King Saud University
Department Of Mathematics
M-203
(Differential and Integral Calculus)

First Mid-Term Examination
(II-Semester 1431/32)

Max. Marks: 20 Time: 90 Minutes

Q. No: 1 Determine whether or not the sequence $\left\{\frac{n^2}{3n-1}-\frac{n^2}{3n+1}\right\}_{n=1}^{\infty}$ converges, and if it converges find its limit................................[3]

Q. No: 2 Determine whether the following infinite series converges or diverges. If it converges, find its sum

$$\sum_{n=1}^{\infty}\left[\left(\frac{3}{4}\right)^n+\frac{5}{2^n}\right]$$ ....[4]

Q. No: 3 Use the integral test to determine the convergence or divergence of the Series $\sum_{n=1}^{\infty}\frac{1}{(n+1)[\ln(n+1)]^2}$ ....[3]

Q, No: 4 Find the power series representation of the function $f(x)=\frac{1}{1+4x^2}$ for $|x|<\frac{1}{2}$, and use it to deduce the power series representation of the function $\tan^{-1}(2x)$ .... [3]

Q. No: 5 Find the interval of convergence and radius of convergence of the power series $\sum_{n=1}^{\infty}\frac{(x+2)^n}{n4^n}$ .... [4]

Q. No: 6 Find the first three non-zero terms of a Taylor series for $f(x)=x^2e^x$ at $c=-1$ ....[3].

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I Mid- term Exam. (II sem. 1431/1432)

Q # 1) Determine whether or not the sequence $\left\{ \frac{n^2}{3n-1} - \frac{n^2}{3n+1} \right\}$ converges, and if it converges find its limit [3]

Soln.
$$a_n = \frac{n^2}{3n-1} - \frac{n^2}{3n+1}$$
$$= \frac{n^2(3n+1) - n^2(3n-1)}{9n^2-1}$$
$$\therefore a_n = \frac{2n^2}{9n^2-1} \quad (1)$$
$$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n^2}{9n^2-1} = \frac{2}{9}, \text{ converges. } (1)$$
(1)

Q # 2) Determine whether the following infinite series converges or diverges. If it converges, find its sum.
$$\sum_{n=1}^{\infty}\left[\left(\frac{3}{4}\right)^n + \frac{5}{2^n}\right] \qquad \text{[Mark: 4]}$$

Soln. $\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n$ is a conv. Geom. series with the common ratio $r_1 = \frac{3}{4} < 1$ and its sum $S_1 = \frac{a_1}{1-r_1}$
$$= \frac{\frac{3}{4}}{1-\frac{3}{4}} = \frac{\frac{3}{4}}{\frac{1}{4}} = 3$$
$(1\frac{1}{2})$

$(1\frac{1}{2})$ $\sum_{n=1}^{\infty} \frac{5}{2^n} = 5\sum_{n=1}^{\infty}\frac{1}{2^n}$ is also a conv. Geom. Series with c.r. $r_2 = \frac{1}{2} < 1$ $\therefore S_2 = 5\left(\frac{a_2}{1-r_2}\right) = 5\left(\frac{\frac{1}{2}}{\frac{1}{2}}\right) = 5.$

Hence $\sum_{n=1}^{\infty}\left[\left(\frac{3}{4}\right)^n + \frac{5}{2^n}\right]$ converges and its sum: S

$\therefore$ Sum: $S = S_1 + S_2 = 3 + 5 = \underline{\underline{8}}$ (1)

Q #3) Use the Integral test to determine the convergence of the Series $\sum_{n=1}^{\infty} \frac{1}{(n+1)[\ln(n+1)]^2}$ [Mark: 3]

Soln. Check that $f(x) = \frac{1}{(x+1)[\ln(x+1)]^2}$

is continuous, positive valued and decreasing, we apply the Integral test: (1)

$$\int_1^{\infty} \frac{1}{(x+1)[\ln(x+1)]^2}\,dx$$

$$= \lim_{t\to\infty} \int_1^{t} \frac{1}{(x+1)[\ln(x+1)]^2}\,dx$$

$$= \lim_{t\to\infty} \Big[-[\ln(x+1)]^{-1}\Big]_1^t$$

$$= \lim_{t\to\infty} \left[-\frac{1}{\ln(t+1)} + \frac{1}{\ln 2}\right] = +\frac{1}{\ln 2};\ \text{con}$$

(1) (1)

Q#4) Find the power Series rep^n. of the function $f(x) = \frac{1}{1+4x^2}$ for $|x| < \frac{1}{2}$, and use it to deduce the power Series Rep^n. of the function $\tan$ [Mark: 3]

Soln. we know $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$

Replacing x by $4x^2$, we get

$$\frac{1}{1+4x^2} = 1 - 4x^2 + (4x^2)^2 - \cdots \quad (1)$$

Now, we know $\tan^{-1}(2x) = \int_0^x \frac{1}{1+4t^2}\,dt$ (1)

$$= \int_0^x [1 - (4t^2) + 16t^4 - \cdots]\,dt$$

$$= \left[t - 4\frac{t^3}{3} + 16\frac{t^5}{5} - \cdots\right]_0^x$$

$$= x - \frac{4x^3}{3} + \frac{16x^5}{5} - \cdots \quad (1)$$

Q#5) Find the interval of convergence and the radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{(x+2)^n}{n4^n}$.

[Mark: 4]

Soln. $\lim_{n\to\infty} \left| \frac{(x+2)^{n+1}}{(n+1)4^{n+1}} \times \frac{n4^n}{(x+2)^n} \right|$

$= \frac{1}{4}|x+2|$

$\lim_{n\to\infty} \frac{n}{n+1} = 1$

For abs. conv. $\frac{1}{4}|x+2| < 1$

$\Leftrightarrow |x+2| < 4$

$\Leftrightarrow -4 < x+2 < 4$

$\Leftrightarrow -6 < x < 2$ (1)

At $x=-6$, we have $\sum_{n=1}^{\infty} \frac{(-4)^n}{n4^n} = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n}$ which is conv. by Alternating Series test (AST) (1)

At $x=2$, we have $\sum_{n=1}^{\infty} \frac{4^n}{n4^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ which is a divergent harmonic series. (1)

Hence interval of conv: $[-6, 2)$

Radius of conv: $r = \frac{2-(-6)}{2} = \underline{\underline{4}}$ (1)

Q#6) Find the first three non-zero terms of a Taylor Series for $f(x) = x^2 e^x$ at $c=-1$ [Mark: 3]

Soln. $f(-1) = (-1)^2 e^{-1} = e^{-1}$

$f'(x) = 2x e^x + x^2 e^x \Rightarrow f'(-1) = 2(-1)e^{-1} + (-1)^2 e^{-1}$

$= -e^{-1}$ (1)

$f''(x) = 2e^x + 2xe^x + 2xe^x + x^2 e^x$

$\Rightarrow f''(-1) = 2e^{-1} + 2(-1)e^{-1} + 2(-1)e^{-1} + (-1)^2 e^{-1} = -e^{-1}$

Hence $f(x) = e^{-1} + (x+1)(-e^{-1}) + \frac{(x+1)^2}{2!}(-e^{-1}) + \cdots$ (2)