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| Student Name:**Old Mid Term 02**  | Mid Term 02 |
| Student Roll Number:Max. Time: 60min |  **Date:** **Max. Marks: 20** |

**Instructions:**

* Attempt all questions.
* Write in the spaces provided (you can use the back side of the page as well).
* No extra time will be given.

**Question No. 1 (Fast Fourier Transform)**

**[4+1 marks]**

Given a sequence$ x\left(n\right)$ for$ 0\leq n\leq 3$, where $x\left(0\right)=1, x\left(1\right)=2, x\left(2\right)=3$, and$ x\left(3\right)=4$,

1. Determine its DFT$ X\left(k\right)$ using decimation-in-frequency FFT method ?
2. Determine the number of complex multiplication in doing part (a).

**Solution:**

1. According to the FFT (decimation-in-frequency method) for calculating DFT of a give sequence

$$DFT\left\{x\left(n\right) with N points\right\}=\left\{\begin{matrix}DFT\left\{a\left(n\right) with \frac{N}{2} points\right\}\\DFT\left\{b\left(n\right) W\_{N}^{n} with \frac{N}{2} points\right\}\end{matrix}\right.$$

where, $W\_{N}^{n}=e^{-j2πn/N}=cos \left(\frac{2πn}{N}\right)+j sin \left(\frac{2πn}{N}\right)$ and

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| $$a\left(n\right)=x\left(n\right)+x\left(n+\frac{N}{2}\right), for n=0, 1, 2,…,\frac{N}{2}-1$$ |
| $$b\left(n\right)=x\left(n\right)-x\left(n+\frac{N}{2}\right), for n=0, 1, 2,…,\frac{N}{2}-1$$ |

The DFT of the given sequence, using FFT (decimation-in-frequency method) is given in the following diagram:

Therefore, the DFT coefficients are $X\left(0\right)=10, X\left(1\right)=-2+2j, X\left(2\right)=-2$, and$ X\left(3\right)=-2-2j$.

1. Number of complex multiplications in $DFT=\frac{N}{2} log\_{2} \left(N\right)=\frac{4}{2} log\_{2} \left(4\right)=4$.

**Question No. 2 (z-Transform using Table of z-Transform Pairs) [5 marks]**

Using Table 1 (given at the end of the paper) find out the z-transform for the following sequences:

1. $x\left(n\right)=15 u(n)$
2. $x\left(n\right)=10 sin\left(0.25πn\right) u(n)$
3. $x\left(n\right)=\left(0.5\right)^{n} u(n)$
4. $x\left(n\right)=\left(0.5\right)^{n} sin\left(0.25πn\right)u(n)$
5. $x\left(n\right)=e^{-0.1n} cos\left(0.25πn\right) u(n)$

**Solution:**

1. From Line 3 in Table 1, we get

$$X\left(z\right)=Z\left(x(n)\right)=Z\left(15 u(n)\right)=\frac{15}{z-1}$$

1. From Line 9 in Table 1, we obtain

$$X\left(z\right)=Z\left(x(n)\right)=Z\left(10 sin\left(0.25πn\right) u(n)\right)=\frac{10 sin\left(0.25π\right) z }{z^{2}-2zcos\left(0.25πn\right)+1}=\frac{7.07 z }{z^{2}-1.414z+1}$$

1. From Line 6 in Table 1, we get

$$X\left(z\right)=Z\left(x(n)\right)=Z\left(\left(0.5\right)^{n} u(n)\right)=\frac{z }{z-0.5}$$

1. From Line 11 in Table 1, we get

$$X\left(z\right)=Z\left(x(n)\right)=Z\left(\left(0.5\right)^{n} sin\left(0.25πn\right)u(n)\right)=\frac{0.5×sin\left(0.25π\right) z }{z^{2}-2×0.5×z cos\left(0.25πn\right)+\left(0.5\right)^{2}}=\frac{0.3536 z }{z^{2}-1.4142z+0.25}$$

1. From Line 14 in Table 1, we get

$$X\left(z\right)=Z\left(x(n)\right)=Z\left(e^{-0.1n} cos\left(0.25πn\right) u(n)\right)=\frac{z\left(z-e^{-0.1}cos\left(0.25π\right)\right) }{z^{2}-2 z e^{-0.1}cos\left(0.25π\right)+\left(e^{-0.1}\right)^{2}}=\frac{z\left(z-0.6397\right) }{z^{2}-1.279 z+0.8187}$$

**Question No. 3 (inverse z-Transform using partial fractions) [5 marks]**

Find the inverse of the following z-transform

$$X\left(z\right)=\frac{1}{\left(1-z^{-1}\right)\left(1-0.5z^{-1}\right)}$$

**Solution:**

Multiplying the numerator and the denominator by $z^{2}$ we get

$$X\left(z\right)=\frac{1}{\left(1-z^{-1}\right)\left(1-0.5z^{-1}\right)}×\frac{z^{2}}{z^{2}}=\frac{z^{2}}{\left(z-1\right)\left(z-0.5\right)}$$

Dividing both sides by $z$, we have

$$\frac{X\left(z\right)}{z}=\frac{z}{\left(z-1\right)\left(z-0.5\right)}$$

We notice that the right hand side of the above equation is a proper rational polynomial of $z$. Also we notice that the denominator of the right hand side has distinct poles, therefore, we right into partial fractions as,

$$\frac{X\left(z\right)}{z}=\frac{A}{\left(z-1\right)}+\frac{B}{\left(z-0.5\right)}$$

To find out the unknown constants$ A$ and$ B$, we use:

$$A=\left[\left(z-1\right)×\frac{X\left(z\right)}{z}\right]\_{z=1}=\left[\left(z-1\right)×\frac{z}{\left(z-1\right)\left(z-0.5\right)}\right]\_{z=1}=\left[\frac{z}{\left(z-0.5\right)}\right]\_{z=1}=\frac{1}{\left(1-0.5\right)}=2$$

$$B=\left[\left(z-0.5\right)×\frac{X\left(z\right)}{z}\right]\_{z=0.5}=\left[\left(z-0.5\right)×\frac{z}{\left(z-1\right)\left(z-0.5\right)}\right]\_{z=0.5}=\left[\frac{z}{\left(z-1\right)}\right]\_{z=0.5}=\frac{0.5}{\left(0.5-1\right)}=-1$$

Substituting the values, we have,

$$\frac{X\left(z\right)}{z}=\frac{2}{\left(z-1\right)}+\frac{\left(-1\right)}{\left(z-0.5\right)}$$

Or it can be written as (by multiplying both sides by $z$)

$$X\left(z\right)=\frac{2z}{\left(z-1\right)}-\frac{z}{\left(z-0.5\right)}$$

Taking the inverse z-transform of both sides and using Table 1, we have

$$x\left(n\right)=Z^{-1}\left(X\left(z\right)\right)=2Z^{-1}\left(\frac{z}{\left(z-1\right)}\right)-Z^{-1}\left(\frac{z}{\left(z-0.5\right)}\right)=2 u\left(n\right)-\left(0.5\right)^{n} u(n)$$

**Question No. 4 (Difference Equations Solution using z-Transform) [5 marks]**

A relaxed (zero initial conditions) DSP system is described by the difference equation

$$y\left(n\right)+0.1y\left(n-1\right)-0.2y\left(n-2\right)=x\left(n\right)+x(n-1)$$

Determine the impulse response$ y\left(n\right)$ due to the impulse sequence$ x\left(n\right)=δ(n)$.

**Solution:**

Taking z-transform of both sides of the given equation, we get

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| $$Z\left(y\left(n\right)\right)+0.1Z\left(y\left(n-1\right)\right)-0.2Z\left(y\left(n-2\right)\right)=Z\left(x\left(n\right)\right)+Z\left(x(n-1)\right)$$ | (1) |

We have

$$Z\left(y\left(n\right)\right)=Y(z)$$

$$Z\left(x\left(n\right)\right)=X(z)$$

Using shift theorem, we have

$$Z\left(x\left(n-1\right)\right)=z^{-1}X(z)$$

Also we can apply sift theorem for$ y$ in case of zero initial conditions, i.e.,

$$Z\left(y\left(n-1\right)\right)=z^{-1}Y(z)$$

$$Z\left(y\left(n-2\right)\right)=z^{-2}Y(z)$$

Putting these values in Equation (1), we have

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| $$Y(z)+0.1z^{-1}Y(z)-0.2z^{-2}Y(z)=X(z)+z^{-1}X(z)$$ |  |
| $$⟹ Y\left(z\right)\left(1+0.1z^{-1}-0.2z^{-2}\right)=X\left(z\right)\left(1+z^{-1}\right)$$ |  |

As$ x\left(n\right)=δ(n)$ therefore, (from Table 1), $X\left(z\right)=1.$ The above equation can now be written as

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| $$Y\left(z\right)=\frac{\left(1+z^{-1}\right)}{\left(1+0.1z^{-1}-0.2z^{-2}\right)}$$ |  |

Multiplying both the numerator and the denominator with$z^{2}$ we get

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| $$Y\left(z\right)=\frac{z\left(z+1\right)}{\left(z^{2}+0.1z-0.2\right)}$$ |  |

The denominator can be factorized as

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| $$Y\left(z\right)=\frac{z\left(z+1\right)}{\left(z^{2}+0.5z-0.4z-0.2\right)}=\frac{z\left(z+1\right)}{\left(z(z+0.5)-0.4(z+0.5)\right)}=\frac{z\left(z+1\right)}{(z+0.5)(z-0.4)}$$ |  |
| $$⟹ \frac{Y\left(z\right)}{z}=\frac{\left(z+1\right)}{(z+0.5)(z-0.4)}$$ | (2) |

The right hand side of the above equation is a proper rational polynomial, with the denominator polynomial having distinct poles, therefore, it can be written into partial fractions as

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| $$\frac{Y\left(z\right)}{z}=\frac{A}{(z+0.5)}+\frac{B}{(z-0.4)}$$ | (3) |

To find out the unknown constants$ A$ and$ B$, we use:

$$A=\left[\left(z+0.5\right)×\frac{X\left(z\right)}{z}\right]\_{z=-0.5}=\left[\left(z+0.5\right)×\frac{\left(z+1\right)}{\left(z+0.5\right)\left(z-0.4\right)}\right]\_{z=-0.5}=\left[\frac{\left(z+1\right) }{\left(z-0.4\right)}\right]\_{z=-0.5}=\frac{\left(-0.5+1\right)}{\left(-0.5-0.4\right)}=\frac{0.5}{-0.9}=-0.5556$$

$$B=\left[\left(z-0.4\right)×\frac{X\left(z\right)}{z}\right]\_{z=0.4}=\left[\left(z-0.4\right)×\frac{\left(z+1\right)}{\left(z+0.5\right)\left(z-0.4\right)}\right]\_{z=0.4}=\left[\frac{\left(z+1\right) }{\left(z+0.5\right)}\right]\_{z=0.4}=\frac{\left(0.4+1\right)}{\left(0.4+0.5\right)}=\frac{1.4}{0.9}=1.5556$$

Equation (3) becomes:

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| $$\frac{Y\left(z\right)}{z}=\frac{-0.5556}{(z+0.5)}+\frac{1.5556}{(z-0.4)}$$ |  |
| $$Y\left(z\right)=\frac{-0.5556 z}{(z+0.5)}+\frac{1.5556 z}{(z-0.4)}$$ |  |

Taking inverse z-transform of both sides

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| $$y\left(n\right)=Z^{-1}\left(Y\left(z\right)\right)=Z^{-1}\left(\frac{-0.5556 z}{(z+0.5)}\right)+Z^{-1}\left(\frac{1.5556 z}{(z-0.4)}\right)=\left(-0.5556\right)Z^{-1}\left(\frac{ z}{(z-(-0.5))}\right)+\left(1.5556\right)Z^{-1}\left(\frac{ z}{(z-0.4)}\right)=\left(-0.5556\right)\left(-0.5\right)^{n} u\left(n\right)+\left(1.5556\right)\left(0.4\right)^{n} u(n)$$ |  |

Thus the output signal is

$$y(n)=\left(-0.5556\right)\left(-0.5\right)^{n} u\left(n\right)+\left(1.5556\right)\left(0.4\right)^{n} u(n)$$

**Table 5.1 Table of z-transform pairs (for causal sequences)**

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| Line No. | Signal$$x\left(n\right), n\geq 0$$ | z-Transform$$Z\left(x\left(n\right)\right)=X(z)$$ | Region of Convergence |
| 1 | $$x(n)$$ | $$\sum\_{n=0}^{\infty }x\left(n\right) z^{-n}$$ |  |
| 2 | $$δ(n)$$ | **1** | **Entire z-plane** |
| 3 | $$a u(n)$$ | $$\frac{az}{z-1}$$ | $$\left|z\right|>1$$ |
| 4 | $$n u(n)$$ | $$\frac{z}{\left(z-1\right)^{2}}$$ | $$\left|z\right|>1$$ |
| 5 | $$n^{2} u(n)$$ | $$\frac{z(z+1)}{\left(z-1\right)^{3}}$$ | $$\left|z\right|>1$$ |
| 6 | $$a^{n} u(n)$$ | $$\frac{z}{z-a}$$ | $$\left|z\right|>\left|a\right|$$ |
| 7 | $$e^{-na} u(n)$$ | $$\frac{z}{z-e^{-a}}$$ | $$\left|z\right|>e^{-a}$$ |
| 8 | $$n a^{n} u(n)$$ | $$\frac{az}{\left(z-a\right)^{2}}$$ | $$\left|z\right|>\left|a\right|$$ |
| 9 | $$sin\left(an\right) u(n)$$ | $$\frac{z sin\left(a\right)}{z^{2}-2z cos\left(a\right)+1}$$ | $$\left|z\right|>\left|1\right|$$ |
| 10 | $$cos\left(an\right) u(n)$$ | $$\frac{z \left(z-cos\left(a\right)\right)}{z^{2}-2z cos\left(a\right)+1}$$ | $$\left|z\right|>\left|1\right|$$ |
| 11 | $$a^{n} sin\left(bn\right) u(n)$$ | $$\frac{ \left[a sin\left(b\right)\right] z}{z^{2}-\left[2a cos\left(b\right)\right]z +a^{2}}$$ | $$\left|z\right|>\left|a\right|$$ |
| 12 | $$a^{n} cos\left(bn\right) u(n)$$ | $$\frac{z \left[z-a cos\left(b\right)\right] }{z^{2}-\left[2a cos\left(b\right)\right]z +a^{2}}$$ | $$\left|z\right|>\left|a\right|$$ |
| 13 | $$e^{-an} sin\left(bn\right) u(n)$$ | $$\frac{ \left[e^{-a} sin\left(b\right)\right] z}{z^{2}-\left[2e^{-a} cos\left(b\right)\right]z +e^{-2a}}$$ | $$\left|z\right|>e^{-a}$$ |
| 14 | $$e^{-an} cos\left(bn\right) u(n)$$ | $$\frac{z \left[z-e^{-a} cos\left(b\right)\right] }{z^{2}-\left[2e^{-a} cos\left(b\right)\right]z +e^{-2a}}$$ | $$\left|z\right|>e^{-a}$$ |
| 15 | $2\left|A\right|\left|P\right|^{n} cos\left(nθ+ϕ\right) u\left(n\right)$ **where**$ P$ **and**$ A$ **are complex constants defined by**$P=\left|P\right|∠θ$**,** $A=\left|A\right|∠ϕ$ | $$\frac{Az}{z-P}+\frac{A^{\*}z}{z-P^{\*}}$$ |  |

**Shift Theorem:** $ Z\left(x\left(n-m\right)\right)=z^{-m} Z\left(x\left(n\right)\right)$