# Unconstrained Optimization:

→ function of one variable

$f(x) \qquad f: R \longrightarrow R$

* local extrema (min. / max.)
* intervals (increasing / decreasing)
* $\lim_{x \to \infty} f(x) \qquad \lim_{x \to -\infty} f(x)$

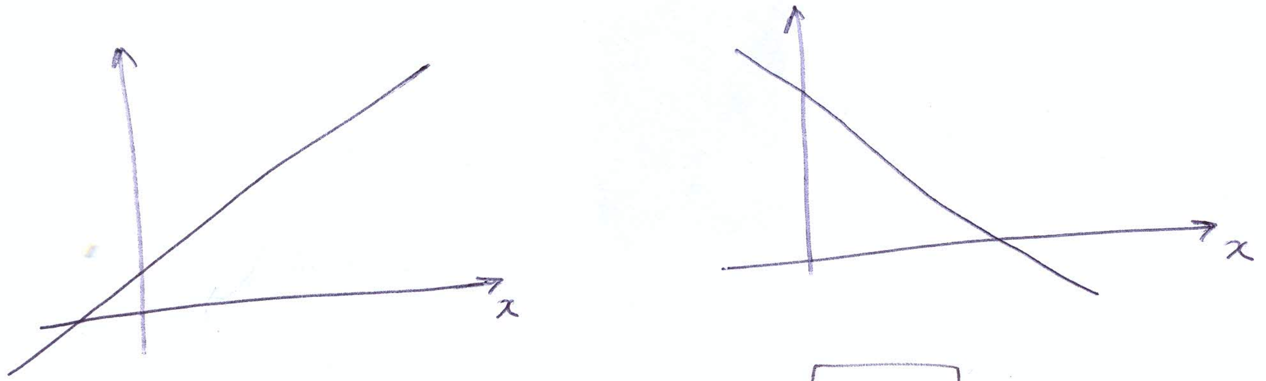
* linear fun.

$f(x) = ax + b$

$a \longrightarrow$ slope (coeff. of $x$)

$(+)$ increasing $\quad x \uparrow \quad f(x) \uparrow$

$(-)$ decreasing $\quad x \uparrow \quad f(x) \downarrow$



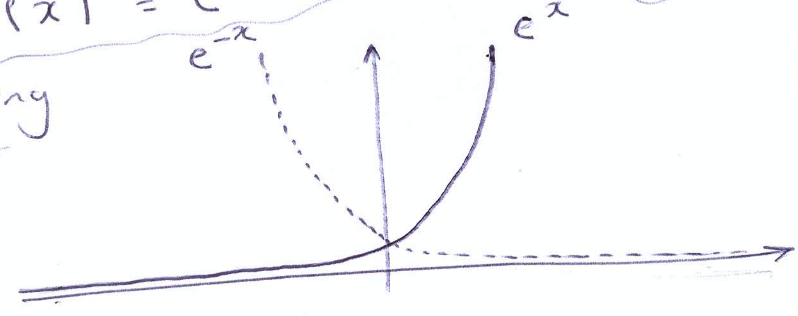

* exponential fun.

$\boxed{\neq 0}$

$f(x) = e^{x}$ increasing

$f(x) = e^{-x}$ decreasing

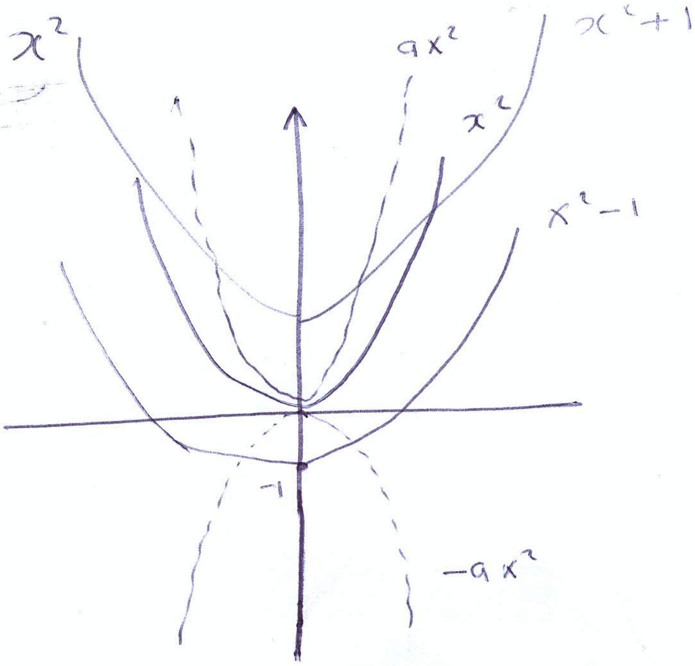
$f(0) = 1$




$\lim_{x \to -\infty} e^{x} = 0 \qquad \lim_{x \to -\infty} e^{-x} = \infty$

$\lim_{x \to \infty} e^{x} = \infty \qquad \lim_{x \to \infty} e^{-x} = 0$

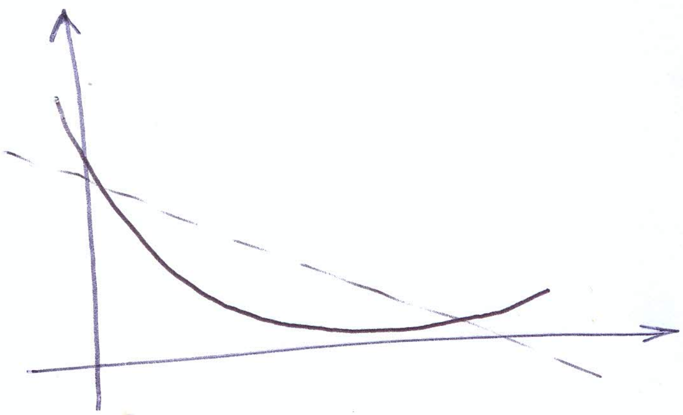
* $f(x) = x^2$




$x^2-1$

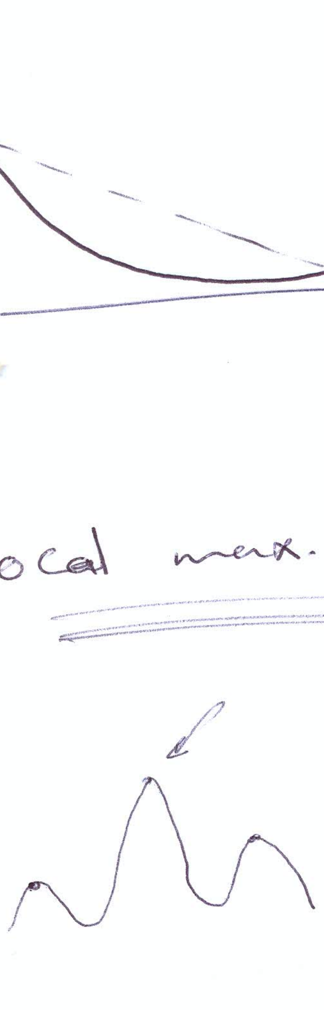
$x^2+1$

convex function & concave function



local max./min.

Global max./min.



**Ex 1**

Let $f(x) = \begin{cases} 2-(x-1)^2, & 0 \leq x \leq 3 \\ -3+(x-4)^2, & 3 \leq x \leq 6 \end{cases}$

find max. of $f(x)$ on $0 \leq x \leq 6$

1st derivative

$f'(x) = \begin{cases} -2(x-1) & 0 \leq x \leq 3 \\ 2(x-4) & 3 \leq x \leq 6 \end{cases}$

critical points: $\boxed{f'(x) = 0}$

$-2(x-1) = 0 \Rightarrow x^* = 1 \in [0,3]$ ✓

$2(x-4) = 0 \Rightarrow x^* = 4 \in [3,6]$ ✓

Evaluate $f$ at critical points & at endpts

$f(0) = 2-(0-1)^2 = 1$

$f(1) = 2 - 0 = 2$

$f(3^-) = 2-(3-1)^2 = -2$

$f(3^+) = -3+(3-4)^2 = -2$

$f(4) = -3+0 = -3$

$f(6) = -3+(6-4)^2 = 1$

max. of $f = 2$ at $x = 1$

## Ex 2

$$f(x) = \frac{1}{x} + 2\ln x - x$$

find critical points & determine their kind.

$$f'(x) = -\frac{1}{x^2} + \frac{2}{x} - 1 = 0$$

$x \neq 0$ $\quad * x^2 \quad -1 + 2x - x^2 = 0$

$$x^2 - 2x + 1 = 0$$

$$(x-1)^2 = 0$$

$$x = 1$$

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$f'(x) > 0 \Rightarrow$ fun. increasing

$f'(x) < 0 \Rightarrow$ ~ decreasing

$x^*$ → max: inc. ↗ * ↘ dec.

$x^*$ → min: dec. ↘ * ↗ inc.

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$f''(x) > 0 \Rightarrow$ fun. concave up (convex)

$f''(x) < 0 \Rightarrow$ ~ concave down (concave)

if $f'(x^*) = 0$ and $f''(x^*) > 0 \Rightarrow x^*$ min.

if $f'(x^*) = 0$ and $f''(x^*) < 0 \Rightarrow x^*$ max.

$$f''(x) = \frac{2}{x^3} - \frac{2}{x^2}$$

$f''(1) = 0$ $\quad$ $x = 1 \longrightarrow$ saddle point

Ex: 3 find the extreme points of

$f(x) = 1 - e^{-x^2}$ on $[-1, 1]$

$\rightarrow$ $f'(x) = 2x\, e^{-x^2} = 0$

$\boxed{x_0 = 0}$ critical point

$f''(x) = -4x^2 e^{-x^2} + 2e^{-x^2}$

$f''(0) = 2 > 0$ $\quad \boxed{x = 0}$ local min.

$f''(x_0) > 0$ $\quad \boxed{x_0 =}$ local min.

$f''(x_0) < 0$ $\quad \boxed{x_0 =}$ local max.

at end points: $\quad$ $f(0) = 0$ $\quad$ $f(x_0)$

$f(1) = ??$ $\quad 1 - e^{-1} = 0.63$

$f(-1) = ??$ $\quad 1 - e^{-1} = 0.63$

$x = 0 \longrightarrow$ global min.

$x = 1,\ x = -1 \longrightarrow$ global max.

## Ex: 4

Draw a curve for the function:

$$f(x) = x^2 e^{-x}$$

$\boxed{f(0) = 0}$

$f(2) = 4e^{-2}$

→ x-intercept $\quad f(x) = 0 \Rightarrow x = 0$ $\quad (0,0)$

→ y-intercept $\quad x = 0 \Rightarrow y = 0$

$$f'(x) = -x^2 e^{-x} + 2x e^{-x}$$

$f'(x) = 0$ $\quad -x^2 e^{-x} + 2x e^{-x} = 0$

$$x e^{-x}(-x + 2) = 0$$

$x = 0$ , $x = 2$ $\quad$ critical points

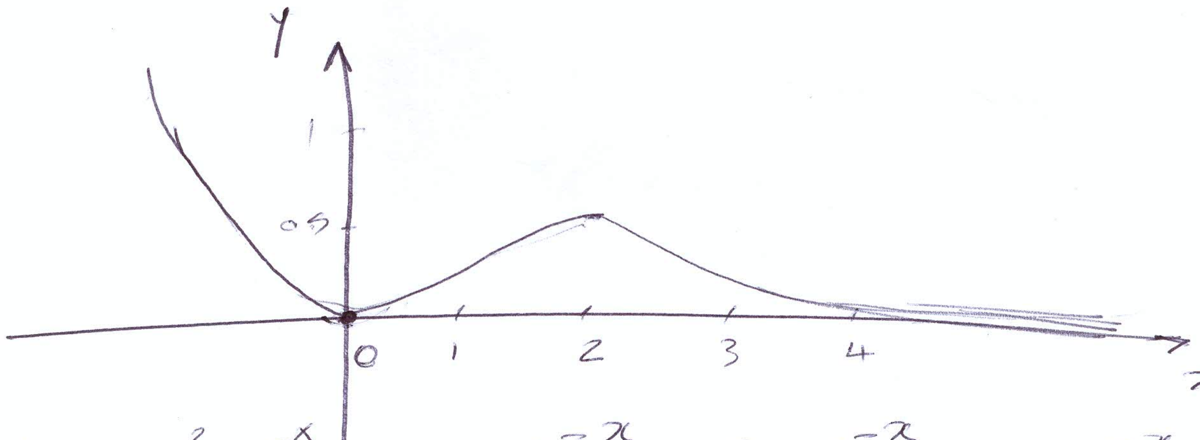
$f'(x)$ $\quad$ -ve $\quad 0 \quad$ +ve $\quad 2 \quad$ -ve

$x = 0$ $\quad$ local min.

$x = 2$ $\quad$ local max.

$\lim_{x \to \infty} x^2 e^{-x} = 0$

$\lim_{x \to -\infty} x^2 e^{-x} = \infty$




$$f''(x) = x^2 e^{-x} - 2x e^{-x} - 2x e^{-x} + 2e^{-x}$$

$$= (x^2 - 4x + 2) e^{-x} \qquad \frac{4 \pm \sqrt{8}}{2} \qquad x_2, x_1$$