King Saud University
Department Of Mathematics
M-203
(Differential and Integral Calculus)

First Mid-Term Examination
(II-Semester 1432/33)

Max. Marks: 20 Time: 90 Minutes

Q. No: 1 Determine whether or not the sequence $\left\{\dfrac{\tan^{-1} n}{n}\right\}_{n=1}^{\infty}$ converges, and if it converges find its limit……………………………………………………..[3]

Solution: $0 \le \left|\tan^{-1} n\right| < \pi/2 \quad \Rightarrow \quad \dfrac{0}{n} \le \left|\dfrac{\tan^{-1} n}{n}\right| < \dfrac{\pi}{2}\left(\dfrac{1}{n}\right)$

$$\Rightarrow \quad 0 = \lim_{n\to\infty} \frac{0}{n} \le \lim_{n\to\infty}\left|\frac{\tan^{-1} n}{n}\right| < \lim_{n\to\infty} \frac{\pi}{2}\left(\frac{1}{n}\right) = 0$$

Hence by Sandwich theorem $\lim\limits_{n\to\infty}\left|\dfrac{\tan^{-1} n}{n}\right| = 0 \Rightarrow \lim\limits_{n\to\infty} \dfrac{\tan^{-1} n}{n} = 0$

$\Rightarrow$ Given sequence is convergent.

Q. No: 2 Determine whether the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\ln(n)}{n}$ **converges absolutely**, **converges con**ditionally, or **diverges**.………………………………[4]

Solution: It is an alternating series with $a_n = \dfrac{\ln(n)}{n} > 0.$

Also, $f(x) = \dfrac{\ln(x)}{x} \Rightarrow f'(x) = \dfrac{1-\ln(x)}{x^2} < 0$ for $x > e$ and

$\lim\limits_{n\to\infty} \dfrac{\ln(n)}{n} = 0.$ Hence, by AST, $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{\ln(n)}{n}$ is convergent.

<u>Now let us check Absolute Convergence</u>

$\displaystyle\sum_{n=1}^{\infty}\left|(-1)^n \frac{\ln(n)}{n}\right| = \sum_{n=1}^{\infty} \frac{\ln(n)}{n}$. Comparing this series with the harmonic series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$. Note $\dfrac{\ln(n)}{n} > \dfrac{1}{n}$ for $n > e$. Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent $\Rightarrow$

$$\sum_{n=1}^{\infty}\frac{\ln(n)}{n} \text{ is divergent.}$$

Note by integral test we can also prove that $\sum_{n=1}^{\infty}\frac{\ln(n)}{n}$ is divergent.

Hence the given series $\sum_{n=1}^{\infty}(-1)^n\frac{\ln(n)}{n}$ is **conditionally convergent**.

Q. No: 3 Find the **interval of convergence** and **radius of convergence** of the power series $\sum_{n=1}^{\infty}(-3)^n\frac{x^n}{\sqrt{n+1}}$ .... [5]

Solution: $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(-3)^{n+1}x^{n+1}}{\sqrt{n+1}}\frac{\sqrt{n}}{(-3)^n x^n}\right| = \lim_{n\to\infty}\frac{3\sqrt{n}}{\sqrt{n+1}}|x| = 3|x|$

For absolute convergence $3|x|<1 \Leftrightarrow |x|<\frac{1}{3} \Leftrightarrow -\frac{1}{3}<x<\frac{1}{3}$.

Check Convergence at $x=-\frac{1}{3}$

$$\sum_{n=1}^{\infty}(-3)^n\frac{(-\frac{1}{3})^n}{\sqrt{n+1}} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n+1}}$$

It is divergent by p-series test

Check Convergence at $x=\frac{1}{3}$

$$\sum_{n=1}^{\infty}(-3)^n\frac{(\frac{1}{3})^n}{\sqrt{n+1}} = \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n+1}}$$

It is convergent by AST.

Hence, **interval of convergence** : $\left(-\frac{1}{3},\frac{1}{3}\right]$

Radius of convergence: $\rho=\frac{1}{3}$.

Q. No: 4 Find Maclaurin series for the function $f(x)=\frac{1}{(1+x)^2}$ ....[4]

Solution:

$$f(x)=\frac{1}{(1+x)^2} \Rightarrow f(0)=1.$$

$$f^{/}(x)=\frac{-2}{(1+x)^3} \Rightarrow f^{/}(0)=-2=-(2!)$$

$$f^{//}(x)=\frac{(2)(3)}{(1+x)^4} \Rightarrow f^{//}(0)=(2)(3)=(3!)$$

$$f^{///}(x)=\frac{-(2)(3)(4)}{(1+x)^5} \Rightarrow f^{////}(0)=-(2)(3)(4)=(4!)$$

$$f^{v}(x)=\frac{(2)(3)(4)(5)}{(1+x)^6} \Rightarrow f^{v}(0)=(2)(3)(4)(5)=(5!)$$

Hence the required Maclaurin series

$$f(0)+xf^{/}(0)+\frac{x^2}{2!}f^{/}(0)+\frac{x^3}{3!}f^{//}(0)+........+\frac{x^n}{n!}f^{(n)}(0)+...$$

$$f(x)=\frac{1}{(1+x)^2}=1-2x+3x^2-4x^3+.................+(-1)^n x^n+......$$

Q. No: 5 Find the power series representation for the function $f(x)=\ln(1+x)$, $|x|<1$ and use it to calculate $\ln(1.2)$ to four decimal places……………[4]

Solution: We know

$$f(x)=\ln(1+x)=\int_0^x \frac{1}{1+t}dt=\int_0^x\left(1-t+t^2-t^3+.....+(-1)^n t^n+...\right)dt \quad \text{if } |t|<1.$$

$$=\left[\ t-\frac{t^2}{2}+\frac{t^3}{3}-.....(-1)^n\frac{t^{n+1}}{n+1}+.......\ \right]_0^x$$

$$=x-\frac{x^2}{2}+\frac{x^3}{3}+.....+(-1)^n\frac{x^{n+1}}{n+1}+....$$

$$=\sum_{n=1}^{\infty}(-1)^n\frac{x^n}{n},......if\ |x|<1.$$

Now, $\ln(1.2)=\ln(1+0.2)=(0.2)-\frac{(0.2)^2}{2}+\frac{(0.2)^3}{3}-\frac{(0.2)^4}{4}+....$

$\ln(1.2)\approx 0.2-0.02+0.22266-....\approx 0.1826.$.