convex & concave functions

if $\underline{H(X)}$ +ve definite — strictly convex fun.

+ve semidefinit

$H(X)$ −ve definite — strictly concave fun.

−ve semi definite

$\underline{Ex}$: $f(x,y) = x^a y^b$ $\quad \underline{a, b > 0}$

gradient vector

$$\nabla f(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix} = \begin{pmatrix} a x^{a-1} y^b \\ b x^a y^{b-1} \end{pmatrix}$$

$$\nabla^2 f(x,y)$$

$$H(X) = \begin{pmatrix} a(a-1) x^{a-2} y^b & ab x^{a-1} y^{b-1} \\ ab x^{a-1} y^{b-1} & b(b-1) x^a y^{b-2} \end{pmatrix}$$

$\underline{a+b<1}$ strictly concave

−ve definite $\quad (-1)^K \quad K=1$ −ve, $K=2$ +ve

h_{11} −ve , h_{21} +ve

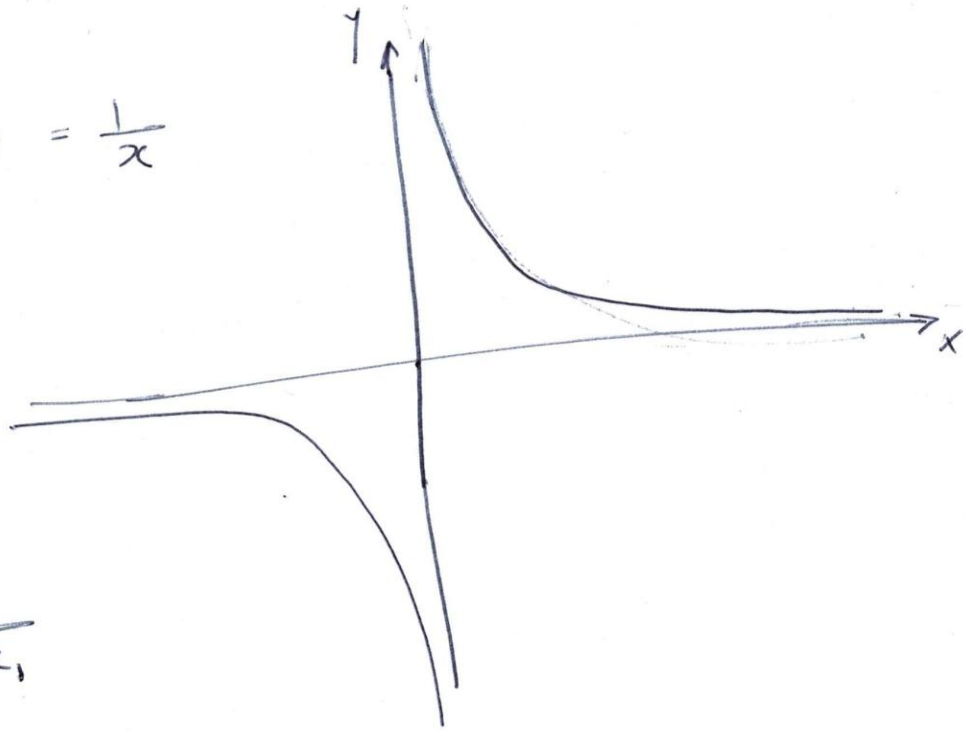
circle:

$$(x - x_c)^2 + (y - y_c)^2 = r^2$$

center (x_c, y_c)

radius r

$$y = f(x) = \frac{1}{x}$$



$$x_2 = \frac{8}{x_1}$$

مربع الكامل

$$(x \mp a)^2$$
$$= x^2 \mp 2ax + a^2$$

x^2 $\quad$ $\underset{2*4}{8}x$ $\quad$ 16

معادلة خط مستقيم

$$y = mx + c$$
$$x_2 = mx_1 + c$$

slope

Constrained Optimization

$X = (x_1, x_2, \cdots, x_n)$

Max/Min $f(X)$

subject to

$$g_i(X) \ (=)(\leq)(\geq) \ 0$$

1- Graphical method:

$$\ast \ \text{Max } Z = f(x_1, x_2) = 2x_1 + 3x_2$$

$$\text{s.t.} \quad x_1^2 + x_2^2 \leq 20$$

$$x_1 x_2 \leq 8 \qquad \geq 8$$

$$x_1, x_2 \geq 0$$

1- Draw constraints

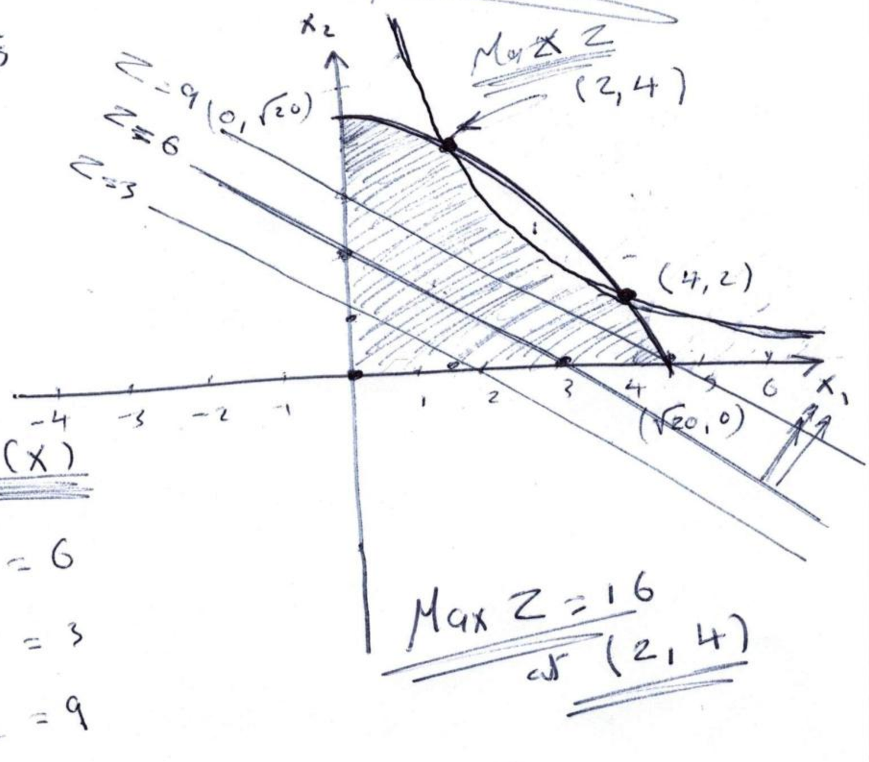
2- Feasible region

$- x_1^2 + x_2^2 = 20$

$x_1 x_2 = 8 , \ x_2 = \frac{8}{x_1}$

$x_2 \leq \frac{8}{x_1}$

level curves of $\underline{f(X)}$



$\underline{Z=6}$ $\quad 2x_1 + 3x_2 = 6$

$\underline{Z=3}$ $\quad 2x_1 + 3x_2 = 3$

$\underline{Z=9}$ $\quad 2x_1 + 3x_2 = 9$

$\underline{\text{Max } Z = 16}$
$\underline{\text{at } (2, 4)}$

$x_1^2 + x_2^2 = 20 \quad , \quad x_2 = \frac{8}{x_1}$

$x_1^2 + \frac{64}{x_1^2} = 20 , \quad \ast \ x_1^2 \qquad \underline{x_1 \neq 0}$

$x_1^4 - 20x_1^2 + 64 = 0 \qquad (x_1^2 - 4)(x_1^2 - 16) = 0$

$x_1^2 = 4 \qquad x_1^2 = 16$

$x_1^2 \quad -4$

$x_1^2 \quad -16$

Ex: Min $Z = f(x_1, x_2)$

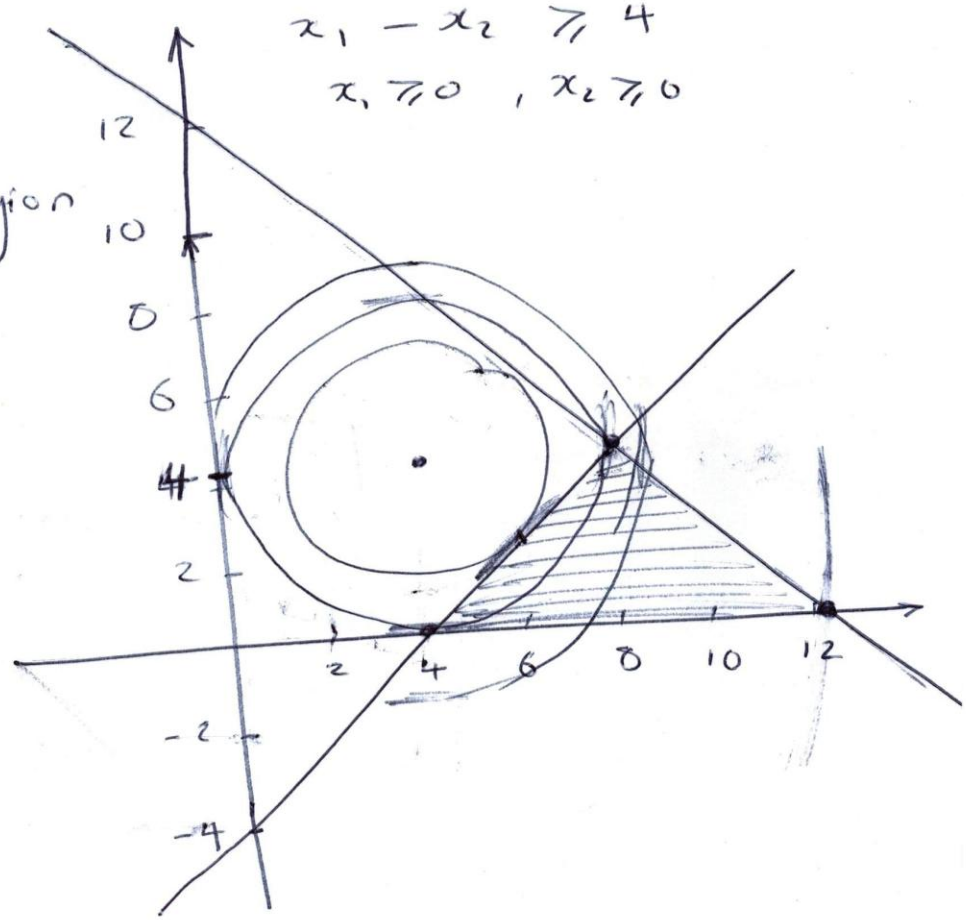
$$= 8x_1 - x_1^2 + 8x_2 - x_2^2$$

subject to

$$x_1 + x_2 \leq 12$$
$$x_1 - x_2 \geq 4$$
$$x_1 \geq 0 \ , \ x_2 \geq 0$$

Draw

(1) Feasible region



(2) level curves of $f(x)$

$$f(x) = 8x_1 - x_1^2 + 8x_2 - x_2^2$$

$$Z - 32 = -(x_1^2 - 8x_1 + 16) - (x_2^2 - 8x_2 + 16)$$

$$\underline{Z} = 32 - (x_1 - 4)^2 - (x_2 - 4)^2$$

$\underline{Z = 7}$ $\quad (x_1 - 4)^2 + (x_2 - 4)^2 = 25$

$\underline{Z = 16}$ $\quad (x_1 - 4)^2 + (x_2 - 4)^2 = 16$

$\underline{Z = 23}$ $\quad (x_1 - 4)^2 + (x_2 - 4)^2 = 9$

Min Z $\Rightarrow$ greatest radius

Min Z = -48

at $(12,0)$

radius = $\sqrt{80}$

Max Z $\Rightarrow$ smallest radius

line $x_1 - x_2 = 4$

tangent to the circle

$$\frac{d}{dx_1}\left((x_1-4)^2 + (x_2-4)^2\right)$$

$$\cancel{2}(x_1-4) + \cancel{2}(x_2-4)\cdot\frac{dx_2}{dx_1} = 0$$

$$\frac{dx_2}{dx_1} = \frac{4-x_1}{x_2-4}$$

slope of $x_1 - x_2 = 4 \Rightarrow x_2 = x_1 - 4$

$m = 1$

$$\frac{4-x_1}{x_2-4} = 1 \Rightarrow 4 - x_1 = x_2 - 4$$

$x_1 + x_2 = 8$ (1)

$x_1 - x_2 = 4$ (2)

$x_1 = 6$, $x_2 = 2$

Max Z = 24

at $(6,2)$

radius = $\sqrt{8}$

Ex3 Min $z = (x_1 - 2)^2 + (x_2 - 2)^2$

sub. to $x_1 + 2x_2 = 4$

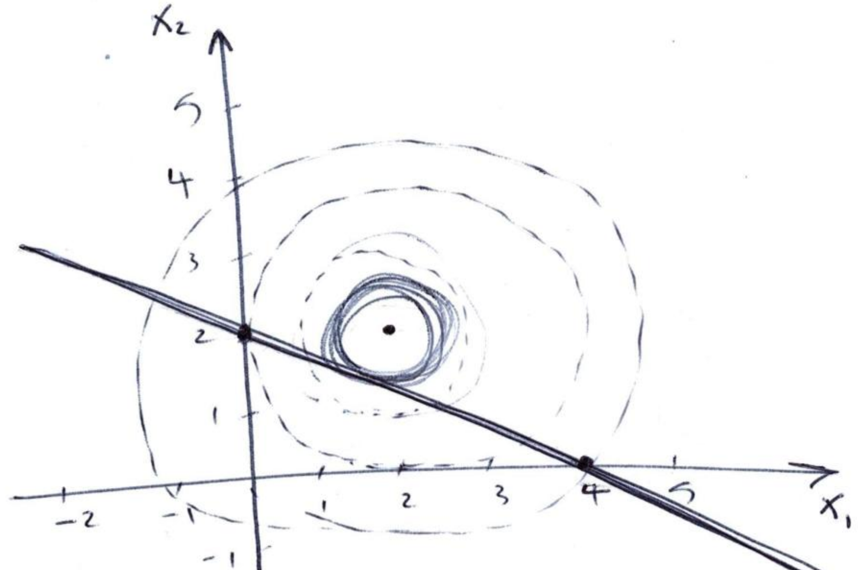
$x_1, x_2 \geq 0$

① feasible region

$x_1 + 2x_2 = 4$

$x_1 = 0 \Rightarrow x_2 = 2$

$x_2 = 0 \Rightarrow x_1 = 4$



② level curves

$z = 1$

$(x_1 - 2)^2 + (x_2 - 2)^2 = 1$

$z = 4 \quad (x_1 - 2)^2 + (x_2 - 2)^2 = 4$

$z = 9 \quad (x_1 - 2)^2 + (x_2 - 2)^2 = 9$

line $x_1 + 2x_2 = 4$ tangent to the circle

Min z

$x_2 = m x_1 + c$

slope

$x_1 + 2x_2 = 4$ ①

$2x_2 = -x_1 + 4$

$\boxed{x_2 = -\frac{1}{2} x_1 + 2}$

$\cancel{2}(x_1 - 2) + \cancel{2}(x_2 - 2)\frac{dx_2}{dx_1} = 0$

$\frac{dx_2}{dx_1} = \frac{2 - x_1}{x_2 - 2} = -\frac{1}{2}$

$$4 - 2x_1 = -x_2 + 2$$

$$-2x_1 + x_2 = -2 \quad (2)$$

Solving (1), (2)

$$2 * \quad 2x_1 \quad 4x_2 \quad 8$$

$$x_1 + 2x_2 = 4 \quad (1)$$

$$-2x_1 + x_2 = -2 \quad (2)$$

$$5x_2 = 6 \quad \boxed{x_2 = \frac{6}{5}} = 1.2$$

$$\boxed{x_1 = \frac{8}{5}} = 1.6$$

at $(1.6, 1.2)$

$\left(\frac{8}{5}, \frac{6}{5}\right)$

$$\text{Min } Z = \frac{4}{5}$$