King Saud University
College of Sciences
Department of Mathematics

Final Exam, S2 1444
M 507 - Advanced Operations Research
Time: 3 hours – **Marks: 40**

Answer the following questions.

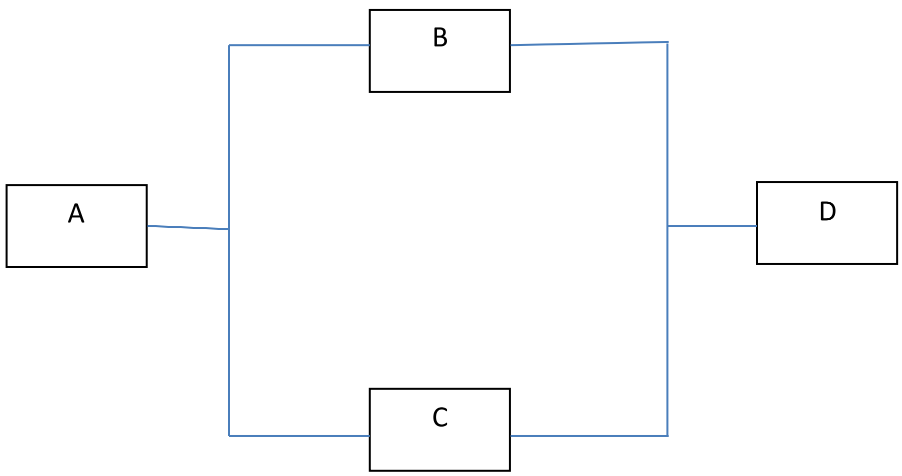
Q1: [4+2+4]

In the reliability diagram below, the reliability of each component is constant and independent. Assuming that each has the same reliability R, compute the system reliability as a function of R using the following methods:

(a) Decomposition using B as the keystone element.

(b) The reduction method.

(c) Compute the importance of each component if $R_A = 0.8$, $R_B = 0.9$, $R_C = 0.95$ and $R_D = 0.98$



Q2: [4+4]

(a) If $X(t)$ represents a size of a population where $X(0)=1$, using the following differential equations

$$\frac{dp_0(t)}{dt} = -\lambda_0 p_0(t) \qquad (1)$$

$$\frac{dp_n(t)}{dt} = \lambda_{n-1} p_{n-1}(t) - \lambda_n p_n(t), \quad n=1,2,3, \ldots \qquad (2)$$

Prove that: $X(t) \sim geom\,(p), \;\; p = e^{-\lambda t}$ when $\lambda_0 = 0$ and $\lambda_n = n\lambda$, and then find the mean and variance of this process.

(b) Suppose that minor defects are distributed over the length of a cable as a Poisson process with rate α, and that, independently, major defects are distributed over the cable according to a Poisson process of rate β. Let $X(t)$ be the number of defects, either major or minor, in the cable up to length t. Argue that $X(t)$ must be a Poisson process of rate $\alpha+\beta$.

Q3: [2.5+2.5]

If X is the life of an item of a product. Find the mean time to failure MTTF, variance, median, failure rate at 500 hours, and also, determine the probability that the item will survive until age 500 hours, in each of the following cases.

(a) $X \sim \text{Weibull}(\eta, \beta)$ where $\beta = 1.5, \; \eta = 1000$

(b) $X \sim \text{Lognormal}(\mu, \sigma^2)$ where $\mu = 6.908, \; \sigma = 0.317$

Q4: [3+3+3]

(a) For the Markov process $\{X_t\}, \; t = 0,1,2,\ldots,n$ with states $i_0, i_1, i_2, \ldots, i_{n-1}, i_n$

Prove that: $\Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n\} = p_{i_0} P_{i_0 i_1} P_{i_1 i_2} \ldots P_{i_{n-1} i_n}$ where $p_{i_0} = \Pr\{X_0 = i_0\}$

(b) If a Markov chain $X_0, X_1, X_2, \ldots$ has the transition probability matrix

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \left\| \begin{array}{ccc} 0.2 & 0.3 & 0.5 \\ 0.4 & 0.2 & 0.4 \\ 0.5 & 0.3 & 0.2 \end{array} \right\| \end{array}$$

and initial distribution $p_0 = 0.5, \; p_1 = 0.2$ and $p_2 = 0.3$ Find $\text{pr}\{X_1 = 1, X_2 = 1, X_3 = 0\}$

(c) Consider the problem of sending a binary message, 0 or 1, through a signal channel consisting of several stages, where transmission through each stage is subject to a fixed probability of error α. Suppose that $X_0 = 0$ is the signal that is sent and let X_n be the signal that is received at the nth stage. Assume that $\{X_n\}$ is a Markov chain with transition probabilities $P_{00} = P_{11} = 1-\alpha$ and $P_{01} = P_{10} = \alpha$, where $0 < \alpha < 1$.

(i) Determine $\Pr\{X_0 = 0, X_1 = 0, X_2 = 0\}$, the probability that no error occurs up to stage $n = 2$.

(ii) Determine the probability that a correct signal is received at stage 2.

Q5: [4+4]

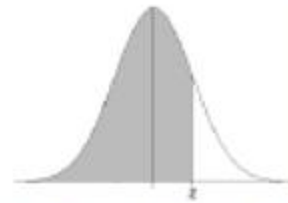
(a) Wild West produces two types of cowboy hats. A type 1 hat requires twice as much labor time as a type 2. If the all available labor time is dedicated to Type 2 alone, the company can produce a total of 400 Type 2 hats a day. The respective market limits for the two types are 150 and 200 hats per day. The profit is \$8 per Type 1 hat and \$5 per Type 2 hat. Determine the number of hats of each type that would maximize profit.

(b) Solve the following linear programming problem by using Simplex method

$$\begin{aligned} \min \quad & z = 4x_1 - x_2 \\ \text{s.t} \quad & 2x_1 + x_2 \le 8 \\ & x_2 \le 5 \\ & x_1 - x_2 \le 4 \\ & x_1,\ x_2 \ge 0 \end{aligned}$$

Table 1

Standard Normal Cumulative Probability Table



Cumulative probabilities for POSITIVE z-values are shown in the following table:

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

Table 2

$B_2 - B_1^2 = \Gamma(2/\beta + 1) - \Gamma^2(1/\beta + 1)$ as a Function of the Shape Parameter β

β	B_1	$B_2 - B_1^2$
1.0	1.0000	1.0000
1.1	0.9649	0.7714
1.2	0.9407	0.6197
1.3	0.9336	0.5133
1.4	0.9114	0.4351
1.5	0.9027	0.3757
1.6	0.8966	0.3292
1.7	0.8922	0.2919
1.8	0.8893	0.2614
1.9	0.8874	0.2360
2.0	0.8862	0.2146
2.5	0.8873	0.1441
3.0	0.8930	0.1053
3.5	0.8997	0.0811
4.0	0.9064	0.0647
5.0	0.9182	0.0442

Model Answer

Q1: [4+2+4]

In the reliability diagram shown in Fig. 1, the reliability of each component is constant and independent. Assuming that each has the same reliability R, compute the system reliability as a function of R using the following methods:

a) Decomposition using B as the keystone element.

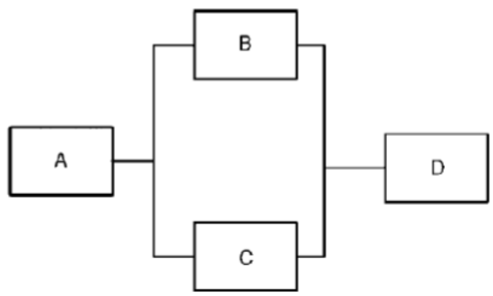


Fig. 1: Reliability diagram

Using B as the keystone element, we have two cases i.e., the case when B functions and the case when it does not.

For the case when B functions, the system reduced to Fig 2.

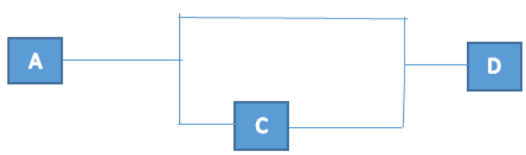


Fig. 2: The case when B functions

Thus the reliability of the system depends only on the reliability of component A and D. Note that $R_A = R_B = R_C = R_D = R$

Therefore,

$$R^+ = R_A R_D = R^2$$

For the case when B fails, the system block is as shown in Fig. 3, which is a series system.

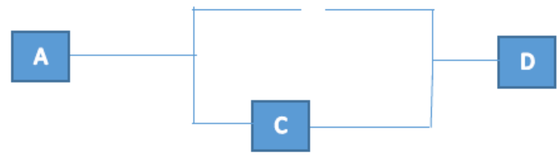


Fig. 3: The case when B fails to work

Thus the reliability of the system depends on A, C, and D, therefore we have:

$$R^{-} = R_A R_C R_D = R^3$$

Thus the reliability of the system using the two decompositions is given as:

$$R_{system} = R_B R^{+} + (1 - R_B) R^{-}$$
$$R_{system} = R(R^2) + (1 - R) R^3$$
$$R_{system} = 2R^3 - R^4$$

b) Using the reduction method

With this method, it can be seen that components B and C are in parallel and jointly in series with A and D. therefore the reduced system is given in Fig. 4.

Fig. 4: Reduced system

For parallel components B and C, we have

$$R_{B\|C} = 1 - \prod_{i=1}^{2} (1 - R_i)$$
$$R_{B\|C} = R_B + R_C - R_B R_C$$
$$R_{B\|C} = 2R - R^2$$

The reliability of the system is thus given as:

$$R_{system} = R_A R_{B\|C} R_D$$
$$R_{system} = R(2R - R^2) R$$
$$R_{system} = 2R^3 - R^4$$

c)

Recall that the reliability of the system is given as:

$$R_{system} = R_A R_D (R_B + R_C - R_B R_C)$$

The importance of each component is computed by taking the partial derivative with respect to each of the component.

Thus the importance of component A is given as:

$$\frac{\delta R_{system}}{\delta R_A} = \frac{\delta(R_A R_D (R_B + R_C - R_B R_C))}{\delta R_A}$$
$$I_A = R_D (R_B + R_C - R_B R_C)$$

$$\Rightarrow \text{ I}_\text{A} = 0.98(0.9 + 0.95 - 0.9 \times 0.95)$$
$$= 0.9751$$

The importance of component B is given as:

$$\frac{\delta R_{system}}{\delta R_B} = \frac{\delta(R_A R_D (R_B + R_C - R_B R_C))}{\delta R_B}$$
$$I_B = R_A R_D - R_A R_D R_C$$

$$\Rightarrow \text{ I}_\text{B} = 0.8(0.98) - 0.8(0.98)(0.95)$$
$$= 0.0392$$

The importance of component C is given as:

$$\frac{\delta R_{system}}{\delta R_C} = \frac{\delta(R_A R_D (R_B + R_C - R_B R_C))}{\delta R_C}$$
$$I_C = R_A R_D - R_A R_B R_D$$

$$\Rightarrow \text{ I}_\text{c} = 0.8(0.98) - 0.8(0.9)(0.98)$$
$$= 0.0784$$

The importance of component D is given as:

$$\frac{\delta R_{system}}{\delta R_D} = \frac{\delta(R_A R_D (R_B + R_C - R_B R_C))}{\delta R_D}$$
$$I_D = R_A (R_B + R_C - R_B R_C)$$

$$\Rightarrow \text{ I}_\text{D} = 0.8(0.9 + 0.95 - 0.9 \times 0.95)$$
$$= 0.796$$

Q2: [4+4]

(a)
$$\frac{dp_0(t)}{dt} = -\lambda_0 p_0(t) \qquad (1)$$
$$\frac{dp_n(t)}{dt} = \lambda_{n-1} p_{n-1}(t) - \lambda_n p_n(t), \; n=1,2,3, \ldots \qquad (2)$$

The initial condition is $X(0)=1 \quad \Rightarrow p_1(0)=1$

$$\Rightarrow p_n(0) = \begin{cases} 1 & , n=1 \\ 0 & , \text{otherwise} \end{cases}$$

$\lambda_0 = 0 \qquad (1) \Rightarrow \dfrac{dp_0(t)}{dt} = 0$

$$\Rightarrow p_0(t) = 0 \qquad (3)$$

$$(2) \Rightarrow \frac{dp_n(t)}{dt} = \lambda_{n-1} p_{n-1}(t) - \lambda_n p_n(t)$$
$$\Rightarrow \frac{dp_n(t)}{dt} + \lambda_n p_n(t) = \lambda_{n-1} p_{n-1}(t), \; n=1,2, \ldots$$

$\because \lambda_n = n\lambda, \;\; \lambda_{n-1} = (n-1)\lambda$

$\therefore \dfrac{dp_n(t)}{dt} + n\lambda p_n(t) = (n-1)\lambda p_{n-1}(t), \;$ n=1,2, ...

Multiply both sides by $e^{n\lambda t}$

$$e^{n\lambda t}\left[\frac{dp_n(t)}{dt} + n\lambda p_n(t)\right] = (n-1)\lambda p_{n-1}(t) e^{n\lambda t}$$
$$\therefore \frac{d}{dt}\left[p_n(t) e^{n\lambda t}\right] = (n-1)\lambda p_{n-1}(t) e^{n\lambda t}$$
$$\Rightarrow \int_0^t d\left[p_n(x) e^{n\lambda x}\right] = (n-1)\lambda \int_0^t p_{n-1}(x) e^{n\lambda x} dx$$
$$\therefore \left[p_n(x) e^{n\lambda x}\right]_0^t = (n-1)\lambda \int_0^t p_{n-1}(x) e^{n\lambda x} dx$$
$$\Rightarrow p_n(t) = e^{-n\lambda t}\left[p_n(0) + (n-1)\lambda \int_0^t p_{n-1}(x) e^{n\lambda x} dx\right], \; n=1,2, \ldots \qquad (4)$$

which is a recurrence relation.

at $n=1$

$$p_1(t) = e^{-\lambda t}\left[p_1(0)+0\right] = e^{-\lambda t} \qquad (5)$$

at $n=2$

$$p_2(t) = e^{-2\lambda t}\left[p_2(0)+\lambda\int_0^t p_1(x)e^{2\lambda x}dx\right]$$

$(5) \Rightarrow\ p_1(x) = e^{-\lambda x}$

$$\therefore\ p_2(t) = e^{-2\lambda t}\left[\lambda\int_0^t e^{-\lambda x}e^{2\lambda x}dx\right]$$

$$\therefore\ p_2(t) = \lambda e^{-2\lambda t}\int_0^t e^{\lambda x}dx$$

$$= e^{-\lambda t}(1-e^{-\lambda t})^1 \qquad (6)$$

Similarly as (5) and (6), we deduce that

$$p_n(t) = e^{-\lambda t}(1-e^{-\lambda t})^{n-1}$$

$$= p(1-p)^{n-1},\quad p = e^{-\lambda t},\ n = 1, 2, \ldots$$

$$\therefore X(t) \sim geom(p),\ p = e^{-\lambda t}$$

$$Mean\left[X(t)\right] = 1/p = e^{\lambda t},$$

$$Variance\left[X(t)\right] = \frac{1-p}{p^2} = \frac{1-e^{-\lambda t}}{e^{-2\lambda t}}$$

(b)

For minor defects, let $Z \sim Poisson(\alpha)$ and for major defects, let $Y \sim Poisson(\beta)$,

where Y and Z are independent random variables. $X(t)$ be the number of defects, either major or minor i.e. $X = Y + Z$

The pgf for Y is given by

$$P_Y(t) = \sum_{y=0}^{\infty} t^y \frac{\beta^y e^{-\beta}}{y!}$$

$$= e^{-\beta}\sum_{y=0}^{\infty}\frac{(t\beta)^y}{y!}$$

$$= e^{-\beta}e^{t\beta}$$

$$P_Y(t) = e^{\beta(t-1)}$$

Similarly, the pgf for Z is given by

$$P_Z(t) = e^{\alpha(t-1)}$$

The pgf for the sum $X = Y + Z$ is given by the product of pgfs for both Y and Z

So, $P_X(t) = e^{\beta(t-1)} e^{\alpha(t-1)}$

$$= e^{(\alpha+\beta)(t-1)}$$

$\therefore X(t)$ must be a Poisson process of rate $\alpha + \beta$.

Q3: [2.5+2.5]

(a)

For Weibull distribution

MTTF

$$MTTF = \eta\ \Gamma\left(\frac{1}{\beta}+1\right)$$
$$= 1000\ \Gamma\left(\frac{1}{1.5}+1\right)$$
$$= \eta\ B_1$$
$$= 1000 \times 0.9027$$
$$= 902.7$$

Variance

$$Var(X) = \eta^2 [B_2 - B_1^2]$$
$$= 1000^2 [0.3757]$$
$$= 375700$$

The median $x_{0.50}$

$$x_p = \left(\ln\left(\frac{1}{1-p}\right)\right)^{1/\beta} \times \eta$$
$$x_{0.50} = \left(\ln\left(\frac{1}{1-0.50}\right)\right)^{1/1.5} \times 1000$$
$$= (\ln 2)^{1/1.5} \times 1000$$
$$= 783.2198$$

The failure rate at 500 hours $\lambda(500)$,

$$\lambda(x)=\frac{f(x)}{R(x)}$$

$$=\frac{\frac{\beta}{\eta}\left[\frac{x}{\eta}\right]^{\beta-1}\exp[-(\tfrac{x}{\eta})^{\beta}]}{\exp[-(\tfrac{x}{\eta})^{\beta}]}$$

$$=\frac{\beta}{\eta}\left[\frac{x}{\eta}\right]^{\beta-1}$$

$$\lambda(500)\ =\frac{1.5}{1000}\times\left[\frac{500}{1000}\right]^{1.5-1}$$

$$=\frac{1.5}{1000}(0.5)^{0.5}$$

$$=1.0607\times10^{-3}\ hours$$

The reliability at 500 hours $R(500)$,

$$R(x)=\exp[-(\tfrac{x}{\eta})^{\beta}]$$

$$=e^{-(\frac{500}{1000})^{1.5}}$$

$$=\exp\left[-(0.5)^{1.5}\right]$$

$$=0.70219$$

(b)

For Lognormal distribution

MTTF

$$MTTF\ =\ e^{\mu+\sigma^2/2}$$

$$=e^{[6.908+0.5(0.317)^2]}$$

$$=1051.785526$$

Variance

$$Var(X)=\exp(2\mu+\sigma^2)[\exp(\sigma^2)-1]$$

$$=\exp(2\times6.908+0.317^2)[\exp(0.317^2)-1]$$

$$=116943.6187$$

The median $x_{0.50}$

$$x_p = \exp(\mu + z_p \sigma)$$

$$x_{0.50} = \exp(\mu + 0 \times \sigma)$$

$$= e^{\mu}$$

$$= e^{6.908}$$

$$= 1000.244751$$

The failure rate at 500 hours $\lambda(500)$,

$$\lambda(x) = \frac{f(x)}{R(x)}$$

$$= \frac{\dfrac{1}{\sigma x}\varphi\left[\dfrac{\ln x - \mu}{\sigma}\right]}{1 - \Phi\left[\dfrac{\ln x - \mu}{\sigma}\right]}$$

$$= \frac{\dfrac{1}{\sigma x}\dfrac{1}{\sqrt{2\pi}\sigma}e^{-(\ln x - \mu)^2/2\sigma^2}}{1 - \Phi\left[\dfrac{\ln x - \mu}{\sigma}\right]}$$

$$\lambda(500) = \frac{7.258965119 \times 10^{-4}}{\Phi(2.19)}$$

$$= \frac{7.258965119 \times 10^{-4}}{0.9857}$$

$$= 7.3643 \times 10^{-4} \text{ hours}$$

The reliability at 500 hours $R(500)$,

$$R(x) = 1 - \Phi\left[\frac{\ln x - \mu}{\sigma}\right]$$

$$= 1 - \Phi\left[\frac{\ln 500 - 6.908}{0.317}\right]$$

$$= 1 - \Phi[-2.19]$$

$$= \Phi(2.19)$$

$$\therefore R(500) = 0.9857$$

Q4: [3+3+3]

(a)

$$\because \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n\}$$
$$= \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_{n-1} = i_{n-1}\}.\Pr\{X_n = i_n | X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_{n-1} = i_{n-1}\}$$
$$= \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_{n-1} = i_{n-1}\}.P_{i_{n-1}i_n} \quad \text{Definition of Markov}$$

By repeating this argument $n-1$ times

$$\therefore \Pr\{X_0 = i_0, X_1 = i_1, X_2 = i_2, \ldots, X_n = i_n\}$$
$$= p_{i_0} P_{i_0 i_1} P_{i_1 i_2} \ldots P_{i_{n-2} i_{n-1}} P_{i_{n-1} i_n}$$ where $p_{i_0} = \Pr\{X_0 = i_0\}$ is obtained from the initial distribution of the process.

(b)

$$\text{pr}\{X_1 = 1, X_2 = 1, X_3 = 0\} = p_1 P_{11} P_{10}, \quad p_1 = \text{pr}\{X_1 = 1\}$$
$$\text{pr}\{X_1 = 1\} = \Pr(X_1 = 1 | X_0 = 0)\Pr(X_0 = 0) + \Pr(X_1 = 1 | X_0 = 1)\Pr(X_0 = 1) + \Pr(X_1 = 1 | X_0 = 2)\Pr(X_0 = 2)$$
$$= P_{01}p_0 + P_{11}p_1 + P_{21}p_2$$
$$= 0.3(0.5) + 0.2(0.2) + 0.3(0.3) = 0.28$$
$$\therefore pr\{X_1 = 1, X_2 = 1, X_3 = 0\} = 0.28(0.2)(0.4) = 0.0224$$

(c)

The transition probability matrix can be written as

$$\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \end{array} \begin{array}{c} \begin{array}{cc} 0 & 1 \end{array} \\ \begin{Vmatrix} 1-\alpha & \alpha \\ \alpha & 1-\alpha \end{Vmatrix} \end{array}$$

(i) The probability that no error occurs up to stage $n = 2$ is given as follows.

$$\Pr\{X_0 = 0, X_1 = 0, X_2 = 0\} = p_0 P_{00} P_{00}$$
$$= 1 \times (1-\alpha) \times (1-\alpha)$$
$$= (1-\alpha)^2$$

where $p_0 = pr(X_0 = 0) = 1$

(ii) The probability that a correct signal is received at stage 2 is given as follows.

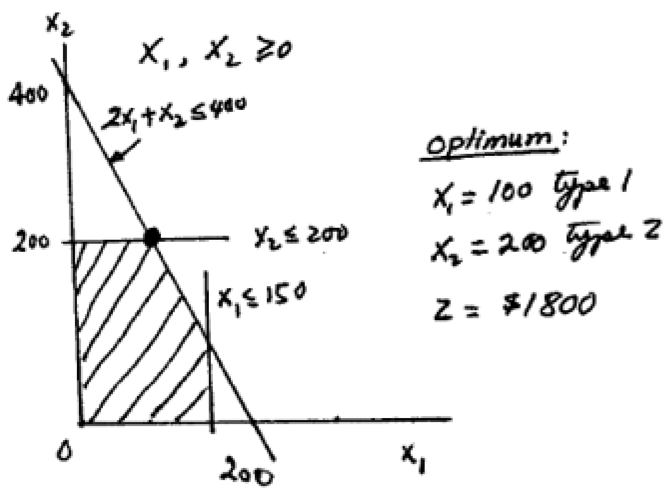
$\Pr\{X_0=0, X_1=0, X_2=0\}+\Pr\{X_0=0, X_1=1, X_2=0\}$
$=p_0 P_{00} P_{00}+p_0 P_{01} P_{10}$
$=(1-\alpha)^2+\alpha^2$
$=1-2\alpha+2\alpha^2$

Q5: [4+4]

(a)

Let x_1 is the daily # of type 1 hat and x_2 is the daily # of type 2 hat
The LP problem will be as follows:

$$\max \quad z = 8x_1 + 5x_2$$
$$\text{s.t} \quad 2x_1 + x_2 \le 400$$
$$x_1 \le 150$$
$$x_2 \le 200$$
$$x_1 \ge 0\ , \quad x_2 \ge 0$$



$\therefore$ The optimal solution is $x_1 = 100,\ x_2 = 200$ where $\max z = \$1800$

(b)

Ans: The optimal solution is $x_1 = 0,\ x_2 = 5$ where $\min z = -5$