-4)} = \frac{A_1}{x+2} + \frac{A_2}{x-4}$$

$3x = A_1(x-4) + A_2(x+2)$

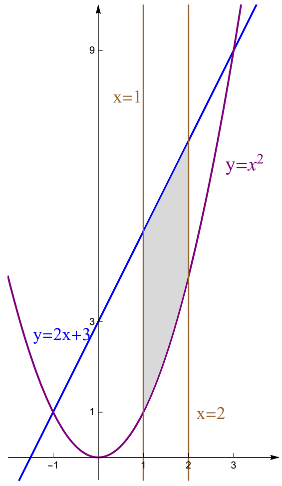
Put $x = -2$: then $-6 = -6A_1 \implies A_1 = 1$

Put $x = 4$: then $12 = 6A_2 \implies A_2 = 2$

$$\int \frac{3x}{x^2-2x-8} \; dx = \int \left(\frac{1}{x+2} + \frac{2}{x-4}\right) \; dx$$

$$= \int \frac{1}{x+2} \; dx + 2 \int \frac{1}{x-4} \; dx = \ln|x+2| + 2 \ln|x-4| + c$$

(b)



$y = x^2$ is a parabola opens upwards with vertex $(0, 0)$.

$y = 2x + 3$ is a straight line passing through $(0, 3)$ and with slope 2 .

$x = 1$ is a straight line parallel to the y-axis and passing through $(1, 0)$.

$x = 2$ is a straight line parallel to the y-axis and passing through $(2, 0)$.

Points of intersection of $y = x^2$ and $y = 2x + 3$

$x^2 = 2x + 3 \implies x^2 - 2x - 3 = 0$

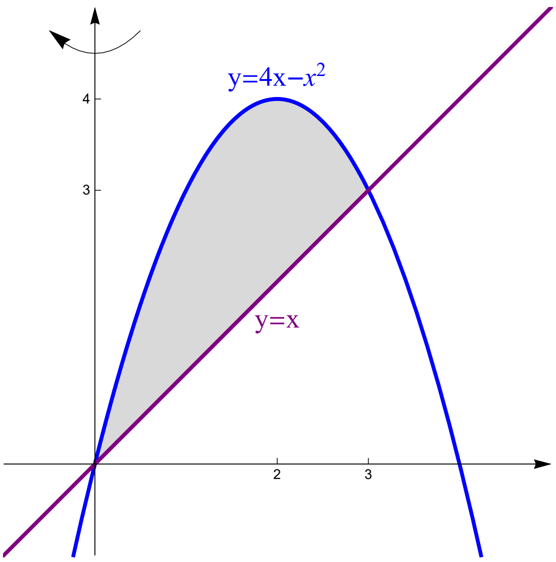
$\implies (x + 1)(x - 3) = 0 \implies x = -1 \ , \ x = 3$

$$\text{Area} = \int_1^2 \left[(2x + 3) - x^2\right] \ dx = \int_1^2 \left(-x^2 + 2x + 3\right) \ dx = \left[-\frac{x^3}{3} + x^2 + 3x\right]_1^2$$

$$= \left(-\frac{2^3}{3} + 2^2 + 3(2)\right) - \left(-\frac{1^3}{3} + 1^2 + 3(1)\right) = \left(-\frac{8}{3} + 4 + 6\right) - \left(-\frac{1}{3} + 1 + 3\right)$$

$$= -\frac{8}{3} + 10 - \left(-\frac{1}{3} + 4\right) = 10 - 4 - \frac{8}{3} + \frac{1}{3} = 6 - \frac{7}{3} = \frac{18 - 7}{3} = \frac{11}{3}$$

(c)



$y = 4x - x^2 = -x^2 + 4x = -(x^2 - 4x) = -(x^2 - 4x + 4) + 4 = -(x - 2)^2 + 4.$

$y = 4x - x^2$ is a parabola opens downwards with vertex $(2, 4)$.

$y = x$ is a straight line passing through $(0, 0)$ and with slope 1.

Points of intersection of $y = 4x - x^2$ and $y = x$

$x = 4x - x^2 \implies x^2 - 3x = 0 \implies x(x-3) = 0 \implies x = 0\ ,\ x = 3\ .$

Using Cylindrical Shells method :

$$\text{Volume} = 2\pi \int_0^3 x\left[(4x - x^2) - x\right]\ dx = 2\pi \int_0^3 x\left(3x - x^2\right)\ dx$$

$$= 2\pi \int_0^3 \left(3x^2 - x^3\right)\ dx = 2\pi \left[x^3 - \frac{x^4}{4}\right]_0^3$$

$$= 2\pi \left[\left(3^3 - \frac{3^4}{4}\right) - \left(0^3 - \frac{0^4}{4}\right)\right] = 2\pi\left(27 - \frac{81}{4}\right) = \frac{27\pi}{2}$$

(d) $M\left(\sqrt{2}, \frac{\pi}{4}\right)\ :\ r = \sqrt{2}$ and $\theta = \frac{\pi}{4}$

$$x = r\ \cos\theta = \sqrt{2}\ \cos\left(\frac{\pi}{4}\right) = \sqrt{2}\ \frac{1}{\sqrt{2}} = 1$$

$$y = r\ \sin\theta = \sqrt{2}\ \sin\left(\frac{\pi}{4}\right) = \sqrt{2}\ \frac{1}{\sqrt{2}} = 1$$

The Cartesian coordinates of M is $(1, 1)$.

$N(2, \pi)\ :\ r = 2$ and $\theta = \pi$

$x = r\ \cos\theta = 2\ \cos(\pi) = 2(-1) = -2$

$y = r\ \cos\theta = 2\ \sin(\pi) = 2(0) = 0$

The Cartesian coordinates of N is $(2, 0)$.

Q.4 **(a)** Let $z = xy^2 + \sin(xy)$, where $x = s^2 t$ and $y = \frac{t}{s}$. Use the chain rule to compute the partial derivatives $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$. [3]

(b) Solve the differential equation : $xy' + y = 3x^2 + 1$. [4]

Solution :

(a) $$\frac{\partial z}{\partial x} = y^2(1) + \cos(xy)\ y = y^2 + y\cos(xy)$$

$$\frac{\partial z}{\partial y} = x(2y) + \cos(xy)\ x = 2xy + x\cos(xy)$$

$$\frac{\partial x}{\partial s} = t(2s) = 2st\ ,\ \frac{\partial y}{\partial s} = t(-s^{-2}) = \frac{-t}{s^2}$$

$$\frac{\partial x}{\partial t} = s^2(1) = s^2 \ , \ \frac{\partial y}{\partial t} = \frac{1}{s} \ (1) = \frac{1}{s}$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$= \left[y^2 + y\cos(xy)\right](2st) + \left[2xy + x\cos(xy)\right]\left(\frac{-t}{s^2}\right)$$

$$= \left[\frac{t^2}{s^2} + \frac{t}{s} \ \cos(st^2)\right](2st) + \left[2st^2 + s^2t \ \cos(st^2)\right]\left(\frac{-t}{s^2}\right)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

$$= \left[y^2 + y\cos(xy)\right](s^2) + \left[2xy + x\cos(xy)\right]\left(\frac{1}{s}\right)$$

$$= \left[\frac{t^2}{s^2} + \frac{t}{s} \ \cos(st^2)\right]s^2 + \left[2st^2 + s^2t \ \cos(st^2)\right]\left(\frac{1}{s}\right)$$

(b) $xy' + y = 3x^2 + 1$

$$y' + \left(\frac{1}{x}\right)y = 3x + \frac{1}{x}$$

It is a First-order differential equation .

$P(x) = \frac{1}{x}$ and $Q(x) = 3x + \frac{1}{x}$

The integrating factor is :

$u(x) = e^{\int P(x) \ dx} = e^{\int \frac{1}{x} \ dx} = e^{\ln|x|} = x.$

The general solution of the differential equation is :

$$y = \frac{1}{u(x)}\int u(x)Q(x) \ dx = \frac{1}{x} \ \int x \ \left(3x + \frac{1}{x}\right) \ dx = \frac{1}{x} \ \int \left(3x^2 + 1\right) \ dx$$

$$= \frac{1}{x} \ \left(x^3 + x + c\right) = x^2 + 1 + \frac{c}{x}$$

M 104 - GENERAL MATHEMATICS -2-
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Solution of the Mid-Term Exam
Second Term 1444 H

Q.1 Find the elements of the conic section of equation

$y^2 - 4y - 8x - 12 = 0$, then sketch it. [4]

Solution :

$y^2 - 4y - 8x - 12 = 0$

$y^2 - 4y = 8x + 12$

By completing the square.

$y^2 - 4y + 4 = 8x + 12 + 4$

$(y-2)^2 = 8x + 16$

$(y-2)^2 = 8(x+2)$

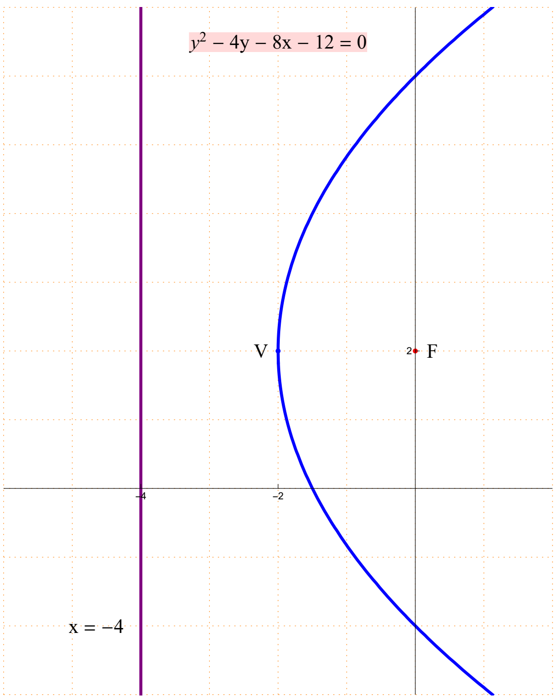
The conic section is a parabola opens to the right.

The vertex is $V(-2, 2)$

$4a = 8 \implies a = \frac{8}{4} = 2$

The focus is $F(0, 2)$

The directrix is : $x = -2 - 2 = -4$



Q.2 Find the standard equation of the ellipse with foci at $(1,5)$, $(1,-3)$ and vertex $(1,6)$, then sketch it. [4]

Solution :

Note that the two foci lies on a line parallel to the $y - axis$.

The standard equation of the ellipse is $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$, where $b > a$.

The center is $P(h,k) = \left(\dfrac{5+(-3)}{2}, \dfrac{1+1}{2}\right) = (1,1)$

c is the distance between $F_1(1,5)$ and $P(1,1)$, hence $c = 4$.

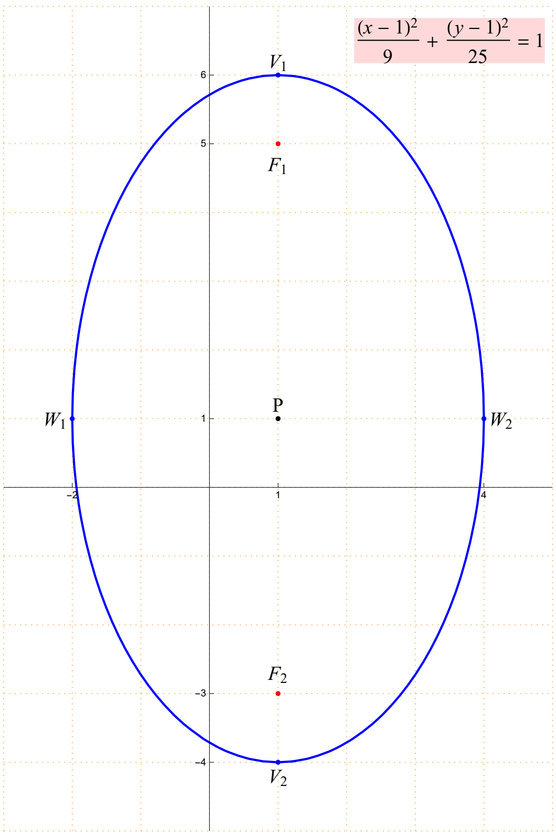
b is the distance between $V_1(1,6)$ and $P(1,1)$, hence $b = 5$.

$c^2 = b^2 - a^2 \implies 16 = 25 - a^2 \implies a^2 = 25 - 16 = 9 \implies a = 3$

The standard equation of the ellipse is $\dfrac{(x-1)^2}{9} + \dfrac{(y-1)^2}{25} = 1$

The other vertex $V_2(1,-4)$.

The end-points of the minor axis are $W_1(-2,1)$ and $W_2(4,1)$



Q.3 Calculate, whenever it is possible, $\mathbf{AB}$ and $2\mathbf{A}+\mathbf{B}^T$ for matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 1 \end{pmatrix}.$$ [4]

Solution :

$$\mathbf{AB} = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+0+1 & 0+0+1 \\ -1+0+1 & 0+2+1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$$

$$2\mathbf{A}+\mathbf{B}^T = 2\begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 & 2 \\ -2 & 2 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 2+1 & 0+0 & 2+1 \\ -2+0 & 2+2 & 2+1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 3 \\ -2 & 4 & 3 \end{pmatrix}$$

Q.4 Consider the system of the linear equations:

$$\begin{cases} x - y + z = 5 \\ 2x + y + 5z = 1 \\ 2y + 3z = -6 \end{cases}$$

(a) Solve this system using Cramer's rule. [4]

(b) Solve this system using Gauss elimination method. [4]

Solution :

(a) Using Cramer's rule :

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 5 \\ 0 & 2 & 3 \end{pmatrix}$$

$$\begin{matrix} 1 & -1 & 1 & 1 & -1 \\ 2 & 1 & 5 & 2 & 1 \\ 0 & 2 & 3 & 0 & 2 \end{matrix}$$

$$|\mathbf{A}| = (3+0+4) - (0+10-6) = 7-4 = 3 \neq 0$$

$$\mathbf{A}_x = \begin{pmatrix} 5 & -1 & 1 \\ 1 & 1 & 5 \\ -6 & 2 & 3 \end{pmatrix}$$

$$\begin{matrix} 5 & -1 & 1 & 5 & -1 \\ 1 & 1 & 5 & 1 & 1 \\ -6 & 2 & 3 & -6 & 2 \end{matrix}$$

$|\mathbf{A}_x| = (15+32) - (-6+50-3) = 47 - 41 = 6$

$x = \frac{|\mathbf{A}_x|}{|\mathbf{A}|} = \frac{6}{3} = 2$

$\mathbf{A}_y = \begin{pmatrix} 1 & 5 & 1 \\ 2 & 1 & 5 \\ 0 & -6 & 3 \end{pmatrix}$

$$\begin{matrix} 1 & 5 & 1 & 1 & 5 \\ 2 & 1 & 5 & 2 & 1 \\ 0 & -6 & 3 & 0 & -6 \end{matrix}$$

$|\mathbf{A}_y| = (3+0-12) - (0-30+30) = -9 - 0 = -9$

$y = \frac{|\mathbf{A}_y|}{|\mathbf{A}|} = \frac{-9}{3} = -3$

$\mathbf{A}_z = \begin{pmatrix} 1 & -1 & 5 \\ 2 & 1 & 1 \\ 0 & 2 & -6 \end{pmatrix}$

$$\begin{matrix} 1 & -1 & 5 & 1 & -1 \\ 2 & 1 & 1 & 2 & 1 \\ 0 & 2 & -6 & 0 & 2 \end{matrix}$$

$|\mathbf{A}_z| = (-6+0+20) - (0+2+12) = 14 - 14 = 0$

$z = \frac{|\mathbf{A}_z|}{|\mathbf{A}|} = \frac{0}{3} = 0$

(b) Using Gauss elimination method: The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 2 & 1 & 5 & 1 \\ 0 & 2 & 3 & -6 \end{array}\right) \xrightarrow{-2R_1+R_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 3 & 3 & -9 \\ 0 & 2 & 3 & -6 \end{array}\right)$$

$$\xrightarrow{\frac{1}{3}\ R_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 2 & 3 & -6 \end{array}\right) \xrightarrow{-2R_2+R_3} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 5 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$z = 0$

$y + z = -3 \implies y + 0 = -3 \implies y = -3$

$x - y + z = 5 \implies x - (-3) + 0 = 5 \implies x + 3 = 5 \implies x = 2$

The solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ 0 \end{pmatrix}$

Q.5 Evaluate the integrals:

(a) $\int \left(2e^x + \frac{3}{x} - 4\sin x\right)\ dx$ [2]

(b) $\int 6\cos x\ (\sin x)^5\ dx$ [2]

(c) $\int \frac{9x^2}{(x^3+1)^4}\ dx$ [3]

(d) $\int (3x^2+2x+1)\ \ln|x|\ dx$ [3]

Solution :

(a) $\int \left(2e^x + \frac{3}{x} - 4\sin x\right)\ dx = \int 2e^x\ dx + \int \frac{3}{x}\ dx - \int 4\sin x\ dx$

$= 2\int e^x\ dx + 3\int \frac{1}{x}\ dx - 4\int \sin x\ dx = 2e^x + 3\ln|x| - 4(-\cos x) + c$

$= 2e^x + 3\ln|x| + 4\cos x + c$

(b) $\int 6\cos x\ (\sin x)^5\ dx = 6\int (\sin x)^5\ \cos x\ dx$

$= 6\ \frac{(\sin x)^6}{6} + c = (\sin x)^6 + c$

Using the formula $\int [f(x)]^n\ \ f'(x)\ dx = \frac{[f(x)]^{n+1}}{n+1} + c$, where $n \neq 1$

(c) $\int \frac{9x^2}{(x^3+1)^4}\ dx = 3\int (x^3+1)^{-4}(3x^2)\ dx$

$= 3\ \frac{(x^3+1)^{-3}}{-3} + c = -(x^3+1)^{-3} + c$

Using the formula $\int [f(x)]^n\ \ f'(x)\ dx = \frac{[f(x)]^{n+1}}{n+1} + c$, where $n \neq 1$

(d) $\int (3x^2+2x+1)\ \ln|x|\ dx$

Using integration by parts:

$u = \ln|x| \qquad dv = (3x^2+2x+1)\ dx$

$du = \frac{1}{x}\ dx \qquad v = x^3 + x^2 + x$

$$\int (3x^2+2x+1)\ \ln|x|\ dx = (x^3+x^2+x)\ \ln|x| - \int (x^3+x^2+x)\ \frac{1}{x}\ dx$$

$$= (x^3 + x^2 + x)\ \ln|x| - \int \frac{x^3 + x^2 + x}{x}\ dx$$

$$= (x^3 + x^2 + x)\ \ln|x| - \int \left(\frac{x^3}{x} + \frac{x^2}{x} + \frac{x}{x}\right)\ dx$$

$$= (x^3 + x^2 + x)\ \ln|x| - \int \left(x^2 + x + 1\right)\ dx$$

$$= (x^3 + x^2 + x)\ \ln|x| - \left(\frac{x^3}{3} + \frac{x^2}{2} + x\right) + c$$

$$= (x^3 + x^2 + x)\ \ln|x| - \frac{x^3}{3} - \frac{x^2}{2} - x + c$$

M 104 - GENERAL MATHEMATICS -2-
Dr. Tariq A. AlFadhel
Solution of the Final Exam
Second Term 1444 H

Q.1 **(a)** Let $\mathbf{A} = \begin{pmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 0 \end{pmatrix}$. Compute (if possible) $\mathbf{AB}$ and $2\mathbf{A} + \mathbf{B}^T$. [3]

(b) Compute the determinant $\begin{vmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 0 & 2 \end{vmatrix}$. [2]

(c) Solve by using Gauss Elimination Method the linear system

$$\begin{cases} 2x + y + 2z = 4 \\ x - 2y + 3z = 4 \\ 2x + y + 3z = 5 \end{cases} \quad [4]$$

Solution :

(a) $\mathbf{AB} = \begin{pmatrix} 1 & 2 & 0 \\ -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \\ 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 1-2+0 & 1+2+0 \\ -1+1+1 & -1-1+0 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 1 & -2 \end{pmatrix}$

$2\mathbf{A} + \mathbf{B}^T = \begin{pmatrix} 2 & 4 & 0 \\ -2 & -2 & 2 \end{pmatrix} + \begin{pmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

$= \begin{pmatrix} 2+1 & 4-1 & 0+1 \\ -2+1 & -2+1 & 2+0 \end{pmatrix} = \begin{pmatrix} 3 & 3 & 1 \\ -1 & -1 & 2 \end{pmatrix}$

(b) Using Sarrus Method

$$\begin{matrix} 2 & 3 & 4 & 2 & 3 \\ 1 & 2 & 3 & 1 & 2 \\ -1 & 0 & 2 & -1 & 0 \end{matrix}$$

$\begin{vmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 0 & 2 \end{vmatrix} = (8-9+0) - (-8+0+6) = -1 - (-2) = -1 + 2 = 1$

(c) Using Gauss Elimination Method :

$$\left(\begin{array}{ccc|c} 2 & 1 & 2 & 4 \\ 1 & -2 & 3 & 4 \\ 2 & 1 & 3 & 5 \end{array}\right) \xrightarrow{R_1 \longleftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 4 \\ 2 & 1 & 2 & 4 \\ 2 & 1 & 3 & 5 \end{array}\right)$$

$$\xrightarrow{-2R_1+R_2} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 4 \\ 0 & 5 & -4 & -4 \\ 2 & 1 & 3 & 5 \end{array}\right) \xrightarrow{-2R_1+R_3} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 4 \\ 0 & 5 & -4 & -4 \\ 0 & 5 & -3 & -3 \end{array}\right)$$

$$\xrightarrow{-R_2+R_3} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 4 \\ 0 & 5 & -4 & -4 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$z = 1$

$5y - 4z = -4 \implies 5y - 4 = -4 \implies 5y = 0 \implies y = 0$

$x - 2y + 3z = 4 \implies x - 2(0) + 3(1) = 4 \implies x + 3 = 4 \implies x = 1$

The solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Q.2 **(a)** Find the standard equation of the hyperbola with foci $(-1, 1)$ and $(9, 1)$, and the distance between the two vertices is 8, and then sketch its graph. [4]

(b) Find the elements of the conic section $y^2 - 6y + 4x + 17 = 0$ and then sketch it. [3]

Solution :

(a) The two foci are located on a line parallel to the x-axis.

The standard equation of the hyperbola is $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$.

$P(h, k) = \left(\dfrac{-1+9}{2}, \dfrac{1+1}{2}\right) = (4, 1)$, hence $h = 4$ and $k = 1$

$2c$ is the distance between its two foci , hence $2c = 10 \implies c = 5$

$2a$ is the distance between the two vertices, hence $2a = 8 \implies a = 4$

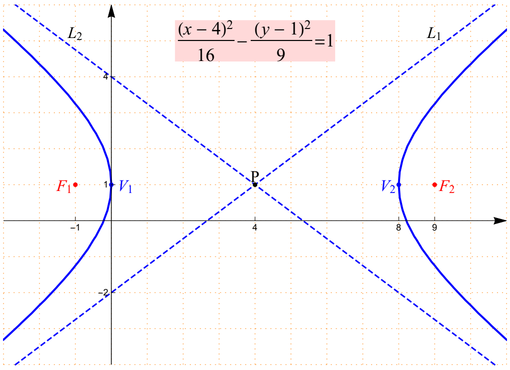
$c^2 = a^2 + b^2 \implies 25 = 16 + b^2 \implies b^2 = 9 \implies b = 3$

The standard equation of the hyperbola is $\dfrac{(x-4)^2}{16} - \dfrac{(y-1)^2}{9} = 1$

The vertices are $V_1(0, 1)$ and $V_2(8, 1)$.

The equations of the asymptotes are :

$L_1 \ : \ y - 1 = \dfrac{3}{4}(x-4)$ and $L_2 \ : \ y - 1 = -\dfrac{3}{4}(x-4)$.



(b) $y^2 - 6y + 4x + 17 = 0$

$y^2 - 6y = -4x - 17$

By completing the square

$y^2 - 6y + 9 = -4x - 17 + 9$

$(y-3)^2 = -4x - 8$

$(y-3)^2 = -4(x+2)$

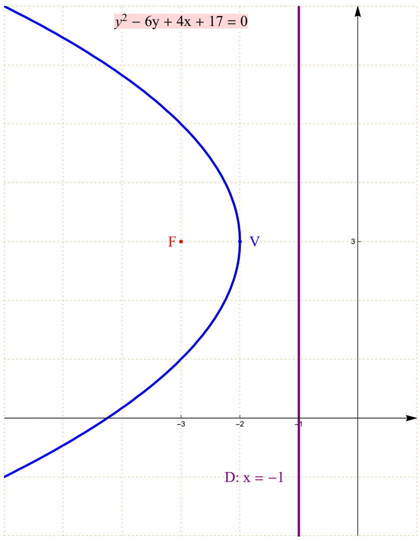
The conic section is a parabola opens to the left.

The vertex is $V(-2, 3)$.

$-4a = -4 \implies a = 1.$

The Focus is $V(-2-1, 3) = (-3, 3)$

The equation of the directrix is : $x = -2 + 1 = -1$



Q.3 **(a)** Compute the integrals : [2,2,4]

(i) $\displaystyle\int \left(3x^2 + 2\cos x - \frac{1}{x^2}\right)\, dx$ (ii) $\displaystyle\int x\; e^{(x^2-3)}\, dx$ (iii) $\displaystyle\int \frac{1}{x^2(x+1)}\, dx$

(b) Sketch the region bounded by the curves $y = x^2 - 4$ and $y = 4 - x^2$ and compute its area. [3]

(c) The region bounded by the curves curves $y = \sqrt{x}$, $y = 0$ and $x = 4$ is rotated about the y-axis to form a solid S . Use the method of cylindrical shells to find the volume of S. [4]

(d) Give the polar coordinates of the points in Cartesian coordinates :

$M\left(-1, \sqrt{3}\right)$ and $N(0, 2)$ [2]

Solution :

(a) (i) $\displaystyle\int \left(3x^2 + 2\cos x - \frac{1}{x^2}\right)\, dx = \int 3x^2\, dx + 2\int \cos x\, dx - \int x^{-2}\, dx$

$\displaystyle = x^3 + 2\sin x - \frac{x^{-1}}{-1} + c = x^3 + 2\sin x + \frac{1}{x} + c$

(ii) $\displaystyle\int x\; e^{(x^2-3)}\, dx = \frac{1}{2}\int e^{(x^2-3)}\,(2x)\, dx = \frac{1}{2}\, e^{(x^2-3)} + c$

(iii) $\displaystyle\int \frac{1}{x^2(x+1)}\, dx$

Using the method of partial fractions

$$\frac{1}{x^2(x+1)} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{A_3}{x+1}$$

$$\frac{1}{x^2(x+1)} = \frac{A_1 x(x+1)}{x^2(x+1)} + \frac{A_2(x+1)}{x^2(x+1)} + \frac{A_3 x^2}{x^2(x+1)}$$

$1 = A_1 x(x+1) + A_2(x+1) + A_3 x^2$

$1 = A_1 x^2 + A_1 x + A_2 x + A_2 + A_3 x^2$

$1 = (A_1 + A_3)\, x^2 + (A_1 + A_2)\, x + A_2 + A_2$

$A_1 + A_3 = 0 \longrightarrow (1)$
$A_1 + A_2 = 0 \longrightarrow (2)$
$A_2 = 1 \longrightarrow (3)$

From equation (2) : $A_1 + 1 = 0 \implies A_1 = -1$

From equation (1) : $-1 + A_3 = 0 \implies A_3 = 1$

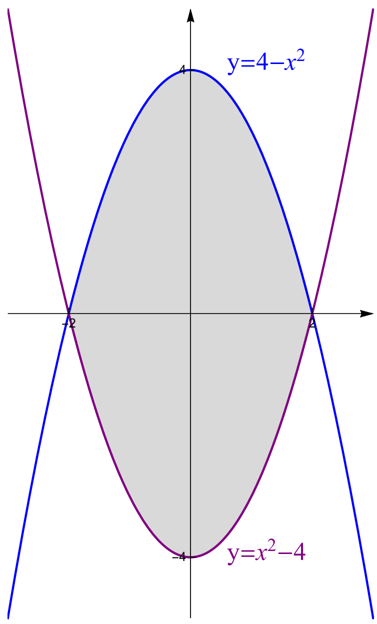
$$\int \frac{1}{x^2(x+1)}\ dx = \int \left(\frac{-1}{x} + \frac{1}{x^2} + \frac{1}{x+1} \right)\ dx$$

$$= -\int \frac{1}{x}\ dx + \int x^{-2}\ dx + \int \frac{1}{x+1}\ dx$$

$$= -\ln|x| - \frac{x^{-1}}{-1} + \ln|x+1| + c = -\ln|x| + \frac{1}{x} + \ln|x+1| + c$$

(b) $y = x^2 - 4$ is a parabola opens upwards with vertex $(0, -4)$.

$y = 4 - x^2$ is a parabola opens downwards with vertex $(0, 4)$.



Points of intersection of $y = x^2 - 4$ and $y = 4 - x^2$

$x^2 - 4 = 4 - x^2 \implies 2x^2 - 8 = 0 \implies x^2 - 4 = 0$

$\implies (x-2)(x+2) = 0 \implies x = -2\ ,\ x = 2$

$$\text{Area} = \int_{-2}^{2} \left[(4 - x^2) - (x^2 - 4)\right]\ dx = \int_{-2}^{2} \left(8 - 2x^2\right)\ dx = \left[8x - 2\ \frac{x^3}{3}\right]_{-2}^{2}$$

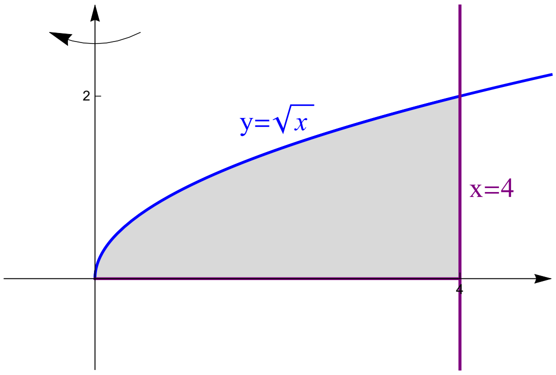
$$= \left[8(2) - 2\ \frac{(2)^3}{3}\right] - \left[8(-2) - 2\ \frac{(-2)^3}{3}\right] = 16 - \frac{16}{3} - \left(-16 + \frac{16}{3}\right)$$

$$= 32 - \frac{32}{3} = \frac{64}{3}$$

(c) $y = \sqrt{x}$ is the upper half of a parabola opens to the right and with vertex $(0, 0)$.

$y = 0$ is the x-axis.

$x = 4$ is a straight line parallel to the y-axis and passing through $(4, 0)$.



Using Cylindrical Shells method :

$$\text{Volume} = 2\pi \int_0^4 x\sqrt{x}\ dx = 2\pi \int_0^4 x^{\frac{3}{2}}\ dx = 2\pi \left[\frac{2}{5}\ x^{\frac{5}{2}}\right]_0^4$$

$$= 2\pi \left[\frac{2}{5}\ (4)^{\frac{5}{2}} - \frac{2}{5}\ (0)^{\frac{5}{2}}\right] = 2\pi \left(\frac{2}{5}\ (32)\right) = \frac{128\pi}{5}$$

(d) $M\left(-1, \sqrt{3}\right)$: $x = -1$ and $y = \sqrt{3}$

$$r = \sqrt{x^2 + y^2} = \sqrt{(-1)^2 + \left(\sqrt{3}\right)^2} = \sqrt{1+3} = \sqrt{4} = 2$$

$\cos\theta = \dfrac{x}{r} = \dfrac{-1}{2}$ and $\sin\theta = \dfrac{y}{r} = \dfrac{\sqrt{3}}{2}$

So, $\theta = \dfrac{2\pi}{3}$. (Note that this point is located in the second quadrant).

The polar coordinates of M is $(r, \theta) = \left(2, \dfrac{2\pi}{3}\right)$.

$N\,(0, 2)$: $x = 0$ and $y = 2$

$$r = \sqrt{x^2 + y^2} = \sqrt{(0)^2 + (2)^2} = \sqrt{0+4} = \sqrt{4} = 2$$

$\cos\theta = \dfrac{x}{r} = \dfrac{0}{2} = 0$ and $\sin\theta = \dfrac{y}{r} = \dfrac{2}{2} = 1$

So, $\theta = \dfrac{\pi}{2}$. (Note that this point is located on the y-axis).

The polar coordinates of M is $(r, \theta) = \left(2, \dfrac{\pi}{2}\right)$.

Q.4 **(a)** We define $z(x, y)$ implicitly by the equation $x^2y + z^2 + \sin(xyz) = 0$.
Compute the partial derivatives $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$. [3]

(b) Solve the differential equation : $\frac{4}{x}y' - \frac{1}{y^3}e^x = 0$. [4]

Solution :

(a) Let $F(x,y,z) = x^2y + z^2 + \sin(xyz)$, then $F(x,y,z) = 0$.

$$F_x = \frac{\partial F}{\partial x} = y(2x) + 0 + \cos(xyz)\ yz(1) = 2xy + yz\cos(xyz)$$

$$F_y = \frac{\partial F}{\partial y} = x^2(1) + 0 + \cos(xyz)\ xz(1) = x^2 + xz\cos(xyz)$$

$$F_z = \frac{\partial F}{\partial z} = 0 + 2z + \cos(xyz)\ xy(1) = 2z + xy\cos(xyz)$$

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2xy + yz\cos(xyz)}{2z + xy\cos(xyz)}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{x^2 + xz\cos(xyz)}{2z + xy\cos(xyz)}$$

(b) $\frac{4}{x}y' - \frac{1}{y^3}e^x = 0$

$$\frac{4}{x}\frac{dy}{dx} - \frac{1}{y^3}e^x = 0$$

$$\frac{4}{x}\frac{dy}{dx} = \frac{1}{y^3}e^x$$

$$4y^3\ dy = xe^x\ dx$$

It is a Separable differential equation .

$$\int 4y^3\ dy = \int xe^x\ dx$$

$$y^4 = xe^x - e^x + c$$

$$y = \sqrt[4]{|xe^x - e^x + c|}$$

M 104 - GENERAL MATHEMATICS -2-

Dr. Tariq A. AlFadhel

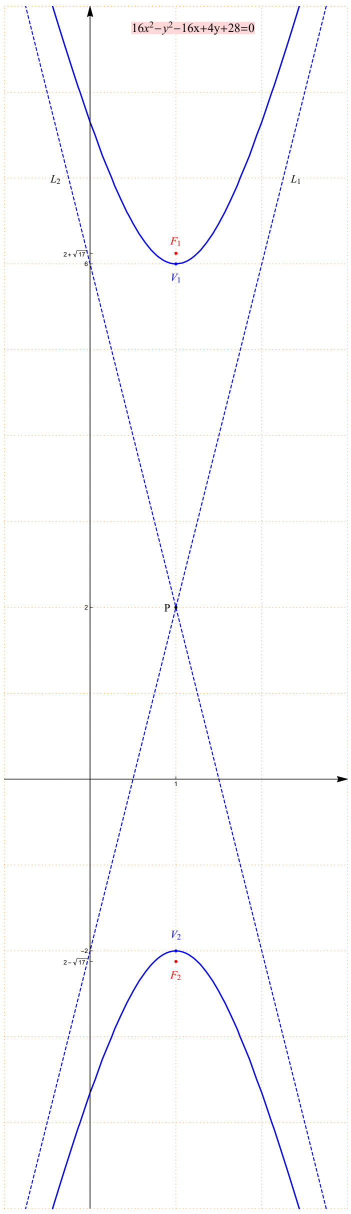
Solution of the Mid-Term Exam

Third Term 1444 H

Q.1 Find the elements of the conic section of equation

$16x^2 - y^2 - 32x + 4y + 28 = 0$, then sketch it. [4]

Solution :



$16x^2 - y^2 - 32x + 4y + 28 = 0$

$16x^2 - 32x - y^2 + 4y = -28$

$16(x^2 - 2x) - (y^2 - 4y) = -28$

By completing the square.

$16(x^2 - 2x + 1) - (y^2 - 4y + 4) = -28 + 16 - 4$

$16(x-1)^2 - (y-2)^2 = -16$

$\dfrac{16(x-1)^2}{-16} - \dfrac{(y-2)^2}{-16} = \dfrac{-16}{-16}$

$-\dfrac{(x-1)^2}{1} + \dfrac{(y-2)^2}{16} = 1$

$\dfrac{(y-2)^2}{4^2} - \dfrac{(x-1)^2}{1^2} = 1$

The conic section is a hyperbola.

The center is $P(1,2)$

$b^2 = 16 \implies b = 4$, and $a^2 = 1 \implies a = 1$

$c^2 = a^2 + b^2 = 1 + 16 = 17 \implies c = \sqrt{17}$

The vertices are : $V_1(1,6)$ and $V_2(1,-2)$

The foci are : $F_1(1, 2+\sqrt{17})$ and $F_2(1, 2-\sqrt{17})$

The equations of the asymptotes are :

$L_1 :\ y-2 = 4(x-1)$ and $L_2 :\ y-2 = -4(x-1)$

Q.2 Find the standard equation of the ellipse with vertices at $(-5, 1)$, $(5, 1)$ and focus at $(-4, 1)$, then sketch it. [4]

Solution :

Note that the two vertices lies on a line parallel to the $x - axis$.

The standard equation of the ellipse is $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$, where $a > b$.

The center is $P(h, k) = \left(\dfrac{5 + (-5)}{2}, \dfrac{1+1}{2} \right) = (0, 1)$

a is the distance between $V_1(-5, 1)$ and $P(0, 1)$, hence $a = 5$.

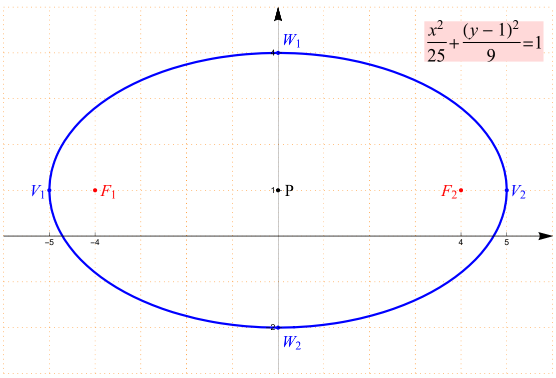
c is the distance between $F_1(-4, 1)$ and $P(0, 1)$, hence $c = 4$.

$c^2 = a^2 - b^2 \implies 16 = 25 - b^2 \implies b^2 = 25 - 16 = 9 \implies b = 3$

The standard equation of the ellipse is $\dfrac{x^2}{25} + \dfrac{(y-1)^2}{9} = 1$

The other focus is $F_2(4, 1)$.

The end-points of the minor axis are $W_1(0, 4)$ and $W_2(0, -2)$



Q.3 Calculate, whenever it is possible, $\mathbf{A} + 2\mathbf{B}$ and $\mathbf{AB}$, for matrices

$\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 1 \\ 3 & 1 & 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$. [4]

Solution :

$$\mathbf{A} + 2\mathbf{B} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 1 \\ 3 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 4 \\ 6 & 2 & 0 \\ 0 & 2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1+2 & 2+0 & -1+4 \\ 0+6 & -1+2 & 1+0 \\ 3+0 & 1+2 & 2+4 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 3 \\ 6 & 1 & 1 \\ 3 & 3 & 6 \end{pmatrix}$$

$$\mathbf{AB} = \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 1 \\ 3 & 1 & 2 \end{pmatrix} . \begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1+6+0 & 0+2-1 & 2+0-2 \\ 0-3+0 & 0-1+1 & 0+0+2 \\ 3+3+0 & 0+1+2 & 6+0+4 \end{pmatrix} = \begin{pmatrix} 7 & 1 & 0 \\ -3 & 0 & 2 \\ 6 & 3 & 10 \end{pmatrix}$$

Q.4 Consider the system of the linear equations:

$$\begin{cases} x & - & y & + & 2z & = & -1 \\ -x & + & 2y & + & z & = & 6 \\ 2x & + & y & + & z & = & 1 \end{cases}$$

(a) Solve this system using Cramer's rule. [4]

(b) Solve this system using Gauss elimination method. [4]

Solution :

(a) Using Cramer's rule :

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

$$\begin{matrix} 1 & -1 & 2 & 1 & -1 \\ -1 & 2 & 1 & -1 & 2 \\ 2 & 1 & 1 & 2 & 1 \end{matrix}$$

$|\mathbf{A}| = (2-2-2) - (8+1+1) = -2 - 10 = -12 \neq 0$

$$\mathbf{A}_x = \begin{pmatrix} -1 & -1 & 2 \\ 6 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\begin{matrix} -1 & -1 & 2 & -1 & -1 \\ 6 & 2 & 1 & 6 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{matrix}$$

$|\mathbf{A}_x| = (-2-1+12) - (4-1-6) = 9 - (-3) = 9 + 3 = 12$

$$x = \frac{|\mathbf{A}_x|}{|\mathbf{A}|} = \frac{12}{-12} = -1$$

$$\mathbf{A}_y = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 6 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

$$\begin{matrix} 1 & -1 & 2 & 1 & -1 \\ -1 & 6 & 1 & -1 & 6 \\ 2 & 1 & 1 & 2 & 1 \end{matrix}$$

$|\mathbf{A}_y| = (6 - 2 - 2) - (24 + 1 + 1) = 2 - 26 = -24$

$y = \dfrac{|\mathbf{A}_y|}{|\mathbf{A}|} = \dfrac{-24}{-12} = 2$

$\mathbf{A}_z = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 2 & 6 \\ 2 & 1 & 1 \end{pmatrix}$

$$\begin{matrix} 1 & -1 & -1 & 1 & -1 \\ -1 & 2 & 6 & -1 & 2 \\ 2 & 1 & 1 & 2 & 1 \end{matrix}$$

$|\mathbf{A}_z| = (2 - 12 + 1) - (-4 + 6 + 1) = -9 - 3 = -12$

$z = \dfrac{|\mathbf{A}_z|}{|\mathbf{A}|} = \dfrac{-12}{-12} = 1$

(b) Using Gauss elimination method: The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & -1 \\ -1 & 2 & 1 & 6 \\ 2 & 1 & 1 & 1 \end{array}\right) \xrightarrow{R_1+R_2} \left(\begin{array}{ccc|c} 1 & -1 & 2 & -1 \\ 0 & 1 & 3 & 5 \\ 2 & 1 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{-2R_1+R_3} \left(\begin{array}{ccc|c} 1 & -1 & 2 & -1 \\ 0 & 1 & 3 & 5 \\ 0 & 3 & -3 & 3 \end{array}\right) \xrightarrow{-3R_2+R_3} \left(\begin{array}{ccc|c} 1 & -1 & 2 & -1 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & -12 & -12 \end{array}\right)$$

$-12z = -12 \implies z = \dfrac{-12}{-12} = 1$

$y + 3z = 5 \implies y + 3 = 5 \implies y = 5 - 3 = 2$

$x - y + 2z = -1 \implies x - 2 + 2 = -1 \implies x = -1$

The solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$

Q.5 Evaluate the integrals:

(a) $\displaystyle\int \left(\frac{4}{x} - 3\sqrt{x} + 3e^x\right)\, dx$ [2]

(b) $\displaystyle\int 6x\cos(x^2 + 1)\, dx$ [2]

(c) $\displaystyle\int 16x^3\ \ln|x|\, dx$ [3]

(d) $\displaystyle\int \frac{4}{(x-1)(x+3)}\, dx$ [3]

Solution :

(a) $\int \left(\frac{4}{x} - 3\sqrt{x} + 3e^x\right)\ dx = \int \frac{4}{x}\ dx - \int 3\sqrt{x}\ dx + \int 3e^x\ dx$

$= 4\int \frac{1}{x}\ dx - 3\int x^{\frac{1}{2}}\ dx + 3\int e^x\ dx = 4\ln|x| - 3\frac{x^{\frac{3}{2}}}{\frac{3}{2}} + 3e^x + c$

$= 4\ln|x| - 2x^{\frac{3}{2}} + 3e^x + c$

(b) $\int 6x\cos(x^2+1)\ dx = 3\int \cos(x^2+1)\ (2x)\ dx = 3\sin(x^2+1) + c$

Using the formula $\int \cos(f(x))\ \ f'(x)\ dx = \sin(f(x)) + c.$

(c) $\int 16x^3\ \ln|x|\ dx$

Using integration by parts:

$u = \ln|x| \qquad dv = 16x^3\ dx$

$du = \frac{1}{x}\ dx \qquad v = 16\left(\frac{x^4}{4}\right) = 4x^4$

$\int 16x^3\ \ln|x|\ dx = 4x^4\ \ln|x| - \int 4x^4\ \frac{1}{x}\ dx$

$= 4x^4\ \ln|x| - \int 4x^3\ dx = 4x^4\ \ln|x| - x^4 + c$

(d) $\int \frac{4}{(x-1)(x+3)}\ dx$

Using the method of partial fractions :

$\frac{4}{(x-1)(x+3)} = \frac{A_1}{x-1} + \frac{A_2}{x+3}$

$\frac{4}{(x-1)(x+3)} = \frac{A_1(x+3) + A_2(x-1)}{(x-1)(x+3)}$

$4 = A_1(x+3) + A_2(x-1)$

Put $x = 1$ then $4 = A_1(1+3) + A_2(1-1) \implies 4A_1 = 4 \implies A_1 = 1$

Put $x = -3$ then $4 = A_1(-3+3) + A_2(-3-1) \implies 4 = -4A_2$

$\implies A_2 = -1$

$\int \frac{4}{(x-1)(x+3)}\ dx = \int \left(\frac{1}{x-1} + \frac{-1}{x+3}\right)\ dx$

$= \int \frac{1}{x-1}\ dx - \int \frac{1}{x+3}\ dx = \ln|x-1| - \ln|x+3| + c$

M 104 - GENERAL MATHEMATICS -2-
Dr. Tariq A. AlFadhel
Solution of the Final Exam
Third Term 1444 H

Q.1 (a) Let $\mathbf{A} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & -2 \end{pmatrix}$. Compute (if possible) $\mathbf{BA}$ and $\mathbf{A}^T - 2\mathbf{B}$. [3]

(b) Compute the determinant $\begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{vmatrix}$. [2]

(c) Solve by using Gauss-Jordan Elimination Method the linear system

$$\begin{cases} x + y - 3z = 6 \\ x + 2y + z = 3 \\ -2x + 3y + z = -2 \end{cases} \quad [4]$$

Solution :

(a) $\mathbf{BA} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & 1 \end{pmatrix}$

$$= \begin{pmatrix} 1+0 & -1-3 & 2-1 \\ 1+0 & -1+0 & 2+0 \\ 1+0 & -1-6 & 2-2 \end{pmatrix} = \begin{pmatrix} 1 & -4 & 1 \\ 1 & -1 & 2 \\ 1 & -7 & 0 \end{pmatrix}$$

$$\mathbf{A}^T - 2\mathbf{B} = \begin{pmatrix} 1 & 0 \\ -1 & 3 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 2 & -2 \\ 2 & 0 \\ 2 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} 1-2 & 0+2 \\ -1-2 & 3-0 \\ 2-2 & 1+4 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -3 & 3 \\ 0 & 5 \end{pmatrix}$$

(b) Using Sarrus Method

$$\begin{matrix} 0 & 1 & 2 & 0 & 1 \\ 1 & 2 & 0 & 1 & 2 \\ 2 & 0 & 1 & 2 & 0 \end{matrix}$$

$$\begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{vmatrix} = (0+0+0) - (8+0+1) = 0 - 9 = -9$$

(c) Using Gauss-Jordan Elimination Method :

$$\left(\begin{array}{ccc|c} 1 & 1 & -3 & 6 \\ 1 & 2 & 1 & 3 \\ -2 & 3 & 1 & -2 \end{array}\right) \xrightarrow{-R_1+R_2} \left(\begin{array}{ccc|c} 1 & 1 & -3 & 6 \\ 0 & 1 & 4 & -3 \\ -2 & 3 & 1 & -2 \end{array}\right)$$

$$\xrightarrow{2R_1+R_3} \left(\begin{array}{ccc|c} 1 & 1 & -3 & 6 \\ 0 & 1 & 4 & -3 \\ 0 & 5 & -5 & 10 \end{array}\right) \xrightarrow{-5R_2+R_3} \left(\begin{array}{ccc|c} 1 & 1 & -3 & 6 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & -25 & 25 \end{array}\right)$$

$$\xrightarrow{\frac{-1}{25}\ R_3} \left(\begin{array}{ccc|c} 1 & 1 & -3 & 6 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & 1 & -1 \end{array}\right) \xrightarrow{-4R_3+R_2} \left(\begin{array}{ccc|c} 1 & 1 & -3 & 6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$$\xrightarrow{3R_3+R_1} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right) \xrightarrow{-R_2+R_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

The solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$

Q.2 **(a)** Find the standard equation of the ellipse with vertices $(1,5)$ and $(1,-1)$, and the length of its minor axis is 4, and then sketch its graph. [4]

(b) Find the elements of the conic section $x^2 - 4x - 8y + 12 = 0$ and then sketch it. [3]

Solution :

(a) The two vertices are located on a line parallel to the y-axis.

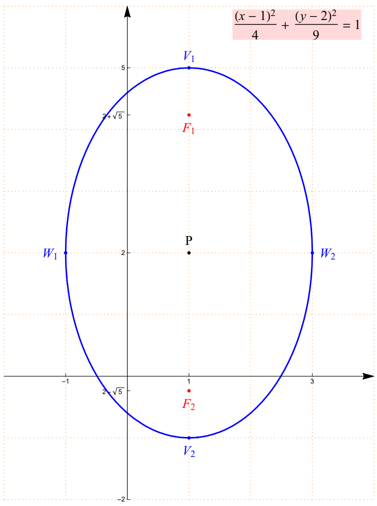
The standard equation of the ellipse is $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1$, where $b > a$.

$P(h,k) = \left(\dfrac{1+1}{2}, \dfrac{5+(-1)}{2}\right) = (1,2)$,
hence $h = 1$ and $k = 2$

$2b$ is the distance between its two vertices ,
hence $2b = 6 \implies b = 3$

The length of its minor axis is 4,
hence $2a = 4 \implies a = 2$

$c^2 = b^2 - a^2 \implies c^2 = 9 - 4 = 5 \implies c = \sqrt{5}$



The standard equation of the ellipse is $\dfrac{(x-1)^2}{4} + \dfrac{(y-2)^2}{9} = 1$

The Foci are $V_1\left(1, 2+\sqrt{5}\right)$ and $V_2\left(1, 2-\sqrt{5}\right)$.

The endpoints of the minor axis are $W_1(-1,2)$ and $W_2(3,2)$.

(b) $x^2 - 4x - 8y + 12 = 0$

$x^2 - 4x = 8y - 12$

By completing the square

$x^2 - 4x + 4 = 8y - 12 + 4$

$(x-2)^2 = 8y - 8$

$(x-2)^2 = 8(y-1)$

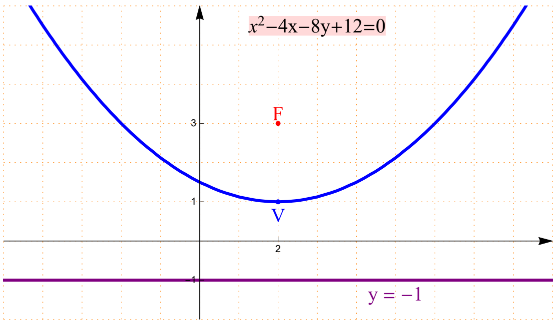
The conic section is a parabola opens upwards.

The vertex is $V(2,1)$.

$4a = 8 \implies a = 2.$

The Focus is $F\,(2, 1+2) = (2,3)$

The equation of the directrix is : $y = 1 - 2 = -1$



Q.3 **(a)** Compute the integrals : [2,3,4]

(i) $\displaystyle\int \frac{4x}{(1+x^2)^3}\,dx$ (ii) $\displaystyle\int 9x\ \cos(3x)\,dx$ (iii) $\displaystyle\int \frac{x^2+1}{x(x-1)(x+1)}\,dx$

(b) Sketch the region bounded by the curves $y = x^2$ and $y = 4x$ and compute its area. [3]

(c) The region bounded by the curves curves $y = x^2$ and $y = 1$ is rotated about the x-axis to form a solid S . Find the volume of S. [4]

Solution :

(a) (i) $\displaystyle\int \frac{4x}{(1+x^2)^3}\,dx = \int 4x(1+x^2)^{-3}\,dx = 2\int (1+x^2)^{-3}\,2x\,dx$

$\displaystyle = 2\,\frac{(1+x^2)^{-2}}{-2} + c = \frac{-1}{(1+x^2)^2} + c$

(ii) $\displaystyle\int 9x\ \cos(3x)\,dx$

Using integration by parts

$$\begin{array}{ll} u = x & dv = 9\ \cos(3x)\ dx \\ du = dx & v = 3\sin(3x) \end{array}$$

$$\int 9x\ \cos(3x)\ dx = x\,(3\sin(3x)) - \int 3\sin(3x)\ dx$$

$$= 3x\sin(3x) - (-\cos(3x)) + c = 3x\sin(3x) + \cos(3x) + c$$

(iii) $\displaystyle\int \frac{x^2+1}{x(x-1)(x+1)}\ dx$

Using the method of partial fractions

$$\frac{x^2+1}{x(x-1)(x+1)} = \frac{A_1}{x} + \frac{A_2}{x-1} + \frac{A_3}{x+1}$$

$$\frac{x^2+1}{x(x-1)(x+1)} = \frac{A_1(x-1)(x+1)}{x(x-1)(x+1)} + \frac{A_2x(x+1)}{x(x-1)(x+1)} + \frac{A_3x(x-1)}{x(x-1)(x+1)}$$

$$x^2+1 = A_1(x-1)(x+1) + A_2x(x+1) + A_3x(x-1)$$

$$x^2+1 = A_1(x^2-1) + A_2(x^2+x) + A_3(x^2-x)$$

$$x^2+1 = A_1x^2 - A_1 + A_2x^2 + A_2x + A_3x^2 - A_3x$$

$$x^2+1 = (A_1+A_2+A_3)\,x^2 + (A_2-A_3)\,x - A_1$$

$$\begin{array}{lll} A_1+A_2+A_3 = 1 & \longrightarrow & (1) \\ A_2 - A_3 = 0 & \longrightarrow & (2) \\ -A_1 = 1 & \longrightarrow & (3) \end{array}$$

From equation (3) : $A_1 = -1$

Equation (1) + Equation (2) : $A_1 + 2A_2 = 1 \implies -1 + 2A_2 = 1$

$\implies 2A_2 = 2 \implies A_2 = 1$

From equation (2) : $1 - A_3 = 0 \implies A_3 = 1$

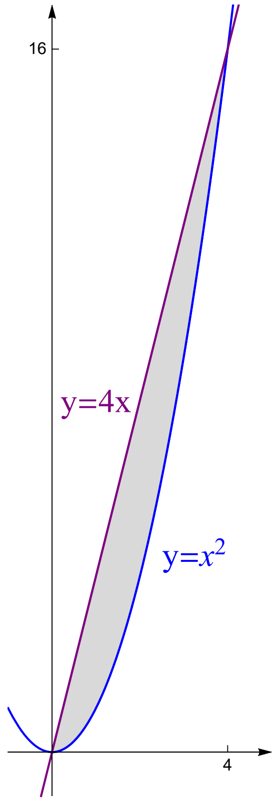
$$\int \frac{x^2+1}{x(x-1)(x+1)}\ dx = \int \left(\frac{-1}{x} + \frac{1}{x-1} + \frac{1}{x+1}\right)\ dx$$

$$= -\int \frac{1}{x}\ dx + \int \frac{1}{x-1}\ dx + \int \frac{1}{x+1}\ dx$$

$$= -\ln|x| + \ln|x-1| + \ln|x+1| + c$$

(b) $y = x^2$ is a parabola opens upwards with vertex $(0, 0)$.

$y = 4x$ is a straight line passing through $(0, 0)$ with slope equals 4 .



Points of intersection of $y = x^2$ and $y = 4x$

$x^2 = 4x \implies x^2 - 4x = 0$

$\implies x(x-4) = 0 \implies x = 0 \ , \ x = 4$

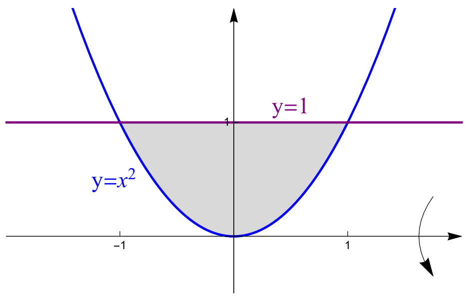
$$\text{Area} = \int_0^4 \left(4x - x^2\right) \ dx = \left[2x^2 - \frac{x^3}{3}\right]_0^4$$

$$= \left(2\ (4^2) - \frac{4^3}{3}\right) - \left(2\ (0^2) - \frac{0^3}{3}\right)$$

$$= 32 - \frac{64}{3} = \frac{96-64}{3} = \frac{32}{3}$$

(c) $y = x^2$ is a parabola opens upwards with vertex $(0, 0)$.

$y = 1$ is a straight line parallel to the x-axis and passing through $(0, 1)$.



Points of intersection of $y = x^2$ and $y = 1$:

$x^2 = 1 \implies x = -1 \ , \ x = 1$

Using Washer method :

$$\text{Volume} = \pi \int_{-1}^{1} \left[(1)^2 - (x^2)^2\right] \ dx = \pi \int_{-1}^{1} \left(1 - x^4\right) \ dx = \pi \left[x - \frac{x^5}{5}\right]_{-1}^{1}$$

$$= \pi \left[\left(1 - \frac{1^5}{5}\right) - \left(-1 - \frac{(-1)^5}{5}\right)\right] = \pi \left[1 - \frac{1}{5} - \left(-1 + \frac{1}{5}\right)\right]$$

$$= \pi \left(1 - \frac{1}{5} + 1 - \frac{1}{5}\right) = \pi \left(2 - \frac{2}{5}\right) = \frac{8\pi}{5}$$

Q.4 **(a)** Let $z = x^2 - 2xy + 2y^2$ with $x = \cos\theta + \sin\theta$ and $y = \sin\theta$. Use the chain rule to compute the partial derivative $\dfrac{dz}{d\theta}$. [4]

(b) Solve the differential equation : $xy' + 2y = 4x^2 + 3x$. [4]

Solution :

(a) $$\frac{dz}{d\theta} = \frac{\partial z}{\partial x}\frac{dx}{d\theta} + \frac{\partial z}{\partial y}\frac{dy}{d\theta}$$

$$\frac{\partial z}{\partial x} = 2x - 2y = 2(x - y)$$

$$\frac{\partial z}{\partial y} = -2x + 4y = 2(2y - x)$$

$$\frac{dx}{d\theta} = -\sin\theta + \cos\theta \text{ and } \frac{dy}{d\theta} = \cos\theta$$

$$\frac{dz}{d\theta} = 2(x - y)(\cos\theta - \sin\theta) + 2(2y - x)\cos\theta$$

$$= 2\cos\theta\ (\cos\theta - \sin\theta) + 2(\sin\theta - \cos\theta)\ \cos\theta$$

$$= 2\cos\theta\ (\cos\theta - \sin\theta) - 2(\cos\theta - \sin\theta)\ \cos\theta = 0$$

(b) $xy' + 2y = 4x^2 + 3x$

$$y' + \left(\frac{2}{x}\right) y = 4x + 3$$

It is a First-order differential equation .

$P(x) = \dfrac{2}{x}$ and $Q(x) = 4x + 3$

The integrating factor is :

$$u(x) = e^{\int P(x)\ dx} = e^{\int \frac{2}{x}\ dx} = e^{2\int \frac{1}{x}\ dx} = e^{2\ln|x|} = e^{\ln x^2} = x^2.$$

The general solution of the differential equation is :

$$y = \frac{1}{u(x)} \int u(x)Q(x)\ dx = \frac{1}{x^2} \int x^2\ (4x + 3)\ dx = \frac{1}{x^2} \int \left(4x^3 + 3x^2\right)\ dx$$

$$= \frac{1}{x^2}\ \left(x^4 + x^3 + c\right) = x^2 + x + \frac{c}{x^2}$$