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| Student Name:**Old Mid Term 01**  | Mid Term 01 |
| Student Roll Number:Max. Time: 60min |  **Date:** **Max. Marks: 20** |

**Instructions:**

* Attempt all questions.
* Write in the spaces provided (you can use the back side of the page as well).
* No extra time will be given.

**Question No. 1 (Analog / Digital Signals / Sampling Theorem)**

**[2+3+3 marks]**

Given an analog signal:

$$x\left(t\right)=5\cos(\left(2π ∙2000t\right)+ 3\cos(\left(2π ∙3000t\right)), for t \geq 0,)$$

sampled at a rate of 8,000 Hz,

1. What is the Nyquist rate for this signal?
2. sketch the spectrum of the sampled signal up to 20 kHz;
3. sketch the recovered analog signal spectrum if an ideal lowpass filter with a cutoff frequency of 4 kHz is used to filter the sampled signal in order to recover the original signal.

**Solution:**

1. For the given signal$ f\_{max}=3000$Hz. According to the Sampling Theorem, the Nyquist rate/limit is$ F\_{N}=2f\_{max}=6000$Hz. To avoid aliasing noise, the given signal should be sampled at a rate greater than$ F\_{N}$.
2. Using Euler’s formula, we get

$$x\left(t\right)=\frac{3}{2}e^{-j2π.3000t}+\frac{5}{2}e^{-j2π.2000t}+\frac{5}{2}e^{j2π2000t}+\frac{3}{2}e^{j2π.3000t}$$

The spectrum of the sampled signal is sketched below:

$$X\_{s}(f)$$

$$2.5/T$$

$f$ **(kHz)**

**-9 -8 -6 -5 -4 -3 -2 0 1 2 3 4 5 6 7 8 9 10 11 13 14 16 18 19**

1. Based on the spectrum in part (b), the sampling theorem condition is satisfied. Hence, we can recover the original spectrum using a reconstruction lowpass filter with cutoff frequency of 4 kHz. The recovered spectrum is sketched below:

$$Y(f)$$

$f$ **(kHz)**

**-6 -5 -4 -3 -2 0 1 2 3 4 5 6 7 8**

**Question No. 2 (Discrete Convolution) [3+3 marks]**

Using the following sequence definitions

$x\left(k\right)=\left\{\begin{matrix}-2,&k=0,1,2\\1,&k=3,4\\0&elsewhere\end{matrix}\right.$ and $h\left(k\right)=\left\{\begin{matrix}2,&k=0,\\-1,&k=1,2\\0&elsewhere\end{matrix}\right.$

evaluate the digital convolution

$$y\left(n\right)=\sum\_{k=-\infty }^{\infty }h\left(k\right)x(n-k)$$

1. using the graphical method;
2. using the table method.

**Solution:**

1. Graphical Method: Sketches of $x\left(k\right)$and$h\left(k\right)$ are given in the following.

$$k$$

$$x\left(k\right)$$

 **2**

 **1**

**-2**

**-4 -3 -2 -1 3 4 5 6 7 8**

$$k$$

$$h\left(k\right)$$

 **2**

 **1**

**-2**

**-4 -3 -2 -1 3 4 5 6 7 8**

$$k$$

$$x\left(-k\right)$$

 **2**

 **1**

**-4 -3 1 2 3 4 5 6 7 8**

To find$ y(0)$, we need the reversed

sequence$ x(-k)$.

$y\left(0\right)=$ **sum of product of** $h\left(k\right)$ **and** $x(-k)$

$⇒$$y\left(0\right)=2×-2=-4$

$$k$$

$$x\left(1-k\right)$$

 **2**

 **1**

**-4 -3 -2 2 3 4 5 6 7 8**

Similarly

$y\left(1\right)=$ **sum of product of** $h\left(k\right)$ **and** $x(1-k)$

$$⇒y\left(1\right)=2×\left(-2\right)+\left(-1\right)×\left(-2\right)=-2$$

$$k$$

$$x\left(2-k\right)$$

 **2**

 **1**

**-4 -3 -2 -1 3 4 5 6 7 8**

$y\left(2\right)=$ **sum of product of** $h\left(k\right)$ **and** $x(2-k)$

$$⇒y\left(2\right)=2×\left(-2\right)+\left(-1\right)×\left(-2\right)$$

$$ +\left(-1\right)×(-2)=0$$

$$k$$

$$x\left(3-k\right)$$

 **2**

 **1**

**-4 -3 -2 -1 4 5 6 7 8**

$y\left(3\right)=$ **sum of product of** $h\left(k\right)$ **and** $x(3-k)$

$$⇒y\left(3\right)=2×\left(1\right)+\left(-1\right)×\left(-2\right)$$

$$ +\left(-1\right)×(-2)=6$$

$$k$$

$$x\left(4-k\right)$$

 **2**

 **1**

**-4 -3 -2 -1 1 5 6 7 8**

$y\left(4\right)=$ **sum of product of** $h\left(k\right)$ **and** $x(4-k)$

$$⇒y\left(4\right)=2×\left(1\right)+\left(-1\right)×\left(1\right)$$

$$ +\left(-1\right)×(-2)=3$$

$$k$$

$$x\left(5-k\right)$$

 **2**

 **1**

**-4 -3 -2 -1 1 2 6 7 8**

$y\left(5\right)=$ **sum of product of** $h\left(k\right)$ **and** $x(5-k)$

$$⇒y\left(5\right)=\left(-1\right)×\left(1\right)+\left(-1\right)×\left(1\right)=-2$$

$$k$$

$$x\left(6-k\right)$$

 **2**

 **1**

**-4 -3 -2 -1 1 2 3 7 8**

$y\left(6\right)=$ **sum of product of** $h\left(k\right)$ **and** $x(6-k)$

$$⇒y\left(5\right)=\left(-1\right)×\left(1\right)=-1$$

And for $n\geq 7$,$y\left(n\right)=0$.

1. Tabular Method

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
| $$k$$ | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |  |
| $$x\left(k\right)$$ |  |  |  |  | -2 | -2 | -2 | 1 | 1 |  |  |  |  |
| $$h\left(k\right)$$ |  |  |  |  | 2 | -1 | -1 |  |  |  |  |  |  |
| $$x\left(-k\right)$$ | 1 | 1 | -2 | -2 | -2 |  |  |  |  |  |  |  | $$y\left(0\right)=2×-2=-4$$ |
| $$x\left(1-k\right)$$ |  | 1 | 1 | -2 | -2 | -2 |  |  |  |  |  |  | $$y\left(1\right)=2×-2+\left(-1\right)×(-2)=-2$$ |
| $$x\left(2-k\right)$$ |  |  | 1 | 1 | -2 | -2 | -2 |  |  |  |  |  | $$y\left(2\right)=2×-2+\left(-1\right)×(-2)+\left(-1\right)×(-2)=0$$ |
| $$x\left(3-k\right)$$ |  |  |  | 1 | 1 | -2 | -2 | -2 |  |  |  |  | $$y\left(3\right)=2×1+\left(-1\right)×(-2)+\left(-1\right)×(-2)=6$$ |
| $$x\left(4-k\right)$$ |  |  |  |  | 1 | 1 | -2 | -2 | -2 |  |  |  | $$y\left(4\right)=2×1+\left(-1\right)×(1)+\left(-1\right)×(-2)=3$$ |
| $$x\left(5-k\right)$$ |  |  |  |  |  | 1 | 1 | -2 | -2 | -2 |  |  | $$y\left(5\right)=\left(-1\right)×(1)+\left(-1\right)×(1)=-2$$ |
| $$x\left(6-k\right)$$ |  |  |  |  |  |  | 1 | 1 | -2 | -2 | -2 |  | $$y\left(6\right)=\left(-1\right)×(1)=-1$$ |
| $$x\left(7-k\right)$$ |  |  |  |  |  |  |  | 1 | 1 | -2 | -2 | -2 | $$y\left(7\right)=0$$ |

**Question No. 3 (Discrete Fourier Transform, DFT) [2+2+2 marks]**

Given a sequence $x\left(n\right)$ for$ 0\leq n\leq 3$**,** where

$$x\left(0\right)=1, x\left(1\right)=2, x\left(2\right)=3, x\left(3\right)=4$$

1. Evaluate its DFT, i.e.$ X(k)$;
2. Evaluate and sketch the resulting two-sided amplitude spectrum, i.e.$ A\_{k}$;
3. Evaluate and sketch the resulting one-sided amplitude spectrum, i.e.$ \overbar{A}\_{k}$.

**Solution:**

1. According to the definition of DFT

$$X\left(k\right)=\sum\_{n=0}^{N-1}x\left(n\right) W\_{N}^{kn}, for k=0,1,2,…,N-1$$

where

$$W\_{N}=e^{-j2π/N}$$

In this case, $N=4$, therefore the DFT can be written as

$$X\left(k\right)=x\left(0\right)+x(1)W\_{4}^{k}+x(2)W\_{4}^{2k}+x(3)W\_{4}^{3k}$$

Therefore,

$$X\left(0\right)=x\left(0\right)+x\left(1\right)+x\left(2\right)+x\left(3\right)=1+2+3+4=10$$

$$X\left(1\right)=x\left(0\right)+x\left(1\right)W\_{4}^{1}+x\left(2\right)W\_{4}^{2}+x\left(3\right)W\_{4}^{3}=x\left(0\right)+x\left(1\right)e^{-j2π/4}+x\left(2\right)e^{-j4π/4}+x\left(3\right)e^{-j6π/4}=1+\left(2\right)\left(-j\right)+3\left(-1\right)+4\left(j\right)=-2+j2$$

$$X\left(2\right)=x\left(0\right)+x\left(1\right)W\_{4}^{2}+x\left(2\right)W\_{4}^{4}+x\left(3\right)W\_{4}^{6}=x\left(0\right)+x\left(1\right)e^{-j4π/4}+x\left(2\right)e^{-j8π/4}+x\left(3\right)e^{-j12π/4}=1+\left(2\right)\left(-1\right)+3\left(+1\right)+4\left(-1\right)=-2$$

$$X\left(3\right)=x\left(0\right)+x\left(1\right)W\_{4}^{3}+x\left(2\right)W\_{4}^{6}+x\left(3\right)W\_{4}^{9}=x\left(0\right)+x\left(1\right)e^{-j6π/4}+x\left(2\right)e^{-j12π/4}+x\left(3\right)e^{-j18π/4}=1+\left(2\right)\left(j\right)+3\left(-1\right)+4\left(-j\right)=-2-j2$$

1. Two-Sided Amplitude Spectrum

The two-sided amplitude spectrum is given by

$$A\_{k}=\frac{1}{N}\left|X(k)\right|=\frac{1}{N}\sqrt{\left(Real\left[X(k)\right]\right)^{2}+\left(Imag\left[X(k)\right]\right)^{2}}$$

Therefore

$A\_{0}=\frac{1}{4}\sqrt{\left(10\right)^{2}+\left(0\right)^{2}}=\frac{10}{4}=2.5$

$$k$$

$$A\_{k}$$

 **3**

 **2**

 **1**

 **0 1 2 3**

$$A\_{1}=\frac{1}{4}\sqrt{\left(-2\right)^{2}+\left(2\right)^{2}}=\frac{\sqrt{8}}{4}=0.7071$$

$$A\_{2}=\frac{1}{4}\sqrt{\left(-2\right)^{2}+\left(0\right)^{2}}=\frac{2}{4}=0.5$$

$$A\_{3}=\frac{1}{4}\sqrt{\left(-2\right)^{2}+\left(-2\right)^{2}}=\frac{\sqrt{8}}{4}=0.7071$$

1. One-Sided Amplitude Spectrum

The one-sided amplitude spectrum is given by

$$\overbar{A}\_{k}=\left\{\begin{matrix}\frac{1}{N}\left|X(k)\right|&k=0\\\frac{2}{N}\left|X(k)\right|& k=1,2,..,N/2\end{matrix}\right.$$

Now$ \frac{N}{2}=\frac{4}{2}=2$**.** Therefore,

$$k$$

$$\overbar{A}\_{k}$$

 **3**

 **2**

 **1**

 **0 1 2 3**

$$\overbar{A}\_{0}=2.5$$

$$\overbar{A}\_{1}=2×0.7071=1.4141$$

$$\overbar{A}\_{2}=2×0.5=1.0$$