TIME: 90 min
M - 107

KING SAUD UNIVERSITY

DEPARTMENT OF MATHEMATICS

II MID TERM EXAM (SEM I) 1433-1434

FULL MARKS:40

Question: 1. (a) Given vectors $r = < 2,1,-3 >$ *and* $s = < 1,3,2 >$ **in space,**

i. **Show that** $r \times s$ **is orthogonal to** r **and** s.

ii. **If vectors** r **and** s **are the edges of a parallelogram, then find the area of the parallelogram.**

(b) For the given points in the space P(2 , 1, 3), Q(8, 4, 2), R(2, 2, 5) and S(6, -1, 8) find the volume of the parallelepiped with edges PQ, PR, and PS

Solution: a (i)

$$r \times s = \begin{vmatrix} i & j & k \\ 2 & 1 & -3 \\ 1 & 3 & 2 \end{vmatrix} = <11, -7, 5>$$

$r \times s$ is orthogonal to r and s if $(r \times s) \cdot s = 0$ and $(r \times s) \cdot r = 0$

$(r \times s) \cdot s = <11,-7,5> \cdot <1,3,2> = 11-21+10 = 21-21 = 0$

$(r \times s) \cdot r = <11,-7,5> \cdot <2,1,-3> = 22-7-15 = 22-22 = 0$

(ii) Area of parallelogram with edges r and s $= \|r \times s\|$

$$= \sqrt{(11)^2 + (-7)^2 + (5)^2} = \sqrt{121+49+25} = \sqrt{195} \text{ unit}^2$$

b

Volume of the parallelepiped with edges $\vec{PQ}$, $\vec{PR}$, and $\vec{RS}$ $= \vec{PQ} \cdot (\vec{PR} \times \vec{PS})$

$\vec{PQ} = <6,3,-1>$, $\vec{PR} = <0,1,2>$, $\vec{PS} = <4,-2,5>$

$$V = \vec{PQ} \cdot (\vec{PR} \times \vec{PS}) = \begin{vmatrix} 6 & 3 & -1 \\ 0 & 1 & 2 \\ 4 & -2 & 5 \end{vmatrix} = 6(9) - 3(-8) + (-1)(-4)$$

$$= 54 + 24 + 4$$

$$= 82 \text{ unit}^3$$

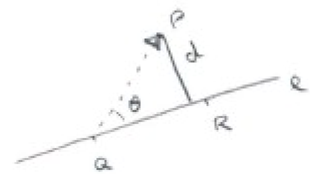
Question: 2 . (a) Find the distance from the point P(1 , 1, 1) to the line through the points Q(0 , 6, 8) and R(-1 , 4, 7).

(b) Find the equation of the plane that passes through the point A(2 , 0, 1) and contains the line $x = 1+2t,\ y = -3 + t,\ z = -t$.

(c) Identify the surface $4x^2 - 9y^2 + z^2 = 36$. Find its traces on the coordinate planes and then sketch the surface.

Solution

(a)



Distance of a point P from line 'ℓ'

$$d = \frac{\|\overrightarrow{QP} \times \overrightarrow{QR}\|}{\|\overrightarrow{QR}\|}$$

$\overrightarrow{QR} = \langle -1, -2, -1 \rangle$

$\overrightarrow{QP} = \langle 1, -5, -7 \rangle$

$$\overrightarrow{QP} \times \overrightarrow{QR} = \begin{vmatrix} i & j & k \\ 1 & -5 & -7 \\ -1 & -2 & -1 \end{vmatrix} = (5-14)i - (-1-7)j + (-2-5)k$$

$$= \langle -9, 8, -7 \rangle$$

$\|\overrightarrow{QP} \times \overrightarrow{QR}\| = \sqrt{81 + 64 + 49} = \sqrt{194}$, $\overrightarrow{QR} = \sqrt{1+4+1} = \sqrt{6}$

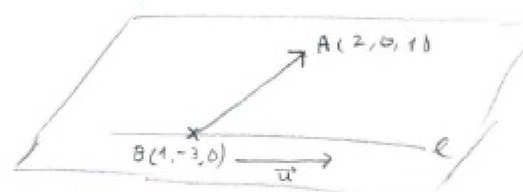
$d = \dfrac{\sqrt{194}}{\sqrt{6}} = 5.56$ units //

(b) To write equation of plane, we need a point and a normal vector

vector $\vec{u}$ is parallel to line ℓ

$\vec{u} = \langle 2, 1, -1 \rangle$

$\overrightarrow{BA} = \langle 1, 3, 1 \rangle$

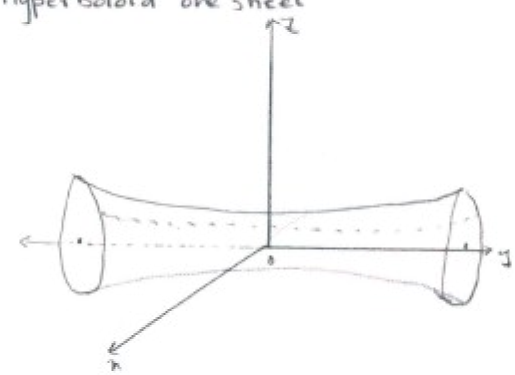


$$n = \vec{u} \times \overrightarrow{BA} = \begin{vmatrix} i & j & k \\ 2 & 1 & -1 \\ 1 & 3 & 1 \end{vmatrix} = \langle 4, -3, 5 \rangle$$

Equation of plane is $4(x-2) - 3(y-0) + 5(z-1) = 0$. //

(c) $\dfrac{x^2}{9} - \dfrac{y^2}{4} + \dfrac{z^2}{36} = 1$, is Hyperboloid one sheet

Trace	Equation	Description
xy-plane	$\frac{x^2}{9} - \frac{y^2}{4} = 1$	Hyperbola
yz-plane	$\frac{z^2}{36} - \frac{y^2}{4} = 1$	Hyperbola
xz-plane	$\frac{x^2}{9} + \frac{z^2}{36} = 1$	Ellipse



Question: 3. (a) The motion of a point moving along the curve is give by

$$r(t) = t^3 i + 3t^2 j + 3tk.$$

Find the velocity, acceleration and speed at time $t = 2$.

(b) The position vector of a moving point at time t is given by

$$r(t) = \cos t i + \sin t j + k.$$

Find the Unit Tangent vector and Principal Normal vector of the curve at time t.

(c) Find the curvature of the space curve $r(t) = 3\sin t i + 3\cos t j + 4tk$.

Solution (a)

velocity, $v(t) = \frac{dr}{dt} = 3t^2 i + 6t j + 3k$

acceleration $a(t) = \frac{d^2r}{dt^2} = 6t i + 6j$

At $t = 2$ $\quad v(2) = 12i + 12j + 3k$

$a(2) = 12i + 6j$

speed $= \|v(2)\| = \sqrt{144 + 144 + 9} = \sqrt{297}$

(b) unit Tangent vector $T(t) = \frac{r'(t)}{\|r'(t)\|}$

$r(t) = \cos t\, i + \sin t\, j + k$

$r'(t) = -\sin t\, i + \cos t\, j \qquad \|r'(t)\| = \sqrt{\sin^2 t + \cos^2 t} = 1$

$T(t) = -\sin t\, i + \cos t\, j$

Principal Normal vector $N(t) = \frac{T'(t)}{\|T'(t)\|}$

$T'(t) = -\cos t\, i - \sin t\, j \qquad \|T'(t)\| = \sqrt{\cos^2 t + \sin^2 t} = 1$

$N(t) = -\cos t\, i - \sin t\, j$

(c) curvature, $\kappa = \frac{|T'(t)|}{|r'(t)|} \qquad r'(t) = 3\cos t\, i - 3\sin t\, j + 4k$

$\|r'(t)\| = \sqrt{25} = 5$

$T(t) = \frac{r'(t)}{\|r'(t)\|} = \frac{3}{5}\cos t\, i - \frac{3}{5}\sin t\, j + \frac{4}{5}k$

$T'(t) = -\frac{3}{5}\sin t\, i - \frac{3}{5}\cos t\, j \qquad \|T'(t)\| = \frac{3}{5}$

$\kappa = \frac{\|T'(t)\|}{\|r'(t)\|} = \frac{1}{5} \cdot \frac{3}{5} = \frac{3}{25}$ units //

Method 2 $\quad \kappa = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3} \qquad r''(t) = -3\sin t\, i - 3\cos t\, j$

$r' \times r'' = \langle 12\cos t, -12\sin t, -9 \rangle$

$\kappa = \frac{\sqrt{225}}{25 \times 5} = \frac{15}{125} = \frac{3}{25}$