Unconstrained Optimization

Min / Max $f(x)$

1. function of one variable

$f(x) \qquad f: R \longrightarrow R$

Ex: Min / Max $f(x) = 2x^3 - 3x^2$

local extrema

$f(x) = 2x^3 - 3x^2$

first derivative $\quad f'(x) = 6x^2 - 6x$

$f'(x) = 0 \qquad 6x^2 - 6x = 0$

$6x(x-1) = 0 \Rightarrow \underline{x=0} \quad \underline{x=1}$

critical points

second derivative

$f''(x) = 12x - 6$

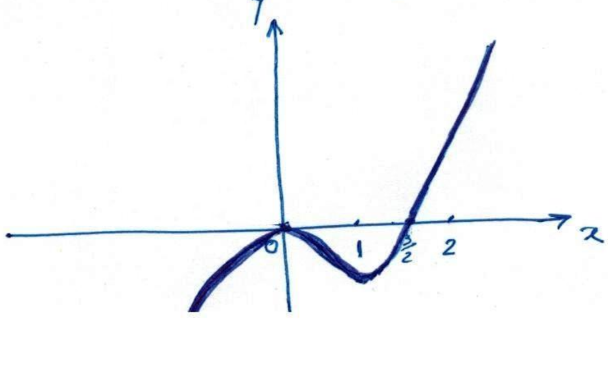
$f''(0) = -6 \Longrightarrow f'(0) = 0, \ f''(0) < 0$

$\underline{x=0}$ local max.

$f''(1) = 6 \Longrightarrow f'(1) = 0, \ f''(1) > 0$

$\underline{x=1}$ local min.

	inc.	dec.	inc.
$f'(x)$	+	−	+
		0	1



x-intercept

$\underline{y=0} \quad 2x^3 - 3x^2 = 0$

$x^2(2x-3) = 0$

$\underline{x=0} \quad x = \frac{3}{2}$

y-intercept

$\underline{x=0} \quad \underline{y=0}$

* function of several variables

$f(X) \qquad f : R^n \longrightarrow R$

$f(x_1, x_2) \qquad R^2 \longrightarrow R$

$f(x_1, x_2, x_3) \qquad R^3 \longrightarrow R$

Gradient vector $\qquad \nabla f(x) = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix} = 0$

vector of first derivatives

x^* ... critical points

$$\nabla^2 f(X) = H(X) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$

Hessian matrix

Matrix of second derivatives

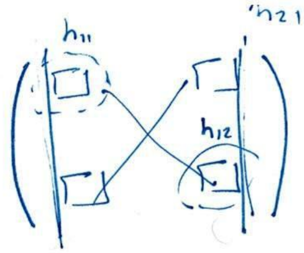
$H(x^*)$ +ve definite $\Rightarrow$ x^* local min.

+ve semi definite ~

$H(x^*)$ −ve definite $\Rightarrow$ x^* local max.

−ve semi definite ~

$H(x^*)$ indefinite $\Rightarrow$ x^* saddle point



(leading) principal minors

$k=1$ h_{11} h_{12}

$k=2$ h_{21}

all leading principal minors > 0

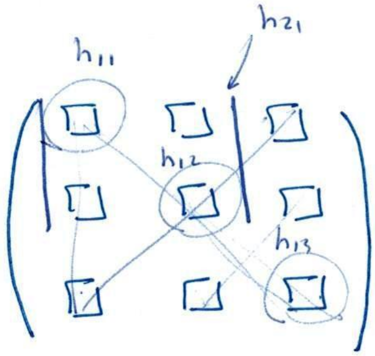
↳ +ve $\Rightarrow$ +ve definite

↳ $(-1)^k$ $\Rightarrow$ h_{11} −ve, h_{21} +ve   −ve definite

all principal minors ≥ 0 +ve semidefinite

~   ~   ~ $(-1)^k$ or 0 −ve semidefinite

Otherwise $\Rightarrow$ indefinite



(l) p. m.'s

$k=1$ (h_{11}) h_{12} h_{13}

$k=2$ (h_{21}) h_{22} h_{23}

$k=3$ (h_{31})

l.p.m.'s   all +ve $\Rightarrow$ +ve definite

$(-1)^k$ − + − $\Rightarrow$ −ve definite

Ex:

$$\text{Max/Min } f(x_1, x_2) = x_1^3 + x_2^4 - 2x_1^2 - 8x_2^2 + 10$$

$$\nabla f(x_1, x_2) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \\ \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 3x_1^2 - 4x_1 \\ \\ 4x_2^3 - 16x_2 \end{bmatrix} = 0$$

$$3x_1^2 - 4x_1 = 0 \Rightarrow x_1(3x_1 - 4) = 0$$

$$\underline{\underline{x_1 = 0}} \qquad \underline{\underline{x_1 = \frac{4}{3}}}$$

$$4x_2^3 - 16x_2 = 0 \Rightarrow 4x_2(x_2^2 - 4) = 0$$

$$\underline{\underline{x_2 = 0}} \qquad x_2^2 = 4 \quad \underline{\underline{x_2 = 2}} \qquad \underline{\underline{x_2 = -2}}$$

critical points: $(0,0)$, $(0,2)$, $(0,-2)$
$(\frac{4}{3}, 0)$, $(\frac{4}{3}, 2)$, $(\frac{4}{3}, -2)$

$$\nabla^2 f(x_1, x_2) = H(X) = \begin{bmatrix} 6x_1 - 4 & 0 \\ \\ 0 & 12x_2^2 - 16 \end{bmatrix}$$

$*$ $H(0,0) = \begin{bmatrix} -4 & 0 \\ 0 & -16 \end{bmatrix}$ l.p.m.'s $h_{11} = -4$

$h_{21} = (-4)(-16) - 0 = 64$

$(-1)^k \Rightarrow$ -ve definite

$\underline{\underline{(0,0) \text{ local max.}}}$

$*$ $H(0,2) = \begin{bmatrix} -4 & 0 \\ 0 & 32 \end{bmatrix}$ l.p.m.'s $h_{11} = -4$

$h_{21} = -4(32) - 0 = -128$

$\underline{\underline{\text{indefinite}}}$

$\underline{\underline{(0,2) \text{ saddle point}}}$

* $H(0,-2) = \begin{bmatrix} -4 & 0 \\ 0 & 32 \end{bmatrix}$ H indefinite

$(0,-2)$ saddle point

* $H(\frac{4}{3}, 0) = \begin{bmatrix} 4 & 0 \\ 0 & -16 \end{bmatrix}$ $h_{11} = 4$

$h_{21} = -64$

$(\frac{4}{3}, 0)$ saddle point indefinite

* $H(\frac{4}{3}, 2) = \begin{bmatrix} 4 & 0 \\ 0 & 32 \end{bmatrix}$ $h_{11} = 4$

$h_{21} = 128$

+ve definite

$(\frac{4}{3}, 2)$ local min

* $H(\frac{4}{3}, -2) = \begin{bmatrix} 4 & 0 \\ 0 & 32 \end{bmatrix}$ +ve definite

$(\frac{4}{3}, -2)$ local min.

Ex 2

$$f(x_1, x_2, x_3) = \frac{x_1^2}{2} + x_2^3 + x_3^2 - x_1 x_2 - 2x_2x_3 + 2x_1 - 2x_2$$

$$\nabla f(x_1, x_2, x_3) = \begin{bmatrix} x_1 - x_2 + 2 \\ 3x_2^2 - x_1 - 2x_3 - 2 \\ 2x_3 - 2x_2 \end{bmatrix} = 0$$

$$x_1 - x_2 + 2 = 0 \quad (1) \longrightarrow \boxed{x_1 = x_2 - 2}$$

$$3x_2^2 - x_1 - 2x_3 - 2 = 0 \quad (2)$$

$$2x_3 - 2x_2 = 0 \quad (3) \longrightarrow \boxed{x_3 = x_2}$$

subs. (2) $\quad 3x_2^2 - (x_2 - 2) - 2x_2 - 2 = 0$

$$3x_2^2 - 3x_2 = 0$$

$$3x_2(x_2 - 1) = 0$$

$x_2 = 0$, $x_2 = 1$

$x_1 = -2$, $x_1 = -1$

$x_3 = 0$, $x_3 = 1$

$(-2, 0, 0)$ → critical points

$(-1, 1, 1)$ ↗

$$H(X) = \nabla^2 f(X) = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 6x_2 & -2 \\ 0 & -2 & 2 \end{bmatrix}$$

$$H(-2,0,0) = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & -2 \\ 0 & -2 & 2 \end{bmatrix}$$

$h_{11} = 1$

$h_{21} = -1$

$h_{31} = -4 - 2 = -6$

(-2, 0, 0)
saddle point

$+$
$-$
$-$

indefinite

$$H(-1,1,1) = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{bmatrix} 1 & -1 & 0 \\ -1 & 6 & -2 \\ 0 & -2 & 2 \end{bmatrix} \end{array}$$

$h_{11} = 1$

$h_{21} = 5$

$h_{31} = 8 - 2 = 6$

+ve definite

(-1, 1, 1) local min.

Max/Min $f(x_1, x_2, x_3) = x_1 x_2 + x_2 x_3 + x_1 x_3$

$$\nabla f(x) = \begin{bmatrix} x_2 + x_3 \\ x_1 + x_3 \\ x_2 + x_1 \end{bmatrix} = 0$$

$x_2 + x_3 = 0 \Rightarrow \underline{x_2 = -x_3}$ $\qquad \underline{x_1 = 0}, \underline{x_2 = 0}, \underline{x_3 = 0}$

$x_2 + x_1 = 0 \Rightarrow \underline{x_2 = -x_1}$

$x_1 + x_3 = 0 \Rightarrow \underline{x_1 = -x_3}$

$$H(X) = \nabla^2 f(x) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$H(0,0,0) = \begin{array}{c} + \\ - \\ + \end{array} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$\ominus$ $h_{11} = 0$ $\quad h_{12} = 0$ $\quad h_{13} = 0$

$\oplus$ $h_{21} = -1$ $\quad h_{22} = -1$ $\quad h_{23} = -1$

$\ominus$ $h_{31} = 2$

$0, (-1)^k, \quad -(-1)^1, \quad +(-1)^2, \quad -(-1)^3$

<u>indefinite</u>