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Numerical Solutions of Stochastic Differential Equations with Jumps and Measurable Drifts

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Abstract: This paper deals with numerical analysis of solutions to stochastic differential equations with jumps (SDEJs) with measurable drifts that may have quadratic growth. The main tool used is the Zvonkin space transformation to eliminate the singular part of the drift. More precisely, the idea is to transform the original SDEJs to standard SDEJs without singularity by using a deterministic real-valued function that satisfies a second-order differential equation. The Euler–Maruyama scheme is used to approximate the solution to the equations. It is shown that the rate of convergence is $\frac{1}{2}$. Numerically, two different methods are used to approximate solutions for this class of SDEJs. The first method is the direct approximation of the original equation using the Euler–Maruyama scheme with specific tests for the evaluation of the singular part at simulated values of the solution. The second method consists of taking the inverse of the Euler–Maruyama approximation for Zvonkin's transformed SDEJ, which is free of singular terms. Comparative analysis of the two numerical methods is carried out. Theoretical results are illustrated and proved by means of an example.

Keywords: stochastic differential equations with jumps; Zvonkin's transformation; numerical approximations; Euler–Maruyama scheme

MSC: 60H10; 60H35; 60J75; 68U20

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1. Introduction

Our aim in the present paper is to explore the numerical schemes of a class of SDEJs that have measurable and integrable drifts. Specifically, the main interest lies in numerically approximating solutions to $\mathbb{R}$-valued SDEJs of the form

$$\begin{aligned} X_t \quad &= \quad X_0 + \int_0^t a(X_s)ds + \int_0^t \kappa(X_s)b^2(X_s)ds - \int_0^t [c]_\Phi(X_s)ds \\ &\quad + \int_0^t b(X_s)dW_s + \int_0^t c(X_{s-})d\widetilde{N}_s, \end{aligned} \tag{1}$$

where X_0 is the initial value, and κ is a bounded, measurable, and integrable function over the entire space $\mathbb{R}$. The function Φ is defined for every $x \in \mathbb{R}$ by

$$\Phi(x) = \int_0^x \exp(2\int_0^y \kappa(t)dt)dy, \tag{2}$$

where, for any ψ,

$$[\psi]_\Phi(x) := \left(\frac{\Phi(x+\psi(x)) - \Phi(x)}{\Phi'(x)} - \psi(x)\right)\lambda.$$

Throughout the paper we will denote Equation (1) as $\text{Eq}(X_0, a, \kappa, b, c)$. The main goal is to apply the transformation function Φ, which has the property of being bilipschitz

(Φ and Φ^{-1} are functions that are Lipschitz continuous), which helps to remove the singular part in the SDEJ (1) and obtain an SDEJ that has "singularity free" coefficients. In [1], the author used phase transformation for SDEs without jumps. We extend this to the jump case and also work beyond the continuity by taking measurable and integrable drifts into consideration.

The EM approximation of the original solution $(X_t)_{t\in[0,T]}$ is defined by two methods. The first one is the direct EM approximation of $(X_t)_{t\in[0,T]}$, and the second one is obtained using the inverse function Φ^{-1} by setting $Y_t = \Phi^{-1}(\overline{Y}_t)$. Further, a comparative analysis is done between the approximate and the exact solution and illustrated through different figures. This paper presents a transformation technique based on the Zvonkin transformation for singular SDEs with jumps to find numerical solutions for SDEJs whose explicit solutions are not otherwise available due to the presence of singularity in the drift coefficient.

SDEJs have been widely used in various fields when modeling different physical and natural problems. Their applications range across physics, ecology, biology, chemistry, astronomy, and so on. Particularly, when modeling events in the finance or insurance industries, unpredictability is the most important factor to be considered, which leads to SDEJs.

In physics, [2] described atomic diffusion, which is usually composed of jumps between vacant lattice sites using SDEJs. On the other hand, [3] has demonstrated quantum jumps in a single atom by using SDEJs. The quantum systems that exhibit random sudden jumps between their states were observed, and results were examined by [4]. Another major contribution in the field of physics was proving existence and uniqueness of stochastic Schrodinger equations of jump type in [5].

Major contributions of SDEJs in the field of chemistry are discussed in [6–8].

Researchers have studied stochastic models obtained by single molecule reactions or coupled reactions. One such example is that of the models of in vivo reactions, whose discrete time approximation for the solution is discussed by Turner in [8].

Pure jump processes have also been used to describe population dynamics, migration models, birth and death phenomena, among the other biological sciences. Stochastic models are used to explain the biochemical processes of living cells. Several studies, such as [9–11], have applied stochastic models to comprehend the relevant phenomena and better describe the true dynamics.

In finance, risky assets, whose dynamics are driven by stochastic processes of the Itô–Lévy type with a singularity in the drift term, were studied in [12]. It is well known that the classical Black–Scholes market model is arbitrage-free. However, it is proved in [13] that, in the case of inclusion of a singular term in the drift of the risky assets, arbitrages exist. Several authors have discussed how the inclusion of a singular term in the drift has the ability to model a case where the asset price is partly controlled by a large company that intercedes when the stock price reaches a fixed lower value, in order to prevent it from going below that certain value. The study was extended in [12] for jump case scenarios, in particular for the financial market model.

In the past few years, financial markets have seen a lot of instabilities caused by external factors. Whether in the 2008 global economic recession, the COVID-19 pandemic that hit the world in 2020, or the present Ukraine–Russia war, stochastic jump models have become more relevant to observing real–life models of financial markets as well as risks linked to several insurance products.

Consider the following SDEJ:

$$dX_t = a(X_t)dt + b(X_t)dW_t + c(X_{t-})d\widetilde{N}_t. \tag{3}$$

The established fact states that if a, b, and c satisfy linear growth and Lipschitz conditions, then (1) admits a strong solution that is pathwise unique (see [14]). The usual assumption of linear growth of the coefficients assures the non–explosion of the solution in finite time with probability 1. However, such an assumption is very limited.

In many models, assuming that the jump coefficient c satisfies the Lipschitz condition is too constricted. For example, for the jump–diffusion process with state–dependent intensities, it is suitable to use a jump coefficient that does not satisfy the Lipschitz continuity condition. In this direction, many results on the existence and uniqueness of the solution to SDEJ for non–Lipschitz continuous jump coefficients have been established. In particular, a solution has been proposed in [14] for the types of SDEs that fail to comply with the Lipschitz condition. Strong solutions to SDEs may not be pathwise unique if the Lipschitz condition is not satisfied. However, in [15], it is specified that, at certain boundaries, one can obtain a unique solution to SDE. Theorem 4.9.1 in [14] demonstrated that the L_2 distance between two solution disappears and the solution becomes pathwise unique if they have the same initial condition. In [16], a one–dimensional case was discussed for the existence of a pathwise unique solution with non–Lipschitz conditions under the specific assumption that the kernel for the compensated Poisson integral term is non-decreasing. Moreover, in [17], conditions were provided for the existence and pathwise uniqueness of solution. Subsequently, the results were extended to a multi-dimensional case in [18].

For the class of SDEJs where the coefficients are Lipschitz, it is well known that SDEJ admits a uniquely strong solution as presented in [19] or [20]. However, as discussed above, there are many applications in all fields of study where jump SDEs with discontinuous drift appear; hence, studying SDEJs for their existence and uniqueness with discontinuous drifts becomes important. For the scalar case with Poisson jumps, [21] proved the existence and uniqueness of strong solutions for discontinuous drifts and possibly degenerate diffusion coefficients. The authors also studied the strong convergence order of the Euler–Maruyama scheme for such cases and proved it to be of order $\frac{1}{2}$. Numerical solutions of SDEs with irregular coefficients in the Brownian setting was studied in [22]. These types of SDEJs appear in optimal control problems in energy markets. This result is further extended in their paper [23], which established the existence and uniqueness of solutions with discontinuous drifts and finite activity jumps.

2. Notations and Preliminary Results

On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the fixed time horizon T, we consider the following stochastic processes:

- $W = \{W_t\}_{t\in[0,T]}$, a one-dimensional standard Brownian motion.
- $N = \{N_t\}_{t\in[0,T]}$, a time-homogeneous Poisson process with compensator λds.

We denote by $d\widetilde{N}_s = dN_s - \lambda ds$ the compensated Poisson jump measure.

We first define functional spaces we are dealing with in this paper. For any $p > 0$, we denote:

a. $\mathcal{S}^p$: the collection of càdlàg and $\mathcal{F}_t$-adapted processes $X.$ satisfying the following integrability condition $\mathbb{E}[\sup_{0\le t\le T}|X_t|^p]$ is finite.
b. $\mathcal{M}^p_W$: the space of $\mathcal{F}_t$-adapted processes $u.$ such that $\mathbb{E}[\int_0^T |u_s|^p ds]$ is finite.
c. $\mathcal{M}^p_N$: the space of $\mathcal{F}_t$-predictable processes $\varphi.$ satisfying $\mathbb{E}[\lambda \int_0^T |\varphi_s|^p ds] < \infty$.

Definition 1. *A stochastic process* $X. = (X_t)_{t\in[0,T]}$ *is said to be a solution to the (SDEJs)* $\text{Eq}(X_0, a, \kappa, b, c)$ *starting from* X_0 *if* $(X_t)_{t\in[0,T]}$ *satisfies the integral Equation (1),* $\mathbb{P}$*-a.s. for all* $t \in [0,T]$*, and the properties below are satisfied:*

- $X.$ *is progressively measurable with respect to the filtration* $\{\mathcal{F}_t\}_{t\in[0,T]}$ *such that* $\mathbb{E}[\sup_{0\le t\le T}|X_t|^2]$ *is finite.*
- $a(X.)$ *belongs to* $\mathcal{M}^1_W$.
- $b(X.)$ *belongs to* $\mathcal{M}^2_W$.
- $c(X.)$ *belongs to* $\mathcal{M}^2_N$.
- $(\kappa(X_s)\frac{d[X_s]^c}{ds})_{0\le s\le T}$ *belongs to* $\mathcal{M}^1_W$.

The following lemma proved in [24] listed important properties satisfied by the space transformation function Φ defined by (2). Moreover, the lemma will be useful for proving Proposition 1.

Lemma 1. *For a given measurable, bounded, and integrable function κ over the whole space $\mathbb{R}$, the positive function Φ defined by (2) satisfies the equation below:*

$$\Phi''(x) - 2\kappa(x)\Phi'(x) = 0, \textit{for a.e. } x, \tag{4}$$

as well as the following properties:

- *Φ and Φ^{-1} satisfy the following properties: $\forall\, x,\ y \in \mathbb{R}$ and $|\kappa|_1 = \int_{\mathbb{R}} |\kappa(x)|dx$,*

$$\begin{array}{rcccl} e^{-2|\kappa|_1}|x-y| & \leq & |\Phi(x) - \Phi(y)| & \leq & e^{2|\kappa|_1}|x-y|, \\ e^{-2|\kappa|_1}|x-y| & \leq & |\Phi^{-1}(x) - \Phi^{-1}(y)| & \leq & e^{2|\kappa|_1}|x-y|. \end{array} \tag{5}$$

- *Φ is a one-to-one function. Both Φ and its inverse Φ^{-1} belong to $\mathcal{W}_1^2(\mathbb{R})$ (space of continuous functions g defined on $\mathbb{R}$ such that both g and its generalized left derivatives $g_l'(x)$ and $g_l''(x)$ are locally integrable on $\mathbb{R}$).*

Assumptions

Assume that we are given three measurable functions, $a : \mathbb{R} \longrightarrow \mathbb{R}$, $b : \mathbb{R} \longrightarrow \mathbb{R}$, and $c : \mathbb{R} \longrightarrow \mathbb{R}$, satisfying the following assumptions:

(A.1) X_0 is $\mathcal{F}_0$-measurable, and $\mathbb{E}|X_0|^2$ is finite;
(A.2) There exists a $K > 0$ such that, for all $x,\ y \in \mathbb{R}$

$$|a(x) - a(y)| + |b(x) - b(y)| + |c(x) - c(y)| \leq K|x-y|, \tag{6}$$

(A.3) The functions a, b, and c are bounded functions and belong to the class $\mathcal{C}^1$.

It follows from the previous assumptions that a, b, and c satisfy the global linear growth conditions: $\exists\, K > 0$ such that, for all $x \in \mathbb{R}$,

$$|a(x)| + |b(x)| + |c(x)| \leq K(1 + |x|).$$

Remark 1. *Throughout this paper, we assume that K is a constant which may change from line to line.*

3. Transformed Equation

By applying the transformation (2) to the original Equation (1), we obtain a simple SDEJ without singular terms as follows:

$$\begin{aligned} \Phi(X_t) \quad = \quad & \Phi(X_0) + \int_0^t \Phi'(X_s)a(X_s)ds + \int_0^t \Phi'(X_s)b(X_s)dW_s \\ & + \int_0^t (\Phi(X_{s-} + c(X_{s-})) - \Phi(X_{s-}))d\widetilde{N}_s. \end{aligned} \tag{7}$$

We refer to [24] for more details.

Setting $\overline{X}_t = \Phi(X_t)$ and using the notations below

$$\overline{a}(\Phi(x)) := \Phi'(x)a(x),\ \overline{b}(\Phi(x)) := \Phi'(x)b(x) \ \text{ and } \ \overline{c}(\Phi(x)) := \Phi(x + c(x)) - \Phi(x),$$

Equation (7) can be rewritten as

$$\overline{X}_t = \overline{X}_0 + \int_0^t \overline{a}(\overline{X}_s)ds + \int_0^t \overline{b}(\overline{X}_s)dW_s + \int_0^t \overline{c}(\overline{X}_{s-})d\widetilde{N}_s, \tag{8}$$

where $\overline{X}_0 = \Phi(X_0)$. In what follows, Equation (8) will be denoted as $\mathrm{Eq}(\overline{X}_0, \overline{a}, 0, \overline{b}, \overline{c})$.

Theorem 1. *If κ is a bounded and integrable function, then $(X_t)_{0\le t\le T}$ is a solution to* $\mathrm{Eq}(X_0, a, \kappa, b, c)$ *if and only if $(\overline{X}_t = \Phi(X_t))_{0\le t\le T}$ is a solution to* $\mathrm{Eq}(\overline{X}_0, \overline{a}, 0, \overline{b}, \overline{c})$.

The proof of this theorem can be found in [24], Theorem 3.2.

The existence and uniqueness of the solution of the original equation $\mathrm{Eq}(X_0, a, \kappa, b, c)$ and the new equation $\mathrm{Eq}(\overline{X}_0, \overline{a}, 0, \overline{b}, \overline{c})$ has been established in [24], Corollary 3.1.

4. Numerical Approximations

In this paper, we use the Euler–Maruyama scheme (EM scheme) to approximate the solution to (1). The EM scheme and its rate of convergence have been scrutinized for different modes of convergence by many researchers previously. For the explicit scheme, the strong convergence of the fixed time-step for SDEJs has been established in [25–28], whereas in [29,30], similar cases have been studied for the implicit scheme. It was also proved in [29] that, in the case of non-linear problems that fail to satisfy the global Lipschitz condition, strong convergence can be achieved for the implicit scheme. Under regular assumptions, it was proved in [20] that the strong convergence of the Euler scheme for SDEJs holds true. More precisely, the approximation through the Euler method converges strongly with order 0.5.

The majority of the research results on numerical approximations of SDEJs have made assumptions that the coefficients satisfy basic regularity conditions, such as the Lipschitz continuity property, and sometimes the boundedness of the derivatives up to some order. In paper [31], the authors analyzed a compensated projected EM method for stochastic differential equations with jumps under specific assumptions that admit the superlinearity of the jump and diffusion coefficients. A higher order approximation for jump diffusion SDEJs with discontinuous drifts has been stated in [32]. Additionally, a new simplified weak second-order scheme for solving SDEJs was established in [33].

We will now study the numerical approximation of the singularity-free Equation (8), for which we need the following estimate whose proof can be found in Lemma 5 in [29].

Lemma 2. *If the coefficients $\overline{a}$, $\overline{b}$, and $\overline{c}$ satisfy the following assumptions:*

- $\overline{a}$, $\overline{b}$, $\overline{c} \in \mathcal{C}^1$;
- *The drift coefficient $\overline{a}$ satisfies a one-sided Lipschitz condition*

$$(x-y)(\overline{a}(x)-\overline{a}(y)) \le K|x-y|^2 \ \ \forall\, x, y \in \mathbb{R};$$

- *The diffusion and jump coefficients satisfy global Lipschitz conditions*

$$|\overline{b}(x)-\overline{b}(y)|^2 \le K|x-y|^2 \quad \forall\, x, y \in \mathbb{R},$$
$$|\overline{c}(x)-\overline{c}(y)|^2 \le K|x-y|^2 \quad \forall\, x, y \in \mathbb{R};$$

- *Finite moment bounds for the initial data, that is, for any $p \ge 2$ $\mathbb{E}|\overline{X}_0|^p$, are finite;*

then for each $p \ge 2$, there exists a constant $K = K(p, T)$ such that

$$\mathbb{E}[\sup_{0\le t\le T} |\overline{X}_t|^p] \le K(1+\mathbb{E}|\overline{X}_0|^p).$$

4.1. Euler–Maruyama Scheme (EM)

If we consider the one-dimensional case as will be seen in our numerical example later, the Euler–Maruyama scheme is given by the algorithm

$$\overline{Y}_{t_{n+1}} = \overline{Y}_{t_n} + \overline{a}(\overline{Y}_{t_n})\Delta + \overline{b}(\overline{Y}_{t_n})\Delta W_n + \overline{c}(\overline{Y}_{t_n})\Delta \widetilde{N}_n, \tag{9}$$

for $n \in 0, 1, \ldots, m-1$, with initial value $\overline{Y}_0 = \overline{X}_0$. Here, $\Delta = t_{n+1} - t_n$, $\Delta W_n = W_{t_{n+1}} - W_{t_n}$ is the n^{th} increment of the Wiener process W, and $\Delta \widetilde{N}_n = \widetilde{N}_{t_{n+1}} - \widetilde{N}_{t_n}$ is the n the increment of the compensated Poisson process $\widetilde{N}$.

Given a discrete-time approximation $(\overline{Y}_{t_n})_{0\leq n\leq m}$, we define a continuous-time approximation $(\overline{Y}_t)_{0\leq t\leq T}$ by

$$\overline{Y}_t = \overline{X}_0 + \int_0^t \overline{a}(\overline{Y}_{k_{n(s)}})ds + \int_0^t \overline{b}(\overline{Y}_{k_n(s)})dW_s + \int_0^t \overline{c}(\overline{Y}_{k_{n(s)}})d\widetilde{N}_s, \tag{10}$$

where $k_n(s) = t_n$ for $s \in [t_n, t_{n+1})$.

4.2. Convergence of the Numerical Scheme

The strong convergence of discrete-time approximations is defined in [20] as follows.

Definition 2. *We say that discrete-time approximation $\overline{Y}_t$, corresponding to the discretization $(t)_\Delta$, where Δ is time-step size, converges strongly with order γ to the solution $\overline{X}$ of SDEJ (8) if there exists a positive constant K, independent of Δ, such that*

$$\sqrt{\mathbb{E}\sup_{0\leq t\leq T}|\overline{Y}_t - \overline{X}_t|^2} \leq K\Delta^\gamma.$$

For simplicity, we consider an equidistant time discretization $0 = t_0,\ t_1,\ \ldots,\ t_m = T$, with $t_n = n\Delta$. Then, $\{\overline{Y}_{t_n},\ n \in \{0,\ 1,\ \ldots,\ m\}\}$ stands for the discrete-time approximation of the solution $\overline{X}$ to the SDEJ (8).

Strong Convergence of Euler Scheme

For given Euler–Maruyama approximations for stochastic differential equations with jumps, the order of convergence is established as 0.5, as stated in the following theorem.

Theorem 2. *Let $\overline{Y} = \{\overline{Y}_t, t \in [0,T]\}$ be the Euler–Maruyama approximation corresponding to a time discretization with maximum step size $\Delta \in (0,1)$. Assume (A.1), (A.2), and (A.3) are satisfied and that $\mathbb{E}|\overline{X}_0|^2 < \infty$; then,*

$$\mathbb{E}\sup_{0\leq t\leq T}|\overline{X}_t - \overline{Y}_t|^2 \leq K\Delta,$$

where K is a constant independent from Δ.

Proof. The proof of this theorem can be found in ([20], Theorem 6.4.1). Therefore, the strong order of convergence for the above transformed jump diffusion stochastic differential equation is 0.5. □

We set $Y_t = \Phi^{-1}(\overline{Y}_t)$. Our main aim is to show that the sequence $Y_.$ converges to $X_.$ uniformly in $L^2(\Omega)$, with 0.5 as a rate of convergence.

Proposition 1. *There exists a positive constant K such that*

$$\mathbb{E}\sup_{0\leq t\leq T}|Y_t - X_t|^2 \leq K\Delta.$$

Proof. Thanks to (5), we have

$$|Y_t - X_t| = |\Phi^{-1}(\overline{Y}_t) - \Phi^{-1}(\overline{X}_t)| \leq e^{2|\kappa|_1}|\overline{Y}_t - \overline{X}_t|,$$

and, consequently,

$$\sup_{0\leq t\leq T}|Y_t - X_t|^2 \leq e^{4|\kappa|_1}\sup_{0\leq t\leq T}|\overline{Y}_t - \overline{X}_t|^2.$$

Now, we apply this expectation to both sides of the above inequality and, using Theorem 2, we conclude that $Y_.$ converges to $X_.$ uniformly in $L^2(\Omega)$. Moreover, the rate of convergence is of the order 0.5. □

5. Illustrative Example

We will demonstrate the application of our result through the following example. Consider the following SDEJ:

$$dX_t = \Big(a(X_t) + \kappa(X_t)b^2(X_t) - [c]_\Phi(X_t)\Big)dt + b(X_t)dW_t + c(X_t)d(N_t - \lambda t), \tag{11}$$

where the coefficients are given as below:

$$a(x) = \begin{cases} -5x & \text{if } x < 0 \\ -2.5 + 2.5e^{-2x} & \text{if } 0 \le x \le 1 \\ -5x + 2.5(e^{-2} + 1) & \text{if } 1 < x \le 2 \\ -10e^{2x-4} + 2.5e^{2x-6} + 2.5 & \text{if } 2 < x \le 4 \\ -5x - 10e^4 + 2.5e^2 + 22.5 & \text{if } x > 4, \end{cases} \tag{12}$$

$$b(x) = \begin{cases} x & \text{if } x < 0 \\ 0.5 - 0.5e^{-2x} & \text{if } 0 \le x \le 1 \\ x - 0.5e^{-2} - 0.5 & \text{if } 1 < x \le 2 \\ 2e^{2x-4} - 0.5e^{2x-6} - 0.5 & \text{if } 2 < x \le 4 \\ x + 2e^4 - 0.5e^2 - 4.5 & \text{if } x > 4, \end{cases} \tag{13}$$

$$c(x) = \begin{cases} 2x & \text{if } x < 0 \\ \frac{1}{2}\ln(3e^{2x} - 2) - x & \text{if } 0 \le x \le 1 \\ 2x - e^{-2} - 1 & \text{if } 1 < x \le 2 \\ 3 - \frac{1}{2}\ln(2 + 3e^{6-2x} - 8e^2) - x & \text{if } 2 < x \le 4 \\ 2x - e^2 + 4e^4 - 9 & \text{if } x > 4, \end{cases} \tag{14}$$

and

$$[c]_\Phi(x) = \left(\frac{\Phi(x + c(x)) - \Phi(x)}{\Phi'(x)} - c(x)\right)\lambda = \begin{cases} 0 & \text{if } x < 0 \\ (\frac{e^2-3}{2})\lambda & \text{if } 0 \le x \le 1 \\ 0 & \text{if } 1 < x \le 2 \\ (\frac{-e^{-2}-1}{2})\lambda & \text{if } 2 < x \le 4 \\ 0 & \text{if } x > 4. \end{cases} \tag{15}$$

Due to the presence of the singular part of the drift, it is difficult to find an explicit solution to Equation (11). However, we can modify this equation into one that has relatively smoother coefficients; thus, its analytic solution is easy to find. By applying the following transformation Φ, we are able to transform the above equation into a linear stochastic differential equation with jumps, which has an exact solution.

Consider a measurable and integrable function $\kappa(t) = \mathbf{1}_{[0,2]}(t) - \mathbf{1}_{[1,4]}(t)$ and define $g(y) = \exp(2\int_0^y \kappa(t)dt)$; then,

$$\int_0^y \kappa(t)\mathrm{d}t = \begin{cases} -\int_y^0 \kappa(t)\mathrm{d}t = 0 & \text{if } y < 0 \\ \int_0^y \mathbf{1}_{[0,2]}(t)\mathrm{d}t = y & \text{if } 0 \le y \le 1 \\ \int_0^1 \mathbf{1}_{[0,2]}(t)\mathrm{d}t + \int_1^y \kappa(t)\mathrm{d}t = 1 & \text{if } 1 < y \le 2 \\ \int_0^1 \mathbf{1}_{[0,2]}(t)\mathrm{d}t + \int_1^2 \kappa(t)\mathrm{d}t - \int_2^y \mathbf{1}_{[1,4]}(t)\mathrm{d}t = 3 - y & \text{if } 2 < y \le 4 \\ \int_0^4 \Big(\mathbf{1}_{[0,2]}(t) - \mathbf{1}_{[1,4]}(t)\Big)\mathrm{d}t = -1 & \text{if } y > 4 \end{cases}$$

and

$$g(y) = \begin{cases} 1 & \text{if} \quad y < 0 \\ e^{2y} & \text{if} \quad 0 \leq y \leq 1 \\ e^2 & \text{if} \quad 1 < y \leq 2 \\ e^{6-2y} & \text{if} \quad 2 < y \leq 4 \\ e^{-2} & \text{if} \quad y > 4, \end{cases}$$

and finally

$$\begin{aligned} \Phi(x) &= \int_0^x g(y)\mathrm{d}y \\ &= \begin{cases} x & \text{if} \quad x < 0 \\ \dfrac{e^{2x}-1}{2} & \text{if} \quad 0 \leq x \leq 1 \\ \Phi(1) + \int_1^x g(y)\mathrm{d}y = \dfrac{e^2-1}{2} + e^2(x-1) & \text{if} \quad 1 < x \leq 2 \\ \Phi(2) + \int_2^x g(y)\mathrm{d}y = \dfrac{3e^2-1}{2} + \dfrac{e^2 - e^{6-2x}}{2} & \text{if} \quad 2 < x \leq 4 \\ \Phi(4) + \int_4^x g(y)\mathrm{d}y = \dfrac{4e^2 - e^{-2} - 1}{2} + e^{-2}(x-4) & \text{if} \quad x > 4. \end{cases} \end{aligned}$$

Now, applying the transformation function Φ as proved in [24], the new modified coefficients are defined as

- $\bar{a}(\Phi(x)) = \Phi'(x)a(x)$,
- $\bar{b}(\Phi(x)) = \Phi'(x)b(x)$,
- $\bar{c}(\Phi(x)) = \Phi(x + c(x)) - \Phi(x)$.

Applying the above transformation and taking $\Phi(X_t) = \overline{X}_t$ and $\Phi(X_0) = \overline{X}_0$, our jump diffusion SDE reduces to

$$d\overline{X}_t = -5\overline{X}_t dt + \overline{X}_t dW_t + 2\overline{X}_t\, d(N_t - \lambda t), \tag{16}$$

where $\bar{a}$, $\bar{b}$, and $\bar{c}$ are given by

$$\bar{a}(x) = -5x, \quad \bar{b}(x) = x \ \text{ and } \bar{c}(x) = 2x.$$

The exact solution to (16) is given by

$$\overline{X}_t = \overline{X}_0(3)^{N_t} \exp((-5.5 + 2\lambda)t + W_t). \tag{17}$$

Clearly, by taking the inverse using the deterministic function Φ, of the exact solution to (16), we obtain the exact solution to the original Equation (11); that is, $X_t = \Phi^{-1}(\overline{X}_t)$, where Φ^{-1} is given by:

$$\Phi^{-1}(x) = \begin{cases} x & \text{if} \quad x < 0 \\ \frac{1}{2}\ln(2x+1) & \text{if} \quad 0 \leq x \leq 3.1945 \\ xe^{-2} + \frac{e^{-2}+1}{2} & \text{if} \quad 3.1945 < x \leq 10.5836 \\ \frac{1}{2}(6 - \ln(4e^2 - 2x - 1)) & \text{if} \quad 10.5836 < x \leq 14.2105 \\ \frac{7}{2} + xe^2 - 2e^4 + \frac{1}{2}e^2 & \text{if} \quad x > 14.2105. \end{cases} \tag{18}$$

5.1. Stability

Stability is an important property in any time-stepping scenario. For SDEJs like SDEs, the two main kinds of stabilities to consider are mean-square stability and asymptotic stability. We will consider mean square stability for our example. The mean square stability for linear SDEJ has been discussed by [30]. This section looks at stability in mean square. We will take our modified Equation (8) as our test equation and determine the value of Δ at which the method shares the stability/instability of the test equation.

Consider the modified linear equation

$$d\overline{X}_t = \overline{a}\,\overline{X}_t dt + \overline{b}\,\overline{X}_t dW_t + \overline{c}\,\overline{X}_t d(N_t - \lambda t), \tag{19}$$

whose explicit solution $\overline{X}_t$ is given by

$$\overline{X}_t = \overline{X}_0(1+\overline{c})^{N_t}\exp\{(\overline{a} - \overline{c}\lambda - \frac{1}{2}\overline{b}^2)t + \overline{b}W_t\}.$$

Using the fact that $\mathbb{E}[(1+\overline{c})^{N_t}] = e^{\lambda t\overline{c}(2+\overline{c})}$, we have

$$\begin{aligned}\mathbb{E}|\overline{X}_t|^2 &= \mathbb{E}|\overline{X}_0|^2 e^{2(\overline{a}-\overline{c}\lambda-\frac{1}{2}\overline{b})t}\mathbb{E}[e^{2\overline{b}W_t}]\mathbb{E}[(1+\overline{c})^{2N_t}] \\ &= \mathbb{E}|\overline{X}_0|^2 e^{(2\overline{a}-2\overline{c}\lambda+\overline{b}^2+\lambda\overline{c}(2+\overline{c}))t}.\end{aligned}$$

Hence, the mean square stability is characterized by

$$\begin{aligned}\lim_{t\to\infty}\mathbb{E}|\overline{X}_t|^2 = 0 \quad &\Leftrightarrow \quad 2\overline{a} - 2\overline{c}\lambda + \overline{b}^2 + \lambda\overline{c}(2+\overline{c}) < 0 \\ &\Leftrightarrow \quad 2\overline{a} + \overline{b}^2 + \lambda\overline{c}^2 < 0.\end{aligned}$$

For our numerical example of modified SDE with jumps (8), we have

$$2(-5) + (1)^2 + \lambda 2^2 = -9 + 4\lambda.$$

Hence, the solution of Equation (16) is mean square stable for $\lambda < 2.25$.

Now, we will check the stability of the original SDEJ (11) by using the bilipschitz property of the function transformation $\Phi(x)$ and $\Phi^{-1}(x)$.

Consider Equation (11), where a, b, and c are given respectively by Equations (12)–(14). Then, for its solution X_t, we have

$$\mathbb{E}|X_t|^2 = \mathbb{E}|\Phi^{-1}(\overline{X}_t)|^2 \le e^{2|\kappa|_1}\mathbb{E}|\overline{X}_t|^2 \le e^{2|\kappa|_1}\mathbb{E}|\overline{X}_0|^2 e^{(2\overline{a}+\overline{b}^2+\lambda\overline{c}^2)t}.$$

We conclude that the mean square stability of the modified Equation (16) implies the mean square stability of original Equation (11).

Stability of the Euler–Maruyama Scheme

In the case of the Euler–Maruyama scheme, we only consider mean square stability and not asymptotic stability. The concept of asymptotic stability for a numerical method for the jump case is discussed in [30], which states that Euler–Maruyama is not asymptotically stable for any value of λ. Hence, our aim now is to analyze the corresponding mean square stability property, $\lim_{n\to\infty}\mathbb{E}|\overline{Y}_{t_n}|^2$, for the Euler–Maruyama scheme.

Here, we examine the value of Δ for which the EM scheme is stable.

The EM scheme for our linear modified equation is given by

$$\overline{Y}_{t_{n+1}} = \overline{Y}_{t_n} + \overline{a}\,\overline{Y}_{t_n}\,\Delta + \overline{b}\,\overline{Y}_{t_n}\,\Delta W_n + \overline{c}\,\overline{Y}_{t_n}\,\Delta\widetilde{N}_n,$$

by squaring both sides of the equation and taking expectation. We are aware that the Brownian increments satisfy $\mathbb{E}(\Delta W_n) = 0$ and $\mathbb{E}(\Delta W_n)^2 = \Delta$ and that the compensated Poisson increments satisfy $\mathbb{E}(\Delta\widetilde{N}_n) = 0$ and $\mathbb{E}(\Delta\widetilde{N}_n)^2 = \lambda^2\Delta^2$. Hence, using the independence of the increments, we have

$$\mathbb{E}[\overline{Y}^2_{t_{n+1}}] = \mathbb{E}[\overline{Y}^2_{t_n}](1 + 2\overline{a}\Delta + \overline{b}^2\Delta + \overline{c}^2\lambda^2\Delta^2).$$

This leads to a simple stability characterization:

$$\lim_{t_n\to\infty}\mathbb{E}[\overline{Y}^2_{t_n}] = 0 \Leftrightarrow \Delta(\overline{c}^2\lambda^2) < -(2\overline{a} + \overline{b}^2)$$

and

$$0 < \Delta < \frac{|2\bar{a} + \bar{b}^2|}{\bar{c}^2 \lambda^2}.$$

For our modified SDE (16) and consequently our original SDE (11), the Euler–Maruyama scheme is stable if Δ satisfies the condition below:

$$0 < \Delta < \frac{|2 \times (-5) + 1^2|}{2^2 \lambda^2} = \frac{9}{4\lambda^2}.$$

5.2. Numerical Simulations for the Exact and Modified Equation

The following tables provide the relationship between the exact and approximate values for the equations for different values of the parameter λ.

For $\lambda = 0.5$:

dt	0.001	0.0001	0.00005
Number of realizations	1000	5000	12,000
Error_1	0.0395	0.0024	0.0000019
Error_2	0.0112	0.000124	0.00000034

For $\lambda = 1.0$:

dt	0.001	0.0001	0.00005
Number of realizations	1000	5000	12,000
Error_1	0.3262	0.0214	0.00012
Error_2	0.045	0.0134	0.00003

For $\lambda = 2.0$:

dt	0.001	0.0001	0.00005
Number of realizations	1000	5000	12,000
Error_1	0.429	0.02647	0.00078
Error_2	0.076	0.025	0.0000156

where

- Error_1 is the expected value of the difference between the approximate solution to the modified equation and the exact solution to the modified equation;
- Error_2 is the expected value of the difference between the approximate solution to the original equation and the inverse function of Φ applied to the exact solution of the modified equation.

6. Illustrations

6.1. Comparative Analysis of the Results

In Figures 1–3 below, comparative plots have been used to compare the solution to the original SDEJ by using two different approaches: the direct one based on EM approximation of the original SDEJ (blue) and the second solution obtained by taking the inverse of the function Φ applied to the EM approximation of the modified SDEJ (red) for different values of the parameter λ.

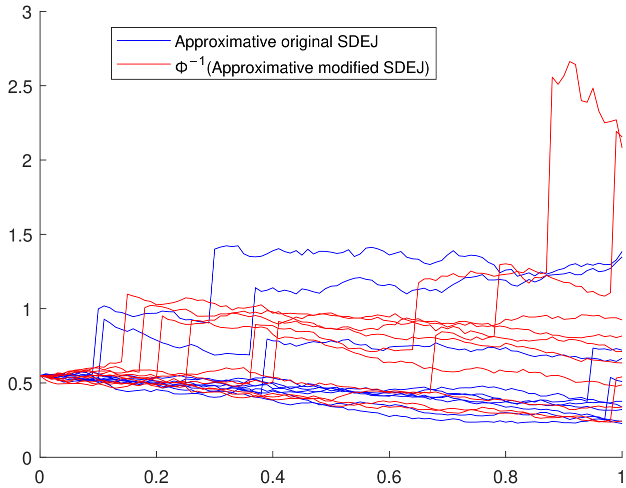

Figure 1. Approximative solution of the original SDEJ by two methods for $\lambda = 0.5$.

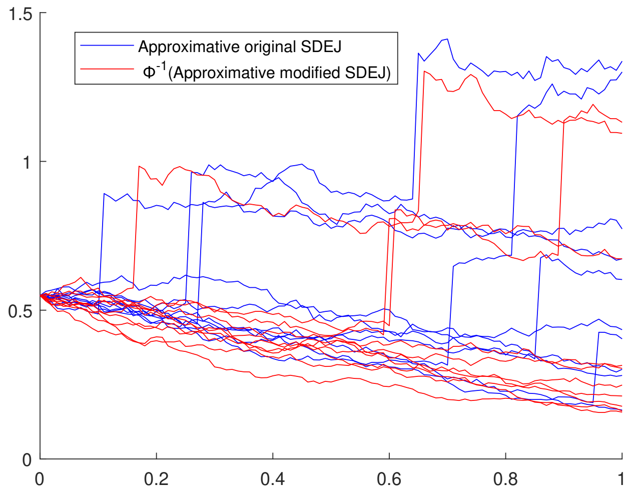

Figure 2. Approximative solution of the original SDEJ by two methods for $\lambda = 1$.

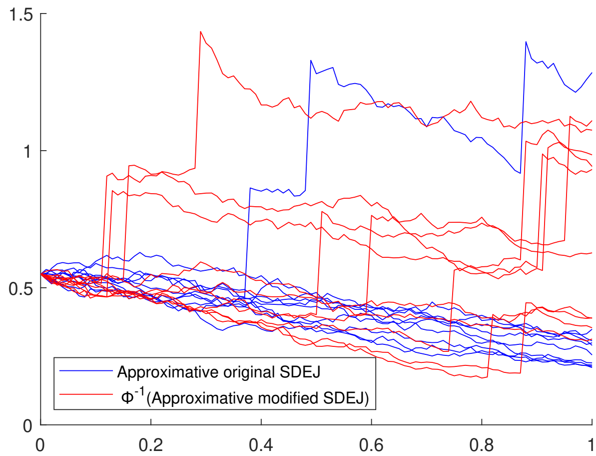

Figure 3. Approximative solution of the original SDEJ by two methods for $\lambda = 2$.

Observation: The numerical analysis illustrates our theoretical results. The solution was obtained after applying inverse function Φ to the modified equation, and we obtained the solution to the original SDEJ as seen in the graphs, where the solutions are seen to overlap. It is also observed that the results are similar when using different values of λ, as long as the value of the step size is kept in range for stable results of EM approximations.

6.2. Analysis of the Error

In this section, we will plot errors (Error_1 and Error_2) for the original SDEJ (11) and the modified SDEJ (16) for different values of the parameter λ.

Observation: The plots in Figures 4–6 below depict the errors for both equations when increasing the number of realizations. It is observed that, as the number of realizations increases, both the Error_1 and Error_2 decrease almost to 0, which justifies our approximation technique for both equations.

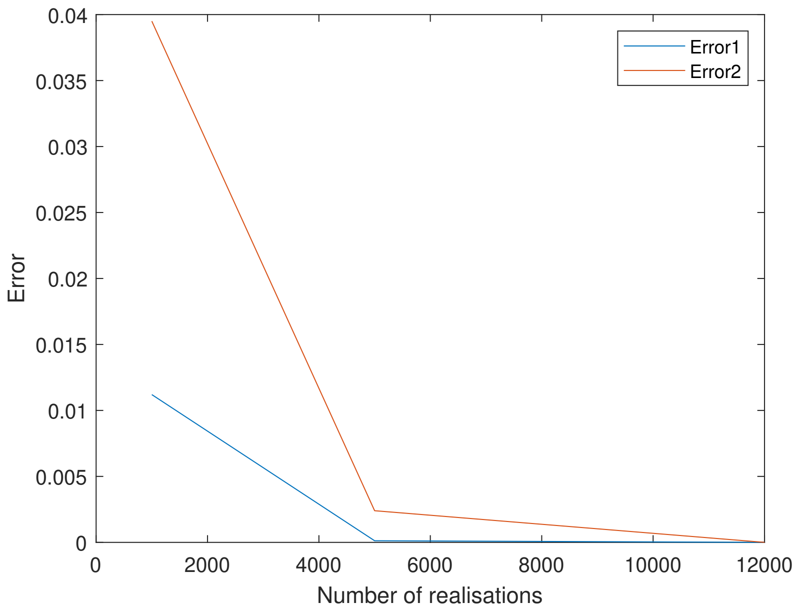

Figure 4. Error plot for $\lambda = 0.5$.

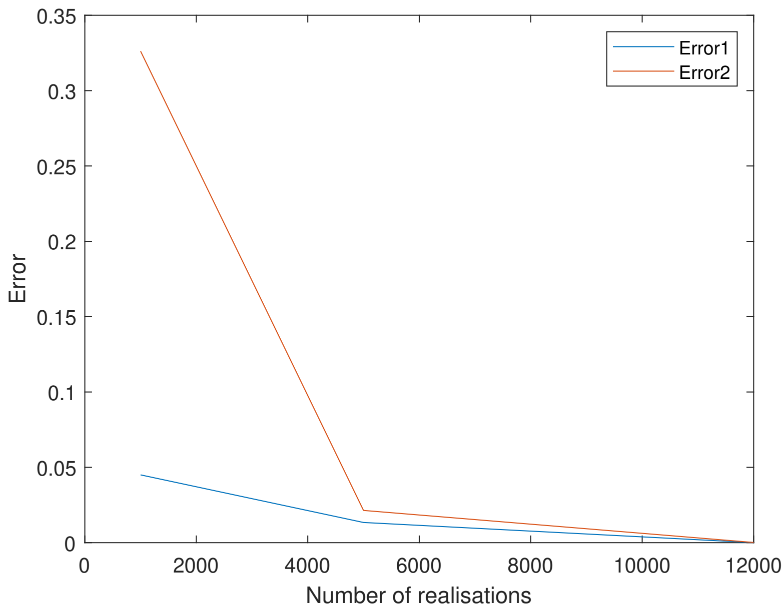

Figure 5. Error plot for $\lambda = 1$.

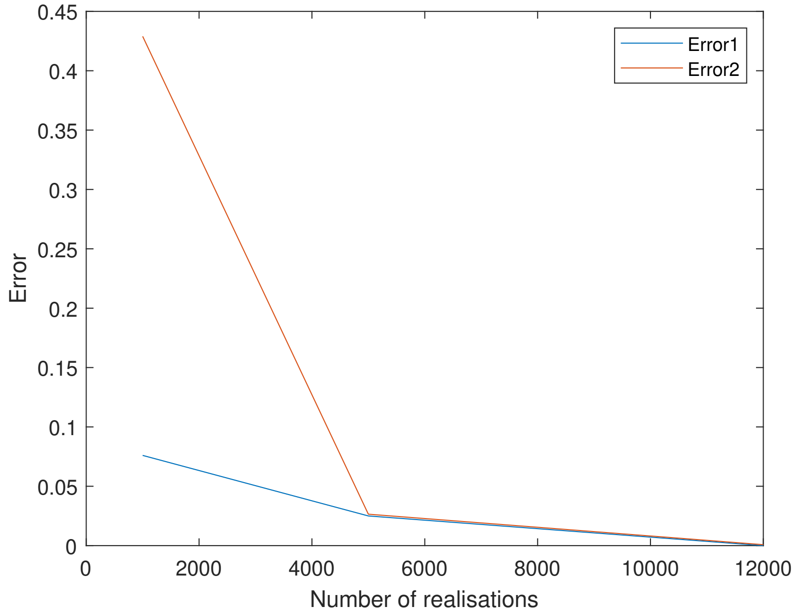

Figure 6. Error plot for $\lambda = 2$.

7. Conclusions

The numerical approach used in this article has applied two different methods to approximate solutions for this class of SDEJs. The first method consisted of a direct approximation using the Euler–Maruyama scheme of the original equation with specific tests for the evaluation of the singular part at simulated values for the solution. The second method was obtained by taking the inverse using Zvonkin's transformation of the Euler–Maruyama approximation of the SDEJ without singular terms. Comparative analysis of the two numerical methods was carried out. Theoretical results were illustrated and tested by an example, and MATLAB was used for numerical simulations.

Introducing this kind of method for approximating solutions to stochastic differential equations with non–smooth coefficients is significant in the sense that this method can be applied to various real life models and will entice readers to use similar methods to solve many other complex models.

For future work, we shall consider this approach for multidimensional SDEJs.

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