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Malliavin Regularity of Non-Markovian Quadratic BSDEs and Their Numerical Schemes

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Abstract: We study both Malliavin regularity and numerical approximation schemes for a class of quadratic backward stochastic differential equations (QBSDEs for short) in cases where the terminal data need not be a function of a forward diffusion. By using the connection between the QBSDE under study and some backward stochastic differential equations (BSDEs) with global Lipschitz coefficients, we firstly prove L^q, $(q \geq 2)$ existence and uniqueness results for QBSDE. Secondly, the L^p-Hölder continuity of the solutions is established for ($q > 4$ and $2 \leq p < \frac{q}{2}$). Then, we analyze some numerical schemes for our systems and establish their rates of convergence. Moreover, our results are illustrated with three examples.

Keywords: quadratic backward stochastic differential equations; Malliavin calculus; explicit scheme; implicit scheme; rate of convergence; Hölder continuity

MSC: 60H07; 60H10; 60H35; 65C30; 91G60

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1. Introduction

Our concern in this paper is to study regularity in the Malliavin sense of the solutions of a class of backward stochastic differential equations that show a quadratic growth of the type below:

$$Y_t = \xi + \int_t^T \left(h(r, Y_r) + h_1(r) Z_r + f(Y_r)|Z_r|^2 \right) \mathrm{d}r - \int_t^T Z_r \mathrm{d}W_r,\ 0 \leq t \leq T, \tag{1}$$

where the process $h : [0,\ T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ is a given bounded and global Lipschitz function in y uniformly in r, $h_1 : [0,\ T] \to \mathbb{R}$ is a bounded function, $f : \mathbb{R} \to \mathbb{R}$ is an integrable function and ξ is a given terminal datum such that ξ need not be a function of a forward diffusion; this means that the random variable ξ can be taken arbitrarily.

Recall that backward stochastic differential equations (BSDEs) were first studied by Bismut [1] in the linear case; Bismut developed the theory of BSDEs and their applications in finance and control theory. Pardoux and Peng [2] published a seminal paper in which they studied the general nonlinear backward stochastic differential equations (BSDEs) where the generator is global Lipschitz and the terminal condition is square integrable. Since then, there has been extensive research on BSDEs; we refer to [3–5] for a more complete presentation of the theory. This type of equation has proven to be a powerful tool for studying stochastic processes and has applications in many fields, including finance, economics, engineering and mathematical biology.

Since it is difficult to find explicit solutions for BSDEs, several researchers have directed their focus towards numerical methods. Generally speaking, numerical approximation schemes are a very important research topic in recent years that many studies have focused on. One of the challenges for solving BSDEs numerically is that the equations are nonlinear and high-dimensional, which makes it difficult to obtain exact solutions. Therefore, researchers have developed a variety of numerical approximation schemes to solve BSDEs, including, Monte Carlo methods, finite difference methods and numerical methods based on partial differential equation theory. In particular, in the Markovian case, Douglas et al. [6] established numerical methods for a class of forward–backward SDEs based on the four-step scheme developed by Ma et al. [7] to solve general FBSDEs requiring the numerical resolution of quasi-linear parabolic PDE. In the non-Markovian case, Bally [8] proposed a time discretization scheme and obtained its convergence rate. Zhang [9] established some L^2-regularity on Z and found that their scheme converges and also derived its rate of convergence. It is worth mentioning the work of Briand et al. [10], where Brownian motion is replaced by a scaled random walk.

In the last few years, many researchers have turned their interest toward QBSDE theory. The very well-known result of the existence of the solution was proved by Kobylanski in [11], when the terminal condition is bounded, the generator coefficient ($h(r,y) + h_1(r)z + f(y)|z|^2$ in our case) is continuous and has a quadratic growth in z. Later, Bahlali et al. studied in [12] one-dimensional QBSDE with a measurable generator in cases where $h(r,y) + h_1(r)z = 0$ and the terminal condition is merely square integrable. QBSDE theory has been developed very remarkably in different perspectives; in particular, in Bahlali et al. [13], the authors studied a BSDE whose generator shows logarithmic growth and provided a relation between this latter and one type of QBSDE. Subsequently, in [14], Madoui et al. focused on solving a class of quadratic BSDEs with Jumps.

As opposed to ordinary BSDEs, there are only a few studies devoted to the numerical study of QBSDEs. Indeed, Imkeller and Dos Reis in [15] gave explicit convergence rates for the difference between the solution of a QBSDE and its truncation; in the same context, Richou in [16] provided a new time discretization scheme with a non-uniform time step for such BSDEs and also obtained an explicit convergence rate for this scheme. Recently, Chassagneux and Richou in [17] introduced a fully implementable algorithm for a QBSDE based on quantization and illustrated their convergence results with numerical examples.

In our case, the concern will be rather somehow singular QBSDE by adding a new term $h(r,y) + h_1(r)z$, so that the generator becomes $h(r,y) + h_1(r)z + f(y)|z|^2$. It is singular in the sense that f is merely a measurable and integrable function. Based on the space transformation $F(x) = \int_0^x \exp(2\int_0^y f(t)\mathrm{d}t)\mathrm{d}y$, we first study the existence and uniqueness of solutions to QBSDE (1). This transformation is used to get rid of the term $f(y)|z|^2$ and obtain a classical BSDE with a global Lipschitz generator. Indeed, applying Itô-Krylov's formula [12] to $F(Y_t)$, one obtains

$$F(Y_t) = F(\xi) + \int_t^T (h(s,Y_s)F'(Y_s) + h_1(s)F'(Y_s)Z_s)\mathrm{d}s - \int_t^T F'(Y_s)Z_s\mathrm{d}W_s. \tag{2}$$

To simplify notations, we put

$$F(Y_t) = \bar{Y}_t,\ \bar{Z}_t = F'(Y_t)Z_t \text{ and } F(\xi) = \bar{\xi}. \tag{3}$$

Thus, setting

$$\bar{h}(s,y,z) = h(s,F^{-1}(y))F'\left(F^{-1}(y)\right) + h_1(s)z \tag{4}$$

Equation (2) reads

$$\bar{Y}_t = \bar{\xi} + \int_t^T \bar{h}(s,\bar{Y}_s,\bar{Z}_s)\mathrm{d}s - \int_t^T \bar{Z}_s\mathrm{d}W_s, \tag{5}$$

Conversely, if $(\bar{Y},\bar{Z})$ is a solution to (5), then by applying Itô-Krylov's formula to $F^{-1}(\bar{Y}_t)$, we show that $(Y = F^{-1}(\bar{Y}),\ Z = \frac{\bar{Z}}{F'(F^{-1}(\bar{Y}))})$ is a solution to QBSDE (1).

Our second aim is to prove, under some extra conditions on f, several statements concerning the path regularity property of the solutions of QBSDE (1) in the sense of

Malliavin calculus and thus establish the following estimates, for any $q > 4$, $2 \le p < \frac{q}{2}$ and $s, t \in [0,\ T]$

$$\mathbb{E}|Y_t - Y_s|^p \ \le\ K|t-s|^{\frac{p}{2}} \quad \text{and} \quad \mathbb{E}|Z_t - Z_s|^p \ \le\ K|t-s|^{\frac{p}{2}},$$

where K is a constant independent of s and t. Moreover, we shall prove that the QBSDE's solution (Y, Z) is Malliavin differentiable and the process Z can be determined as the trace of the Malliavin derivative of Y.

The third aim is to construct different types of numerical schemes for the solution of QBSDE (1) and give the rate of their convergence throughout the solution to BSDE (2). The main tool is to use the Malliavin calculus since the process $\bar{Z}$ can be represented in terms of the Malliavin derivative of $\bar{Y}$, which means that $\bar{Z}_t = D_t\bar{Y}_t$ for a.e. t. Our starting point is to define both explicit and implicit numerical schemes of the associated global Lipschitz BSDE (5) which will be denoted by $(\bar{Y}^\pi, \bar{Z}^\pi)$; this is inspired by the numerical schemes proposed by [18]; then we define those of QBSDE (1) as

$$(Y^\pi, Z^\pi) = \left(F^{-1}(\bar{Y}^\pi), \frac{\bar{Z}^\pi}{F'(F^{-1}(\bar{Y}^\pi))}\right).$$

We first propose the study of the explicit scheme. We prove that the rate of convergence of Y is given by

$$\mathbb{E} \sup_{0\le t\le T} |Y_t - Y_t^\pi|^2 \ \le\ K(|\pi| + \mathbb{E}|\xi - \xi^\pi|^2).$$

while the one of Z takes the following form

$$\int_0^T \mathbb{E}|Z_t - Z_t^\pi|^p \mathrm{d}t \ \le\ K\Big(|\pi| + \mathbb{E}|\xi - \xi^\pi|^2\Big)^{\frac{p}{2}} \quad \forall\, 1 \le p < 2.$$

where ξ^π stands for the approximation of the random variable ξ whenever it is not smooth enough.

For the implicit numerical scheme, the rates of convergence are of the form below: for any $p \ge 2$

$$\mathbb{E} \sup_{0\le t\le T} |Y_t - Y_t^\pi|^p \ \le\ K\Big(|\pi|^{\frac{p}{2}} + \mathbb{E}|\xi - \xi^\pi|^p\Big),$$

and

$$\mathbb{E}\left(\int_0^T |Z_t - Z_t^\pi|^2 \mathrm{d}t\right)^{\frac{p}{2}} \ \le\ K\Big(|\pi|^{\frac{p}{2}} + \max\Big(\mathbb{E}|\xi - \xi^\pi|^p,\ |\pi|^{-\frac{p}{2}}\mathbb{E}|\xi - \xi^\pi|^{2p}\Big)\Big).$$

Notice that both explicit and implicit numerical schemes are not completely discrete, this is due to the use of the integral of the process Z in each iteration of the schemes. However, we suggest a "fully discrete scheme" in the case where the Lipschitz part h of the generator is independent of y and is assumed to be linear in z. More precisely, we will deal with the following form of QBSDE for $0 \le t \le T$

$$Y_t = \xi + \int_t^T \Big(\alpha(r) + \beta(r)Z_r + f(Y_r)|Z_r|^2\Big)dr - \int_t^T Z_r dW_r$$

and prove the following rate of convergence in two cases: the first case when $\alpha \equiv 0$ and the second one when $f \equiv \frac{1}{2}$

$$\mathbb{E} \max_{0\le i\le n} \left\{ \left|Y_{t_i} - Y_{t_i}^\pi\right|^p + \left|Z_{t_i} - Z_{t_i}^\pi\right|^p \right\} \ \le\ C|\pi|^{\frac{p}{2} - \frac{p}{2\ln\frac{1}{|\pi|}}} \left(\ln \frac{1}{|\pi|}\right)^{\frac{p}{2}}.$$

The rest of the paper is organized as follows. In Section 2, we present some auxiliary results about Malliavin calculus and numerical schemes as well as their rate of convergence for BSDE in the global Lipschitz framework. In Section 3, we study results of the existence and uniqueness of solutions to QBSDE (1) along with the L^p-Hölder continuity of the solution. Then, we prove that the solution of the underlying QBSDE is regular in the Malliavin sense. Moreover, our results are illustrated by three examples. In Section 4, we establish the rate of convergence of QBSDE (1) for both explicit and implicit schemes; we conclude this section by giving some totally discrete schemes for some particular cases.

2. Preliminaries and Some Auxiliary Results

2.1. Malliavin Calculus for BSDE

Let $W = \{W_t\}_{0\leq t\leq T}$ be a real-valued Brownian motion defined on a complete filtered probability space $\big(\Omega, \mathcal{F}, \mathrm{P}, \{\mathcal{F}_t\}_{0\leq t\leq T}\big)$, such that $\mathbb{F}$=$\{\mathcal{F}_t\}_{0\leq t\leq T}$ is the natural filtration generated by the Brownian motion W and the P-null sets. Let $\mathcal{F} = \mathcal{F}_T$ and $\mathcal{P}$ be the progressive σ-field defined on the product space $[0,\ T]\times\Omega$.

To begin with, let us define, for any $p\geq 1$, the following spaces which will be used frequently in the sequel:

- $\mathcal{H}^p_{\mathcal{F}}([0,\ T])$ denotes the Banach space of all progressively measurable processes $\varphi : ([0,\ T]\times\Omega, \mathcal{P}) \to (\mathbb{R}, \mathcal{B})$ with norm

$$\|\varphi\|_{\mathcal{H}^p} = \left(\mathbb{E}\left(\int_0^T |\varphi_t|^2 \mathrm{d}t\right)^{\frac{p}{2}}\right)^{\frac{1}{p}} < +\infty.$$

- $\mathcal{S}^p_{\mathcal{F}}([0,\ T])$ denotes the Banach space of all the RCLL (right continuous with left limits) adapted processes $\varphi : ([0,\ T]\times\Omega, \mathcal{P}) \to (\mathbb{R}, \mathcal{B})$ with norm

$$\|\varphi\|_{\mathcal{S}^p} = \left(\mathbb{E}\sup_{0\leq t\leq T}|\varphi_t|^p\right)^{\frac{1}{p}} < +\infty.$$

Next, we present some preliminaries on Malliavin calculus and refer the reader to Nualart's seminal book [19] for more information and details about this subject.

Set $H := L^2([0,\ T])$ and denote by $\langle\cdot,\cdot\rangle_H$ the scalar product of the separable Hilbert space H and by $\|h\|_H$ the norm of any element h of H. For any $h\in H$, we denote the Wiener integral by $\mathcal{W}(h) = \int_0^T h(t)dW_t$.

We denote by $\mathcal{C}^\infty_p(\mathbb{R}^n)$ the set of all infinitely continuously differentiable functions $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that g and all their partial derivatives have polynomial growth. For $(h_1,\ldots,h_n)\in H^{\otimes n}$, set

$$F = g(\mathcal{W}(h_1),\ldots,\mathcal{W}(h_n)). \tag{6}$$

We denote by $\mathcal{S}$ the class of all smooth random variables of the form (6). Denote $\partial_i g := \frac{\partial g}{\partial x_i}$ for $i = 1,\ \ldots,\ n$; then the following H-valued random variable defined by

$$DF = \sum_{i=1}^n \partial_i g(\mathcal{W}(h_1),\ldots,\mathcal{W}(h_n))h_i,$$

is said to be the Malliavin derivative of F. For any $p\geq 1$, the domain of D in $L^p(\omega)$ will be denoted by $\mathbb{D}^{1,p}$, meaning that $\mathbb{D}^{1,p}$ is the closure of the class of smooth variables $\mathcal{S}$ with respect to the norm

$$\|F\|_{1,p} = \left(\mathbb{E}|F|^p + \mathbb{E}\|DF\|^p_H\right)^{\frac{1}{p}}.$$

A definition can be provided for the iteration of the operator D in such a way that for $F \in \mathcal{S}$, the iterated derivative D^kF is an $H^{\otimes k}$-random variable. We can then introduce a semi-norm on $\mathcal{S}$ for every $p \geq 1$ and any natural number $k \geq 1$,

$$\|F\|_{k,p} = \left(\mathbb{E}|F|^p + \sum_{j=1}^{k} \mathbb{E}\left\| D^j F \right\|^p_{H^{\otimes j}} \right)^{\frac{1}{p}}.$$

The completion of the family of smooth random variables $\mathcal{S}$ with respect to the norm $\|\cdot\|_{k,p}$ will denoted by $\mathbb{D}^{k,p}$.

Let μ be the Lebesgue measure on $[0,\ T]$. For any $k \geq 1$ and $F \in \mathbb{D}^{k,p}$, the derivative

$$D^kF = \left\{ D^k_{t_1,\ldots,t_k}F,\ t_i \in [0,\ T],\ i = 1,\ldots,k \right\},$$

is a measurable function on the product space $[0,\ T]^k \times \Omega$, which is defined a.e. with respect to the measure $\mu^k \otimes \mathrm{P}$.

Let $\mathbb{L}^{1,p}_a$ stand for the set of all H-valued processes $\{u_t\}_{0\leq t\leq T}$, which are progressively measurable and have real-valued versions, such that:

(a) For almost all $t \in [0,\ T]$, $u_t \in \mathbb{D}^{1,p}$.
(b) $\mathbb{E}[(\int_0^T |u_t|^2 \mathrm{d}t)^{\frac{p}{2}} + (\int_0^T \int_0^T |D_\theta u_t|^2 \mathrm{d}\theta \mathrm{d}t)^{\frac{p}{2}}] < +\infty$.

Notice also that we can select a progressively measurable version of the H-valued process $\{D.u_t\}_{0\leq t\leq T}$.

2.2. Numerical Schemes for Lipschitz BSDEs

In this subsection, we recall the most important results established in [18] concerning the numerical schemes and their rate of convergence in the case where the generator is global Lipschitz. For this purpose, we consider the following BSDE

$$\bar{Y}_t = \bar{\xi} + \int_t^T \bar{h}(r, \bar{Y}_r, \bar{Z}_r)\mathrm{d}r - \int_t^T \bar{Z}_r \mathrm{d}W_r, \quad 0 \leq t \leq T, \tag{7}$$

where $\bar{h}$:$[0,\ T] \times \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given generator and $\bar{\xi}$ the terminal condition. Note that if for any $q \geq 2$, $\bar{\xi} \in L^q(\Omega)$ and $\bar{h}$ is a uniformly Lipshitz function, then Lemma 2.2 in [18] shows the existence of a unique solution $(\bar{Y}, \bar{Z}) \in S^q_{\mathcal{F}}([0,\ T]) \times \mathcal{H}^q_{\mathcal{F}}([0,\ T])$ to BSDE (7).

Throughout this section, we will make use of the following important assumptions:

Assumption 1. *Fix $q > 4$ and $2 \leq p < \frac{q}{2}$*

(1.i) $\bar{\xi} \in \mathbb{D}^{2,q}$ *and satisfies*

$$\mathbb{E}|D_\theta\bar{\xi} - D_{\theta'}\bar{\xi}|^p \leq L|\theta - \theta'|^{\frac{p}{2}}, \tag{8}$$

$$\sup_{0\leq\theta\leq T} \mathbb{E}|D_\theta\bar{\xi}|^q < +\infty \tag{9}$$

and

$$\sup_{0\leq\theta\leq T}\ \sup_{0\leq u\leq T}\ \mathbb{E}|D_u D_\theta \bar{\xi}|^q < +\infty. \tag{10}$$

where $L > 0$ is a constant and $\theta,\ \theta' \in [0,\ T]$.

(1.ii) $\bar{h}$ has continuous and uniformly bounded first- and second-order partial derivatives with respect to $\bar{y}$ and $\bar{z}$, and $\bar{h}(\cdot,\ 0, 0) \in \mathcal{H}^q_{\mathcal{F}}([0,\ T])$.

(1.iii) $\bar{\xi}$ and $\bar{h}$ satisfy respectively the conditions (1.i) and (1.ii). Let $(\bar{Y}, \bar{Z})$ be the unique solution of (7) with terminal value $\bar{\xi}$ and generator $\bar{h}$ such that $\bar{h}(t,\ \bar{Y}_t, \bar{Z}_t)$, $\partial_{\bar{y}}\bar{h}(t,\ \bar{Y}_t, \bar{Z}_t)$ and

$\partial_{\bar{z}}\bar{h}(t,\bar{Y}_t,\bar{Z}_t)$ belong to $\mathbb{L}_a^{1,q}$ and $D.\bar{h}(t,\ \bar{Y}_t,\bar{Z}_t)$, $D.\partial_{\bar{y}}\bar{h}(t,\ \bar{Y}_t,\bar{Z}_t)$ and $D.\partial_{\bar{z}}\bar{h}(t,\ \bar{Y}_t,\bar{Z}_t)$ satisfy

$$\sup_{0\le\theta\le T}\mathbb{E}\left(\int_\theta^T\left|D_\theta\bar{h}(t,\bar{Y}_t,\bar{Z}_t)\right|^2\mathrm{d}t\right)^{\frac{q}{2}}\ <\ +\infty, \tag{11}$$

$$\sup_{0\le\theta\le T}\mathbb{E}\left(\int_\theta^T\left|D_\theta\partial_{\bar{y}}\bar{h}(t,\bar{Y}_t,\bar{Z}_t)\right|^2\mathrm{d}t\right)^{\frac{q}{2}}\ <\ +\infty, \tag{12}$$

$$\sup_{0\le\theta\le T}\mathbb{E}\left(\int_\theta^T\left|D_\theta\partial_{\bar{z}}\bar{h}(t,\bar{Y}_t,\bar{Z}_t)\right|^2\mathrm{d}t\right)^{\frac{q}{2}}\ <\ +\infty. \tag{13}$$

There exists $L>0$ such that for any $t\in(0,T]$ and for any $0\le\theta,\theta'\le t\le T$

$$\mathbb{E}\left[\left(\int_t^T\left|D_\theta\bar{h}(t,\bar{Y}_t,\bar{Z}_t)-D_{\theta'}\bar{h}(t,\bar{Y}_t,\bar{Z}_t)\right|^2\mathrm{d}t\right)^{\frac{p}{2}}\right]\le L|\theta-\theta|^{\frac{p}{2}}. \tag{14}$$

For each $\theta\in[0,\ T]$, $D_\theta\bar{h}(\cdot,\ \bar{Y}_t,\bar{Z}_t)\in\mathbb{L}_a^{1,q}$ and it has continuous partial derivatives with respect to $\bar{y}$, $\bar{z}$, which are denoted by $\partial_{\bar{y}}D_\theta\bar{h}(\cdot,\ \bar{Y}_t,\bar{Z}_t)$ and $\partial_{\bar{z}}D_\theta\bar{h}(\cdot,\ \bar{Y}_t,\bar{Z}_t)$ and the Malliavin derivative $D_uD_\theta\bar{h}(\cdot,\bar{Y}_t,\bar{Z}_t)$ satisfies

$$\sup_{0\le\theta\le T}\sup_{0\le u\le T}\mathbb{E}\left(\int_{\theta\vee u}^T\left|D_uD_\theta\bar{h}(t,\bar{Y}_t,\bar{Z}_t)\right|^2\mathrm{d}t\right)^{\frac{q}{2}}<+\infty. \tag{15}$$

In this section and in the rest of the paper, we let $\pi=\{0=t_0<t_1<\ldots<t_n=T\}$ stand for an arbitrary partition of the interval $[0,\ T]$ and $|\pi|=\max_{0\le i\le n-1}|t_{i+1}-t_i|$ and we denote $\Delta_i=t_{i+1}-t_i$, $0\le i\le n-1$.

◁ **Explicit scheme:** An explicit scheme has been presented in [18], where the approximate pairs $(\bar{Y}^\pi,\bar{Z}^\pi)$ are defined as follows

$$\begin{cases}\bar{Y}^\pi_{t_n} &= \bar{\xi}^\pi,\quad \bar{Z}^\pi_{t_n}=0,\\ \bar{Y}^\pi_t &= \bar{Y}^\pi_{t_{i+1}}+\bar{h}\left(t_{i+1},\bar{Y}^\pi_{t_{i+1}},\mathbb{E}\left[\frac{1}{\Delta_{i+1}}\int_{t_{i+1}}^{t_{i+2}}\bar{Z}^\pi_r\mathrm{d}r\mid\mathcal{F}_{t_{i+1}}\right]\right)\Delta_i\\ &\quad -\int_t^{t_{i+1}}\bar{Z}^\pi_r\mathrm{d}W_r,\qquad t\in[t_i,t_{i+1}),\end{cases} \tag{16}$$

$i=n-1,n-2,\ldots,0$, where we have by convention,

$$\mathbb{E}\left[\frac{1}{\Delta_{i+1}}\int_{t_{i+1}}^{t_{i+2}}\bar{Z}^\pi_r\mathrm{d}r\mid\mathcal{F}_{t_{i+1}}\right]=0\ \text{ for } i=n-1.$$

Proposition 1. *Consider the explicit scheme (16). Assume that Assumption 1 holds true and the partition π satisfies*

$$\max_{0\le i\le n-1}\frac{\Delta_i}{\Delta_{i+1}}\le L_1,$$

where L_1 is a constant. Assume that a constant $L_2>0$ exists such that

$$\left|\bar{h}(t_2,\bar{y},\bar{z})-\bar{h}(t_1,\bar{y},\bar{z})\right|\le L_2|t_2-t_1|^{\frac{1}{2}} \tag{17}$$

for all t_1, $t_2 \in [0, T]$ and $\bar{y}$, $\bar{z} \in \mathbb{R}$. Then, there exist two positive constants δ and K which are independent from π, such that, if $|\pi| < \delta$, then

$$\mathbb{E} \sup_{0\le t\le T} |\bar{Y}_t - \bar{Y}_t^\pi|^2 + \int_0^T \mathbb{E}|\bar{Z}_t - \bar{Z}_t^\pi|^2 \mathrm{d}t \le K\Big(|\pi| + \mathbb{E}|\bar{\xi} - \bar{\xi}^\pi|^2\Big).$$

$\triangleleft$ **Implicit scheme:** We also recall the numerical scheme in the implicit case, the approximating pair $(\bar{Y}^\pi, \bar{Z}^\pi)$ is defined recursively by

$$\begin{cases} \bar{Y}_{t_n}^\pi & = \bar{\xi}^\pi \\ \bar{Y}_t^\pi & = \bar{Y}_{t_{i+1}}^\pi + \bar{h}\left(t_{i+1}, \bar{Y}_{t_{i+1}}^\pi, \frac{1}{\Delta_i}\int_{t_i}^{t_{i+1}} \bar{Z}_r^\pi \mathrm{d}r\right)\Delta_i - \int_t^{t_{i+1}} \bar{Z}_r^\pi \mathrm{d}W_r, \end{cases} \tag{18}$$

for $t \in [t_i, t_{i+1})$, $i = n-1, n-2, \ldots, 0$, where $\bar{\xi}^\pi$ is an approximation of the terminal value $\bar{\xi}$.

Proposition 2. *Assume that $\bar{h}$ satisfies condition (17) in Proposition 1. If Assumption 1 holds true and $\bar{\xi}^\pi \in L^p(\omega)$, then there exist two positive constants δ and K which are independent from π, such that, whenever $|\pi| < \delta$, we have*

$$\mathbb{E} \sup_{0\le t\le T} |\bar{Y}_t - \bar{Y}_t^\pi|^p + \mathbb{E}\left(\int_0^T |\bar{Z}_t - \bar{Z}_t^\pi|^2 \mathrm{d}t\right)^{\frac{p}{2}} \le K\Big(|\pi|^{\frac{p}{2}} + \mathbb{E}|\xi - \bar{\xi}^\pi|^p\Big).$$

$\triangleleft$ **Totally discrete scheme**: In addition to the two aforementioned types of schemes discussed in [18], the authors propose a totally discrete scheme in the case where the generator $\bar{h}$ takes the following linear form:

$$\bar{h}(t, \bar{y}, \bar{z}) = g(t)\bar{y} + h(t)\bar{z} + f_1(t), \tag{19}$$

where the functions g, h are bounded and $f_1 \in L^2([0,T])$ and the following assumptions are in force:

(H1) $\bar{h}$ is deterministic, which implies $D_\theta \bar{h}(t, \bar{y}, \bar{z}) = 0$.

(H2) The functions g, h, and f_1 are $\frac{1}{2}$-Hölder continuous in t.

(H3) $\mathbb{E}\sup_{0\le\theta\le T} |D_\theta \bar{\xi}|^r < +\infty$, for all $r \ge 1$.

From (7), $\{D_\theta \bar{Y}_t\}_{0\le\theta\le t\le T}$ can be represented as

$$D_\theta \bar{Y}_t = \mathbb{E}\left[\rho_{t,T} D_\theta \bar{\xi} + \int_t^T \rho_{t,r} D_\theta \bar{h}(r, \bar{Y}_r, \bar{Z}_r)\mathrm{d}r \mid \mathcal{F}_t\right], \tag{20}$$

where

$$\rho_{t,r} = \exp\left\{\int_t^r \beta_s \mathrm{d}W_s + \int_t^r \left(h_1(s) - \frac{1}{2}\beta_s^2\right)\mathrm{d}s\right\}, \tag{21}$$

with $\alpha_s = \partial_{\bar{y}} \bar{h}(s, \bar{Y}_s, \bar{Z}_s)$ and $\beta_s = \partial_{\bar{z}} \bar{h}(s, \bar{Y}_s, \bar{Z}_s)$.

Using $\bar{Z}_t = D_t \bar{Y}_t$, $\mu \otimes \mathrm{P}$ a.e., from (7), (20) and (21), we define recursively

$$\begin{cases} \bar{Y}_t^\pi & = \bar{\xi}^\pi, \quad \bar{Z}_{t_n}^\pi = D_T \bar{\xi} \\ \bar{Y}_{t_i}^\pi & = \mathbb{E}\left[\bar{Y}_{t_{i+1}}^\pi + \bar{h}\left(t_{i+1}, \bar{Y}_{t_{i+1}}^\pi, \bar{Z}_{t_{i+1}}^\pi\right)\Delta_i \mid \mathcal{F}_{t_i}\right] \\ \bar{Z}_{t_i}^\pi & = \mathbb{E}\left[\rho_{t_{i+1},t_n}^\pi D_{t_i}\bar{\xi} + \sum_{k=i}^{n-1} \rho_{t_{i+1},t_{k+1}}^\pi D_{t_i}\bar{h}\left(t_{k+1}, \bar{Y}_{t_{k+1}}^\pi, \bar{Z}_{t_{k+1}}^\pi\right)\Delta_k \mid \mathcal{F}_{t_i}\right], \end{cases} \tag{22}$$

$i = n-1, n-2, \ldots, 0$, such that $\rho^{\pi}_{t_i,t_i} = 1$, $i = 0, 1, \ldots, n$ and for $0 \le i < j \le n$,

$$\rho^{\pi}_{t_i,t_j} = \exp\Bigg\{\sum_{k=i}^{j-1}\int_{t_k}^{t_{k+1}} \partial_{\bar{z}}\bar{h}\Big(r, \bar{Y}^{\pi}_{t_k}, \bar{Z}^{\pi}_{t_k}\Big)\mathrm{d}W_r + \sum_{k=i}^{j-1}\int_{t_k}^{t_{k+1}}\bigg(\partial_{\bar{y}}\bar{h}\Big(r, \bar{Y}^{\pi}_{t_k}, \bar{Z}^{\pi}_{t_k}\Big) - \frac{1}{2}\Big[\partial_{\bar{z}}\bar{h}\Big(r, \bar{Y}^{\pi}_{t_k}, \bar{Z}^{\pi}_{t_k}\Big)\Big]^2\bigg)\mathrm{d}r\Bigg\}. \tag{23}$$

Proposition 3. *Let $\bar{\xi}$ satisfy (1.i) in Assumption 1 and $\bar{h}$ be a linear function taking the form (19). Then, under assumptions (H1)–(H3) there exist two positive constants δ and K which are independent from π, such that when $|\pi| < \delta$ we have*

$$\mathbb{E}\max_{0\le i\le n}\Big\{\Big|\bar{Y}_{t_i} - \bar{Y}^{\pi}_{t_i}\Big|^p + \Big|\bar{Z}_{t_i} - \bar{Z}^{\pi}_{t_i}\Big|^p\Big\} \le K|\pi|^{\frac{p}{2} - \frac{p}{\left(2\ln\frac{1}{|\pi|}\right)}}\left(\ln\frac{1}{|\pi|}\right)^{\frac{p}{2}}.$$

3. Malliavin Regularity for QBSDE

In this section, we shall study the regularity in the sense of stochastic calculus of variations for solutions of a class of singular BSDEs that show quadratic growth on the variable z.

Assumption 2. *Fix $q \ge 2$.*

(2.i) $h(\cdot, 0) \in \mathcal{H}^q_{\mathcal{F}}([0,\ T])$ *and h is bounded and uniformly Lipschitz in y.*
(2.ii) $f : \mathbb{R} \longrightarrow \mathbb{R}$ *is a given integrable function.*
(2.iii) ξ is q-integrable.
(2.iv) There exists a constant $L > 0$ such that

$$|h(t_2, y) - h(t_1, y)| \le L|t_2 - t_1|^{\frac{1}{2}}.$$

We recall the following Lemma which will be used frequently in the sequel.

Lemma 1. *The function F defined for every $x \in \mathbb{R}$, by*

$$F(x) = \int_0^x \exp\left(2\int_0^y f(t)\mathrm{d}t\right)\mathrm{d}y$$

has the following property:
(i) F and F^{-1} are quasi-isometry; that is, for any $x,\ y \in \mathbb{R}$ and $|f|_1 = \int_{\mathbb{R}}|f(x)|\mathrm{d}x$

$$\begin{array}{ccccc} e^{-2|f|_1}|x-y| & \le & |F(x) - F(y)| & \le & e^{2|f|_1}|x-y|, \\ e^{-2|f|_1}|x-y| & \le & \left|F^{-1}(x) - F^{-1}(y)\right| & \le & e^{2|f|_1}|x-y|. \end{array}$$

Proof. It suffices to observe that $F'(x) = \exp(2\int_0^x f(t)\mathrm{d}t)$; hence, for all $x \in \mathbb{R}$

$$m =: e^{-2|f|_1} \le F'(x) \le e^{2|f|_1} := M \text{ and } m \le (F^{-1})'(x) \le M.$$

□

3.1. A Priori Estimates

Proposition 4. *Let $\xi \in L^q(\Omega)$. If (Y, Z) satisfies QBSDE (1), then we have:*
(i) $(Z_r)_{0\le r\le T} \in \mathcal{H}^q_{\mathcal{F}}([0,\ T])$,
(ii) $(Y_r)_{0\le r\le T} \in \mathcal{S}^q_{\mathcal{F}}([0,\ T])$,
(iii) $\mathbb{E}|\int_0^T(h(r, Y_r) + h_1(r)Z_r + f(Y_r)|Z_r|^2)\mathrm{d}r|^q$ *is finite.*

Proof of (i) and (ii). Let $(\bar{Y}_t, \bar{Z}_t)$ be a solution to BSDE (7). From Proposition 5.1 in [4] with $(f^1, \xi^1) = (\bar{h}, \bar{\xi})$ and $(f^2, \xi^2) = (0, 0)$, $(\bar{Y}_t, \bar{Z}_t) \in \mathcal{S}^q_{\mathcal{F}}([0,\ T]) \times \mathcal{H}^q_{\mathcal{F}}([0,\ T])$. Furthermore, we have the following estimate

$$\mathbb{E} \sup_{0\leq t\leq T} |\bar{Y}_t|^q + \mathbb{E}\left(\int_0^T |\bar{Z}_t|^2 dt\right)^{\frac{q}{2}} \leq K\left(\mathbb{E}|\bar{\xi}|^q + \mathbb{E}\left(\int_0^T |h(t,0)|^2 dt\right)^{\frac{q}{2}}\right).$$

Since $m|x-y| \leq |F(x) - F(y)|$ and $m \leq F'(x)$, we have

$$\begin{aligned} & m^q \mathbb{E}\left[\sup_{0\leq t\leq T} |Y_t|^q + \left(\int_0^T |Z_t|^2 dt\right)^{\frac{q}{2}}\right] \\ \leq\ & \mathbb{E} \sup_{0\leq t\leq T} |F(Y_t) - F(0)|^q + \mathbb{E}\left(\int_0^T \left|Z_t F'(Y_t)\right|^2 dt\right)^{\frac{q}{2}} \\ \leq\ & \mathbb{E} \sup_{0\leq t\leq T} |\bar{Y}_t|^q + \mathbb{E}\left(\int_0^T |\bar{Z}_t|^2 dt\right)^{\frac{q}{2}} \\ \leq\ & K\left(\mathbb{E}|\bar{\xi}|^q + \mathbb{E}\left(\int_0^T |h(t,0)|^2 dt\right)^{\frac{q}{2}}\right). \end{aligned}$$

□

Proof of (iii). Since $(Y_., Z_.)$ satisfies QBSDE (1),

$$\int_0^T \left(h(r, Y_r) + h_1(r) Z_r + f(Y_r)|Z_r|^2\right) dr = \int_0^T Z_r dW_r + Y_0 - \xi.$$

Now, taking the expectation and using Burkholder–Davis–Gundy inequality, we obtain

$$\begin{aligned} & \mathbb{E}\left|\int_0^T \left(h(r, Y_r) + h_1(r) Z_r + f(Y_r)|Z_r|^2\right) dr\right|^q \\ \leq\ & C\left(\mathbb{E}\left|\int_0^T Z_r dW_r\right|^q + \mathbb{E}(|Y_0|^q + |\xi|^q)\right) \\ \leq\ & C\left(\mathbb{E}|\xi|^q + |Y_0|^q + C_p \mathbb{E}\left(\int_0^T |Z_r|^2 dr\right)^{\frac{q}{2}}\right). \end{aligned}$$

Finally,

$$\mathbb{E}\left|\int_0^T (h(r, Y_r) + h_1(r) Z_r + f(Y_r)|Z_r|^2) dr\right|^q$$

is finite thanks to (i) and (ii). □

3.2. $\mathbb{L}^q$ $(q \geq 2)$ Solutions of QBSDE

The objective of this subsection is to establish an existence and uniqueness result to QBSDE (1) by performing Zvonkin's transformation as mentioned in [12].

Theorem 1. *For any $q \geq 2$, assume that (2.i), (2.ii) and (2.iii) in Assumption 2 are in force. Then, QBSDE (1) has a unique solution that belongs to $\mathcal{S}^q_{\mathcal{F}}([0,\ T]) \times \mathcal{H}^q_{\mathcal{F}}([0,\ T])$.*

Proof. The function F defined for every $x \in \mathbb{R}$ by

$$F(x) = \int_0^x \exp\left(2\int_0^y f(t) dt\right) dy \tag{24}$$

satisfies

$$F''(x) - 2f(x)F'(x) = 0, \text{ for a.e. } x \in \mathbb{R}.$$

It was shown in [12], that both F and its inverse are global Lipschitz, one to one and $\mathcal{C}^2$ functions from $\mathbb{R}$ onto $\mathbb{R}$.

If (Y, Z) is a solution to QBSDE (1), then Itô-Krylov's formula applied to $F(Y_t)$ shows that

$$\begin{aligned} \mathrm{d}F(Y_t) &= F'(Y_t)\mathrm{d}Y_t + \frac{1}{2}F''(Y_t)\mathrm{d}\langle Y.\rangle_t \\ &= -F'(Y_t)(h(t, Y_t) + h_1(t)Z_t)\mathrm{d}t + Z_t F'(Y_t)\mathrm{d}W_t \\ &\quad + \Big(-F'(Y_t)f(Y_t) + \frac{1}{2}F''(Y_t)\Big)|Z_t|^2\mathrm{d}t, \end{aligned}$$

since

$$-F'(x)f(x) + \frac{1}{2}F''(x) = 0, \tag{25}$$

we obtain

$$\begin{aligned} F(Y_t) &= F(\xi) - \int_t^T Z_s F'(Y_s)\mathrm{d}W_s \\ &\quad + \int_t^T \big(h(s, Y_s)F'(Y_s) + h_1(s)F'(Y_s)Z_s\big)\mathrm{d}s. \end{aligned}$$

If we set

$$F(Y_t) = \bar{Y}_t \text{ then } Y_t = F^{-1}(\bar{Y}_t),\ \ Z_t F'(Y_t) = \bar{Z}_t \text{ then } Z_t = \frac{\bar{Z}_t}{F'(Y_t)}, \text{ and } F(\xi) = \bar{\xi}. \tag{26}$$

BSDE (2) becomes:

$$\bar{Y}_t = \bar{\xi} + \int_t^T \Big(h(s,\ F^{-1}(\bar{Y}_s))F'(F^{-1}(\bar{Y}_s)) + h_1(s)\bar{Z}_s\Big)\mathrm{d}s - \int_t^T \bar{Z}_s \mathrm{d}W_s. \tag{27}$$

To simplify statement (27), we use the following notation

$$\bar{h}(s, y, z) = h(s,\ F^{-1}(y))F'(F^{-1}(y)) + h_1(s)z. \tag{28}$$

Then, we have

$$\bar{Y}_t = \bar{\xi} + \int_t^T \bar{h}(s, \bar{Y}_s, \bar{Z}_s)\mathrm{d}s - \int_t^T \bar{Z}_s \mathrm{d}W_s. \tag{29}$$

Since $\bar{\xi}$, $\bar{h}$ satisfy the conditions of Lemma 2.2 in [18], there exists a unique solution pair $(\bar{Y}, \bar{Z}) \in \mathcal{S}^q_{\mathcal{F}}([0,\ T]) \times \mathcal{H}^q_{\mathcal{F}}([0,\ T])$ to (29).

Now, we will find (1) from (29). By applying Itô-Krylov's formula to $F^{-1}(\bar{Y}_t)$, we obtain

$$\mathrm{d}F^{-1}(\bar{Y}_t) = (F^{-1})'(\bar{Y}_t)\mathrm{d}\bar{Y}_t + \frac{1}{2}(F^{-1})''(\bar{Y}_t)\mathrm{d}\langle \bar{Y}.\rangle_t,$$

we know that

$$(F^{-1})'(\bar{Y}_t) = \frac{1}{F'(F^{-1}(\bar{Y}_t))} \ \text{ and } \ (F^{-1})''(\bar{Y}_t) = \frac{-F''(F^{-1}(\bar{Y}_t))}{(F'(F^{-1}(\bar{Y}_t)))^3}. \tag{30}$$

Using notations (26), we have

$$\mathrm{d}Y_t = -(h(t, Y_t) + h_1(t)Z_t + f(Y_t)Z_t^2)\mathrm{d}t + Z_t \mathrm{d}W_t \ \text{ and } \ Y_T = F^{-1}(\bar{Y}_T) = \xi.$$

Then, we obtain (1), thanks to (26) and through easy calculations one can check that $(Y, Z) \in \mathcal{S}^q_{\mathcal{F}}([0,\ T]) \times \mathcal{H}^q_{\mathcal{F}}([0,\ T])$ is the unique solution to (1). □

3.3. $\mathbb{L}^p$-Hölder Continuity of the Solutions of QBSDEs ($2 \le p < \frac{q}{2}$)

In this context, we can obtain some estimates for solutions to QBSDE (1). To begin with, let $(\bar{Y}, \bar{Z})$ be the unique solution of BSDE (5) associated to QBSDE (1).

Below, we specify some assumptions on the coefficients.

Assumption 3. *Fix $q > 4$ and $2 \le p < \frac{q}{2}$.*

(3.i) ξ satisfies (1.i) in Assumption 1,

(3.ii) The first- and second-order partial derivatives of h are continuous and uniformly bounded with respect to y and $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuously differentiable function such that f and f' are bounded functions.

(3.iii) $h(\cdot, Y_t)$ and $\partial_y h(\cdot, Y_t)$ belong to $\mathbb{L}_a^{1,q}$ and we have

$$\sup_{0\le\theta\le T} \mathbb{E}\left(\int_\theta^T |D_\theta h(t, Y_t|^2 \mathrm{d}t\right)^{\frac{q}{2}} < +\infty,$$

$$\sup_{0\le\theta\le T} \mathbb{E}\left(\int_\theta^T \left|D_\theta \partial_y h(t, Y_t)\right|^2 \mathrm{d}t\right)^{\frac{q}{2}} < +\infty,$$

and there exists $L > 0$ such that for any $t \in (0, T]$ and for any $0 \le \theta, \theta' \le t \le T$

$$\mathbb{E}\left(\int_t^T |D_\theta h(r, Y_r) - D_{\theta'} h(r, Y_r)|^2 \mathrm{d}t\right)^{\frac{p}{2}} \le L\left|\theta - \theta'\right|^{\frac{p}{2}}.$$

(3.iv) For each $\theta \in [0,\ T]$, $D_\theta h(\cdot, Y_t) \in \mathbb{L}_a^{1,q}$ and it has continuous partial derivative with respect to y, which is denoted by $\partial_y D_\theta h(\cdot, Y_t)$ and the Malliavin derivatives $D_u D_\theta h(\cdot, Y_t)$ satisfy

$$\sup_{0\le\theta\le T} \sup_{0\le u\le T} \mathbb{E}\left(\int_{\theta\vee u}^T |D_u D_\theta h(t, Y_t)|^2 \mathrm{d}t\right)^{\frac{q}{2}} < +\infty.$$

Remark 1. *Notice that Assumption 3 and (2.i), (2.iii) in Assumption 2 imply that the terminal value $\bar{\xi}$ and the generator $\bar{h}$ satisfy Assumption 1.*

Remark 2. *From Theorem 2.6 in [18], we know that $\left\{(D_\theta \bar{Y}_t, D_\theta \bar{Z}_t)\right\}_{0\le\theta\le t\le T}$ satisfies the following linear BSDE*

$$\begin{aligned} D_\theta \bar{Y}_t &= D_\theta \bar{\xi} - \int_t^T D_\theta \bar{Z}_r \mathrm{d}W_r \\ &\quad + \int_t^T \left[\partial_{\bar{y}} \bar{h}(r, \bar{Y}_r, \bar{Z}_r) D_\theta \bar{Y}_r + \partial_{\bar{z}} \bar{h}(r, \bar{Y}_r, \bar{Z}_r) D_\theta \bar{Z}_r + D_\theta \bar{h}(r, \bar{Y}_r, \bar{Z}_r)\right] \mathrm{d}r \end{aligned} \tag{31}$$

and $\bar{Z}_t = D_t \bar{Y}_t$, $\mu \otimes$ P-a.e. Moreover, since $\bar{\xi}$ and $\bar{h}$ satisfy (9) and (11) in Assumption 1, then, from estimate (2.2) in (Lemma 2.2 in [18]), we deduce that

$$\mathbb{E} \sup_{0\le t\le T} |\bar{Z}_t|^q < \infty \tag{32}$$

for any $q \ge 2$.

Lemma 2. *For any $2 \le p < \frac{q}{2}$, let Assumption 3, (2.i) and (2.iii) in Assumption 2 be satisfied then, there exists a constant K such that, for any $s, t \in [0,\ T]$, we have:*

(i) $\mathbb{E}|Y_t - Y_s|^p \le K|t - s|^{\frac{p}{2}}$,

(ii) $\mathbb{E}|Z_t - Z_s|^p \le K|t - s|^{\frac{p}{2}}$,

(iii) *For any partition* $\pi = \{0 = t_0 < t_1 < \ldots < t_n = T\}$ *of the interval* $[0,\ T]$*, we have*

$$\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \mathbb{E}\Big[|Z_t - Z_{t_i}|^2 + |Z_t - Z_{t_{i+1}}|^2\Big] dt \leq K|\pi|,$$

where $|\pi| = \max_{0\leq i\leq n-1}(t_{i+1} - t_i)$ *and* K *is a constant independent of the partition* π*.*

Proof. Due to the fact that $\bar{\xi}$ and $\bar{h}$ satisfy Assumption 1, then, thanks to Theorem 2.6 and corollary 2.7 in [18] and with Hölder's inequality, there exists a constant K, that may change from line to line, such that for any $s, t \in [0,\ T]$

$$\mathbb{E}|\bar{Y}_t - \bar{Y}_s|^p \leq K|t - s|^{\frac{p}{2}}, \tag{33}$$

and

$$\mathbb{E}|\bar{Z}_t - \bar{Z}_s|^p \leq K|t - s|^{\frac{p}{2}}. \tag{34}$$

Now, we proceed to prove the estimates (i) and (ii).
Since F^{-1} is Lipschitz and by using the previous result (33), the following estimate holds for all $s,\ t \in [0,\ T]$,

$$\mathbb{E}|Y_t - Y_s|^p = \mathbb{E}\Big|F^{-1}(\bar{Y}_t) - F^{-1}(\bar{Y}_s)\Big|^p \leq K|t - s|^{\frac{p}{2}}. \tag{35}$$

where K is a positive constant.

Using the fact that both F' and F^{-1} are Lipschitz functions, $m \leq F' \leq M$, we obtain:

$$\begin{aligned}
|Z_t - Z_s| &= \left|\frac{\bar{Z}_t}{F'(Y_t)} - \frac{\bar{Z}_s}{F'(Y_t)} + \frac{\bar{Z}_s}{F'(Y_t)} - \frac{\bar{Z}_s}{F'(Y_s)}\right| \\
&\leq \frac{1}{m}|\bar{Z}_t - \bar{Z}_s| + \frac{1}{m^2}\left(|\bar{Z}_s||F'(Y_s) - F'(Y_t)|\right) \\
&\leq \frac{1}{m}|\bar{Z}_t - \bar{Z}_s| + \frac{1}{m^2}\Big(|\bar{Z}_s|\Big|F'(F^{-1}(\bar{Y}_t)) - F'(F^{-1}(\bar{Y}_s))\Big|\Big) \\
&\leq \frac{1}{m}|\bar{Z}_t - \bar{Z}_s| + \frac{L}{m^2}\left(|\bar{Z}_s||\bar{Y}_t - \bar{Y}_s|\right),
\end{aligned}$$

where L is the Lipschitz constant of $F'(F^{-1}(\cdot))$.
For $\alpha = \frac{q}{p}$ and $\beta = \frac{q}{q-p}$, such that, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, using Hölder inequality and taking into account the relations (32)–(34), we obtain (ii).

Finally, (iii) is a simple consequence of (ii) with $p = 2$. □

3.4. Smoothness of Solutions of QBSDEs

In this section, we will present some important results concerning QBSDEs, the natural tool we will use is the Malliavin calculus, and previous results for the regularity of the associated global Lipschitz BSDE (29).

Theorem 2. *Let* $(Y_., Z_.) \in \mathcal{S}^q_{\mathcal{F}}([0,\ T]) \times \mathcal{H}^q_{\mathcal{F}}([0,\ T])$ *be the unique solution to QBSDE* (1) *for any* $q \geq 4$*, then*

(i) $\mathbb{E}\sup_{0\leq t\leq T}|Z_t|^q < +\infty$,

(ii) Y *belongs to* $\mathbb{L}^{1,q}_a$ *and* Z *belongs to* $\mathbb{L}^{1,\frac{q}{2}}_a$,

(iii) $Z_t = D_t Y_t$, $\mu \otimes$ P*-a.e.*

(iv) $\sup_{0\leq\theta\leq T}\{\mathbb{E}\sup_{0\leq t\leq T}|D_\theta Y_t|^q + \mathbb{E}(\int_\theta^T |D_\theta Z_t|^2 dt)^{\frac{q}{4}}\} < +\infty.$

Proof of (i). Keeping in mind that F' is bounded and (32), then (i) follows immediately from the fact that $Z_t = \frac{\bar{Z}_t}{F'(F^{-1}(\bar{Y}_t))}$. □

Proof of (ii). We will prove that Y, Z belongs to $\mathbb{L}_a^{1,q}$. To do that we should prove the following two assertions.

(a) Due to the Lipschitz continuity of F^{-1} and $\bar{Y} \in \mathbb{D}^{1,q}$, it is obvious that $Y \in \mathbb{D}^{1,q}$. Since $\bar{Z} \in \mathbb{D}^{1,q}$ and $F' \in \mathcal{C}^1(\mathbb{R})$ with bounded derivative and $Y \in \mathbb{D}^{1,q}$, then $Z \in \mathbb{D}^{1,q}$

(b) First, since (Y, Z) is a solution of QBSDE (1) then we have the following estimates

$$\mathbb{E}\left(\int_0^T |Y_t|^2 \mathrm{d}t\right)^{\frac{q}{2}} + \mathbb{E}\left(\int_0^T |Z_t|^2 \mathrm{d}t\right)^{\frac{q}{2}} < \infty.$$

We want to prove,

$$\mathbb{E}\left(\int_0^T \int_0^T |D_\theta Y_t|^2 \mathrm{d}\theta \mathrm{d}t\right)^{\frac{q}{2}} + \mathbb{E}\left(\int_0^T \int_0^T |D_\theta Z_t|^2 \mathrm{d}\theta \mathrm{d}t\right)^{\frac{q}{4}} < \infty.$$

Since $D_\theta Y_t = \dfrac{D_\theta \bar{Y}_t}{F'(Y_t)}$, $\bar{Y} \in \mathbb{L}_a^{1,q}$ and the relation $m \leq F'(x) \leq M$, we obtain

$$\mathbb{E}\left(\int_0^T \int_0^T |D_\theta Y_t|^2 \mathrm{d}\theta \mathrm{d}t\right)^{\frac{q}{2}} \leq \frac{1}{m^q} \mathbb{E}\left(\int_0^T \int_0^T |D_\theta \bar{Y}_t|^2 \mathrm{d}\theta \mathrm{d}t\right)^{\frac{q}{2}} < \infty.$$

A simple computation shows that

$$D_\theta Z_t = \frac{D_\theta \bar{Z}_t}{F'(Y_t)} - \frac{F''(Y_t)}{F'(Y_t)} Z_t D_\theta Y_t.$$

Thus by the Cauchy–Schwartz inequality and the fact that $\mathbb{E} \sup_{0 \leq t \leq T} |Z_t|^q$ is finite for any $q \geq 4$, we have

$$\mathbb{E}\left(\int_0^T \int_0^T |Z_t D_\theta Y_t|^2 \mathrm{d}\theta \mathrm{d}t\right)^{\frac{q}{4}} \leq \left(\mathbb{E} \sup_{0 \leq t \leq T} |Z_t|^q\right)^{\frac{1}{2}} \left(\mathbb{E}\left(\int_0^T \int_0^T |D_\theta Y_t|^2 \mathrm{d}\theta \mathrm{d}t\right)^{\frac{q}{2}}\right)^{\frac{1}{2}}.$$

and hence,

$$\begin{aligned} \mathbb{E}\left(\int_0^T \int_0^T |D_\theta Z_t|^2 \mathrm{d}\theta \mathrm{d}t\right)^{\frac{q}{4}} \leq\ & \frac{1}{m^{\frac{q}{2}}} \mathbb{E}\left(\int_0^T \int_0^T |D_\theta \bar{Z}_t|^2 \mathrm{d}\theta \mathrm{d}t\right)^{\frac{q}{4}} \\ & + \frac{M^{\frac{q}{2}}}{m^{\frac{q}{2}}} \left(\mathbb{E} \sup_{0 \leq t \leq T} |Z_t|^q\right)^{\frac{1}{2}} \left(\mathbb{E}\left(\int_0^T \int_0^T |D_\theta Y_t|^2 \mathrm{d}\theta \mathrm{d}t\right)^{\frac{q}{2}}\right)^{\frac{1}{2}}. \end{aligned}$$

Proof of (iii). $\{D_t \bar{Y}_t\}_{0 \leq t \leq T}$ gives a version of $\{\bar{Z}_t\}_{0 \leq t \leq T}$, namely, $\bar{Z}_t = D_t \bar{Y}_t$; then

$$Z_t F'(Y_t) = \bar{Z}_t = D_t \bar{Y}_t = D_t(F(Y_t)) = F^{'}(Y_t) D_t Y_t,$$

then, $Z_t = D_t Y_t$. □

Proof of (iv). We know that $\bar{\xi}$ and $\bar{h}$ satisfy conditions (9) and (11) and by invoking the estimate in Lemma 2.2 in [18], we obtain

$$\sup_{0 \leq \theta \leq T} \left\{ \mathbb{E} \sup_{0 \leq t \leq T} |D_\theta \bar{Y}_t|^q + \mathbb{E}\left(\int_\theta^T |D_\theta \bar{Z}_t|^2 \mathrm{d}t\right)^{\frac{q}{4}} \right\} < +\infty. \tag{36}$$

Using (36) and $m \leq F' \leq M$, we have

$$\begin{aligned}
\sup_{0\leq\theta\leq T} \mathbb{E} \sup_{0\leq t\leq T} |D_\theta Y_t|^q &\leq \sup_{0\leq\theta\leq T} \mathbb{E} \sup_{0\leq t\leq T} \left| \frac{1}{F'(Y_t)} D_\theta \bar{Y}_t \right|^q \\
&\leq \frac{1}{m^q} \sup_{0\leq\theta\leq T} \mathbb{E} \sup_{0\leq t\leq T} \left| D_\theta \bar{Y}_t \right|^q < +\infty.
\end{aligned}$$

The Cauchy–Schwartz inequality, F' and F'' are bounded functions and the fact that $\mathbb{E}\sup_{0\leq t\leq T}|Z_t|^q$ is finite, leads to

$$\begin{aligned}
&\sup_{0\leq\theta\leq T} \mathbb{E}\left(\int_\theta^T |D_\theta Z_t|^2 \mathrm{d}t\right)^{\frac{q}{4}} \\
\leq\ & \sup_{0\leq\theta\leq T} \left\{ \mathbb{E}\left(\int_\theta^T \left|\frac{D_\theta \bar{Z}_t}{F'(Y_t)}\right|^2 \mathrm{d}t\right)^{\frac{q}{4}} + \mathbb{E}\left(\int_\theta^T \left|\frac{F''(Y_t)}{F'(Y_t)} Z_t D_\theta Y_t\right|^2 \mathrm{d}t\right)^{\frac{q}{4}} \right\} \\
\leq\ & \frac{1}{m^{\frac{q}{2}}} \sup_{0\leq\theta\leq T} \mathbb{E}\left(\int_\theta^T |D_\theta \bar{Z}_t|^2 \mathrm{d}t\right)^{\frac{q}{4}} \\
& + \frac{M^{\frac{q}{2}}}{m^{\frac{q}{2}}} \sup_{0\leq\theta\leq T} \left\{ \left(\mathbb{E} \sup_{0\leq t\leq T} |Z_t|^q\right)^{\frac{1}{2}} \left(\mathbb{E}\left(\int_\theta^T |D_\theta Y_t|^2 \mathrm{d}t\right)^{\frac{q}{2}}\right)^{\frac{1}{2}} \right\} < +\infty.
\end{aligned}$$

□

Remark 3. *We would like to point out that the estimations for Y and $D_\theta Y$ in Theorem 2 hold also true for $q \geq 2$.*

In what follows, we shall adapt the examples studied in [18] to our setting of quadratic BSDEs for which Assumption 3 is satisfied.

Example 1. *Consider the QBSDE (1) with generator $h(t,y) + h_1(t)z + f(y)|z|^2$.*

(i) Assume that $h : [0, T] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a deterministic function twice continuously differentiable with uniformly bounded first- and second-order partial derivatives with respect to y and $\int_0^T |h(t,0)|^2 dt < +\infty$.

(ii) We define the terminal value ξ as the multiple stochastic integrals of the form

$$\xi = \int_{[0,\ T]^n} g(t_1, \cdots, t_n)\mathrm{d}W_{t_1} \cdots \mathrm{d}W_{t_n},$$

where $n \geq 2$ is an integer and $g(t_1, \ldots, t_n)$ is a symmetric function in $L^2([0,\ T]^n)$, such that

$$\begin{aligned}
D_u\xi &= n \int_{[0,\ T]^{n-1}} g(t_1, \cdots, t_{n-1}, u)\mathrm{d}W_{t_1} \cdots \mathrm{d}W_{t_{n-1}} \\
D_v D_u \xi &= n(n-1) \int_{[0,\ T]^{n-2}} g(t_1, \cdots, t_{n-2}, u, v)\mathrm{d}W_{t_1} \cdots \mathrm{d}W_{t_{n-2}}.
\end{aligned}$$

Then

$$\begin{aligned}
&\sup_{0\leq u\leq T} \mathbb{E}\left[|D_u\xi|^2\right] \\
\leq\ & n \sup_{0\leq u\leq T} \int_{[0,\ T]^{n-1}} (g(t_1, \cdots, t_{n-1}, u))^2 \mathrm{d}t_1 \cdots \mathrm{d}t_{n-1} < +\infty,
\end{aligned}$$

and

$$\sup_{0\le u\le T}\sup_{0\le v\le T}\mathbb{E}\Big[|D_uD_v\xi|^2\Big]$$
$$\le\quad n(n-1)\sup_{0\le u\le T}\sup_{0\le v\le T}\int_{[0,\ T]^{n-2}}(g(t_1,\cdots,t_{n-2},u,v))^2\mathrm{d}t_1\cdots\mathrm{d}t_{n-2}\ <\ +\infty,$$

Moreover, there is a constant $L>0$ such that for any $u,v\in[0,\ T]$

$$\int_{[0,\ T]^{n-1}}|g(t_1,\cdots,t_{n-1},u)-g(t_1,\cdots,t_{n-1},v)|^2\mathrm{d}t_1\cdots\mathrm{d}t_{n-1}\le L|u-v|.$$

Assumptions (i) and (ii) imply Assumption 3, and thus Z satisfies property (ii) of Lemma 2.

Example 2. *Consider the QBSDE (1) with generator $h(r,y)+h_1(r)z+f(y)|z|^2$.*
Let $\Omega=C_0([0,\ T])$ be the classical Wiener space equipped with the Borel σ-field and Wiener measure. Then, Ω is a Banach space with a uniform norm $\|\cdot\|_\infty$ and $W_t=\omega(t)$ is the canonical Wiener process:

(i) *Assume that $h:[0,\ T]\times\mathbb{R}\longrightarrow\mathbb{R}$ is a twice differentiable deterministic function such that their first- and second-order partial derivatives with respect to y are uniformly bounded and $\int_0^T h^2(t,0)dt<+\infty$.*
(ii) *We put $\xi=\varphi(W)$ such that $\varphi:\Omega\to\mathbb{R}$ is twice Fréchet differentiable, assuming further that the Fréchet derivatives $\delta\varphi$ and $\delta^2\varphi$ satisfy for all $\omega\in\Omega$ and two positive constants C_1 and C_2*

$$|\varphi(\omega)|+\|\delta\varphi(\omega)\|+\|\delta^2\varphi(\omega)\|\le C_1\exp\big(C_2\|\omega\|_\infty^r\big),\quad 0<r<2$$

where $\|\cdot\|$ stands for the total variation norm.
(iii) *We associate with $\delta\varphi$ and $\delta^2\varphi$ the signed measure λ on $[0,\ 1]$ and ν on $[0,\ 1]\times[0,\ 1]$, respectively; there exists a constant $L>0$ such that for all $0\le\theta\le\theta'\le1$, for some $p\ge2$*

$$\mathbb{E}\big|\lambda\big((\theta,\theta']\big)\big|^p\le L|\theta-\theta'|^{\frac{p}{2}},$$

we know that $D_\theta\xi=\lambda((\theta,1])$ and $D_uD_\theta\xi=\nu((\theta,1]\times(u,1])$. From (i), (ii), (iii) and Fernique's theorem, we can check that Assumption 3 is satisfied and therefore the Hölder continuity property of Z (ii) of Lemma 2 is established.

Example 3. *Consider the following quadratic forward–backward SDE*

$$\begin{cases} X_t &= X_0+\int_0^t b(r,X_r)\mathrm{d}r+\int_0^t\sigma(r,X_r)\mathrm{d}W_r,\\ Y_t &= \varphi\Big(\int_0^T X_r^2\mathrm{d}r\Big)-\int_t^T Z_r\mathrm{d}W_r\\ &\quad+\int_t^T\Big(h(r,X_r,Y_r)+h_1(r)Z_r+f(Y_r)|Z_r|^2\Big)\mathrm{d}r,\end{cases}\tag{37}$$

where b, σ, φ, h, and f are deterministic functions and $X_0\in\mathbb{R}$.
We make the following assumptions:

(i) *b and σ are twice differentiable and their first- and second-order partial derivatives with respect to x are uniformly bounded; in addition, there is a constant $L>0$, such that, for any $s,\ t\in[0,\ T]$, $x\in\mathbb{R}$*

$$|\sigma(t,x)-\sigma(s,x)|\le L|t-s|^{\frac{1}{2}}$$

(ii) $\sup_{0\le t\le T}\{|b(t,0)|+|\sigma(t,0)|\}<+\infty$.

(iii) *φ is twice differentiable and there exists a positive constant C and integer n such that*

$$\left|\varphi\left(\int_0^T x_t^2 \mathrm{d}t\right)\right| + \left|\varphi'\left(\int_0^T x_t^2 \mathrm{d}t\right)\right| + \left|\varphi''\left(\int_0^T x_t^2 \mathrm{d}t\right)\right| \leq C(1+\|x\|_\infty)^n,$$

where $\|x.\|_\infty = \sup_{0\leq t\leq T}|x_t|$, for any $x. \in \mathcal{C}([0,T])$.

(iv) *The first- and second-order partial derivatives of $h(t,\cdot,\cdot)$ with respect to x and y are continuous and uniformly bounded and $\int_0^T (h(t,0,0))^2 \mathrm{d}t < +\infty$.*

Under assumptions (i) and (iv), equation (37) has a unique solution triple (X,Y,Z). Moreover, the following results hold true; for any real number $r > 0$, there exists a constant $C > 0$ such that, for any $t,s \in [0,\ T]$

$$\mathbb{E}\sup_{0\leq t\leq T}|X_t|^r < +\infty, \quad \mathbb{E}|X_t - X_s|^r \leq C|t-s|^{\frac{r}{2}}. \tag{38}$$

For any fixed $y \in \mathbb{R}$, we have $D_\theta h(t,X_t,y) = \partial_x h(t,X_t,y) D_\theta X_t$. Then, keeping in mind all the assumptions in this example, by Theorem 2.2.1, Lemma 2.2.2 in [19], and the estimates in (38), one can check that the assertions of Assumption 3 are satisfied. Therefore, Z enjoys the Hölder continuity property (ii) of Lemma 2.

4. The Rate of Convergence of QBSDE

From different types of numerical schemes in both explicit (16) and implicit (18) cases for BSDE (7) with Lipschitz generator, we deduce the rate of convergence of the scheme related to QBSDE (1) in the L^p norm, for $p \geq 2$. We assume that h_1 in QBSDE (1) is a constant function.

4.1. An Explicit Scheme for QBSDE

From (29), we know that, when $t \in [t_i, t_{i+1}]$, the global Lipschitz BSDE associated to the QBSDE (1) is given by

$$\bar{Y}_t = \bar{Y}_{t_{i+1}} + \int_t^{t_{i+1}} \bar{h}(r,\bar{Y}_r,\bar{Z}_r)\mathrm{d}r - \int_t^{t_{i+1}} \bar{Z}_r \mathrm{d}W_r.$$

In the same way as defining the alternative scheme (16), the approximating pairs $(\bar{Y}^\pi, \bar{Z}^\pi)$ of the above BSDE are defined recursively by

$$\begin{cases} \bar{Y}^\pi_{t_n} &= \bar{\xi}^\pi, \quad \bar{Z}^\pi_{t_n} = 0 \\ \bar{Y}^\pi_t &= \bar{Y}^\pi_{t_{i+1}} - \int_t^{t_{i+1}} \bar{Z}^\pi_r \mathrm{d}W_r \\ & \quad + h\left(t_{i+1}, F^{-1}\left(\bar{Y}^\pi_{t_{i+1}}\right)\right) F'\left(F^{-1}\left(\bar{Y}^\pi_{t_{i+1}}\right)\right)\triangle_i \\ & \quad + h_1(t_{i+1})\mathbb{E}\left[\frac{1}{\triangle_{i+1}}\int_{t_{i+1}}^{t_{i+2}} \bar{Z}^\pi_r \mathrm{d}r \mid \mathcal{F}_{t_{i+1}}\right]\triangle_i \end{cases} \tag{39}$$

for $t \in [t_i, t_{i+1})$, $i = n-1,\ n-2,\ldots,\ 0$ and $\bar{\xi}^\pi \in L^2(\omega)$ is an approximation of the final condition $\bar{\xi}$ and by convention

$$\mathbb{E}\left[\frac{1}{\triangle_{i+1}}\int_{t_{i+1}}^{t_{i+2}} \bar{Z}^\pi_r \mathrm{d}r \mid \mathcal{F}_{t_{i+1}}\right] = 0 \text{ when } i = n-1.$$

We define the scheme associated with QBSDE (1) as follows: $Y^\pi_t = F^{-1}(\bar{Y}^\pi_t)$ where $Z^\pi_t = \frac{\bar{Z}^\pi_t}{F'(F^{-1}(\bar{Y}^\pi_t))}$. We should point out that (Y^π, Z^π) does not satisfy a QBSDE; the reason is that the numerical scheme (39) is not a BSDE since the non-linear generator $\bar{h}$ contains the information of $\bar{Z}^{\pi}$ on the time interval $[t_{i+1}, t_{i+2}]$ rather than $[t_i, t_{i+1}]$.

Theorem 3. *Let Assumption 3, (2.i), (2.iii) and (2.iv) in Assumption 2 be satisfied and the partition π satisfies $\max_{0\le i\le n-1} \frac{\triangle_i}{\triangle_{i+1}} \le L_1$, where L_1 is a positive constant. Then, there exist two positive constants δ and K which are independent of π, such that, for $|\pi| < \delta$, we have the following estimates*

$$\mathbb{E} \sup_{0\le t\le T} |Y_t - Y_t^{\pi}|^2 \le K\Big(|\pi| + \mathbb{E}|\xi - \xi^{\pi}|^2\Big),$$

and for all $1 \le p < 2$,

$$\mathbb{E} \int_0^T |Z_t - Z_t^{\pi}|^p \mathrm{d}t \le K\Big(|\pi| + \mathbb{E}|\xi - \xi^{\pi}|^2\Big)^{\frac{p}{2}}.$$

Proof. We consider the approximation scheme (39). We have already seen that the inputs $\bar{\xi}$ and $\bar{h}$ of Equation (29) satisfy Assumption 1. Using assertion (iv) of Assumption 2, h_1 is a constant function and the fact that F' is bounded one shows the existence of a constant $L_3 > 0$, such that, for all $t_1, t_2 \in [0,\ T]$ and $y,\ z \in \mathbb{R}$

$$\big|\bar{h}(t_2, y, z) - \bar{h}(t_1, y, z)\big| \le L_3 |t_2 - t_1|^{\frac{1}{2}}. \tag{40}$$

Then, thanks to Proposition 1, there exist positive constants K and δ, independent of the partition π, such that, whenever $|\pi| < \delta$, we have

$$\mathbb{E} \sup_{0\le t\le T} \big|\bar{Y}_t - \bar{Y}_t^{\pi}\big|^2 + \mathbb{E} \int_0^T |\bar{Z}_t - \bar{Z}_t^{\pi}|^2 \mathrm{d}t \le K\Big(|\pi| + \mathbb{E}|\xi - \xi^{\pi}|^2\Big). \tag{41}$$

Firstly, by using (41) and the fact that F and F^{-1} are Lipschitz functions, we have

$$\begin{aligned}
\mathbb{E} \sup_{0\le t\le T} |Y_t - Y_t^{\pi}|^2 &= \mathbb{E} \sup_{0\le t\le T} \Big|F^{-1}(\bar{Y}_t) - F^{-1}(\bar{Y}_t^{\pi})\Big|^2 \\
&\le K\Big[|\pi| + \mathbb{E}|\xi - \xi^{\pi}|^2\Big].
\end{aligned}$$

Now, we shall show that for $1 \le p < 2$

$$\mathbb{E} \int_0^T |Z_t - Z_t^{\pi}|^p \mathrm{d}t \ \le\ K\Big(|\pi| + \mathbb{E}|\xi - \xi^{\pi}|^2\Big)^{\frac{p}{2}}.$$

Applying the Hölder inequality twice and using (41), we have

$$\begin{aligned}
\mathbb{E} \int_0^T |\bar{Z}_t - \bar{Z}_t^{\pi}|^p \mathrm{d}t &\le K\mathbb{E}\bigg(\int_0^T |\bar{Z}_t - \bar{Z}_t^{\pi}|^2 \mathrm{d}t\bigg)^{\frac{p}{2}} \\
&\le K\Big(|\pi| + \mathbb{E}|\xi - \xi^{\pi}|^2\Big)^{\frac{p}{2}},
\end{aligned} \tag{42}$$

With the Hölder inequality and $\mathbb{E}\sup_{0\le t\le T} |\bar{Z}_t|^q < +\infty$, for any $q \ge 2$, we have

$$\begin{aligned}
\mathbb{E} \int_0^T |\bar{Z}_t|^p \big|\bar{Y}_t - \bar{Y}_t^{\pi}\big|^p \mathrm{d}t &\le T\bigg(\mathbb{E} \sup_{0\le t\le T} \big|\bar{Y}_t - \bar{Y}_t^{\pi}\big|^2\bigg)^{\frac{p}{2}} \bigg(\mathbb{E} \sup_{0\le t\le T} |\bar{Z}_t|^{\frac{2p}{2-p}}\bigg)^{\frac{2-p}{2}} \\
&\le K\Big(|\pi| + \mathbb{E}|\xi - \xi^{\pi}|^2\Big)^{\frac{p}{2}},
\end{aligned} \tag{43}$$

and hence, by (42) and (43), we obtain the desired result. □

4.2. An Implicit Scheme for QBSDE

Firstly, we give the numerical scheme for the associated global Lipschitz BSDE (27)

$$\begin{cases} \bar{Y}^{\pi}_{t_n} &= \bar{\xi}^{\pi} \\ \bar{Y}^{\pi}_{t} &= \bar{Y}^{\pi}_{t_{i+1}} - \int_t^{t_{i+1}} \bar{Z}^{\pi}_r \mathrm{d}W_r \\ & \quad + \Big[F'\Big(F^{-1}\Big(\bar{Y}^{\pi}_{t_{i+1}}\Big)\Big) h\Big(t_{i+1}, F^{-1}\Big(\bar{Y}^{\pi}_{t_{i+1}}\Big)\Big) + \frac{h_1(t_{i+1})}{\triangle_i} \int_{t_i}^{t_{i+1}} \bar{Z}^{\pi}_s \mathrm{d}s \Big] \triangle_i, \end{cases} \tag{44}$$

for $t \in [t_i, t_{i+1})$, $i = n-1,\ n-2, \ldots,\ 0$, where the partition π and Δ_i, $i = n-1, \ldots, 0$ are defined as in the previous section and $\bar{\xi}^{\pi}$ is an approximation (if necessary) of the terminal condition $\bar{\xi}$. Then, we define $Y^{\pi}_t = F^{-1}(\bar{Y}^{\pi}_t)$ and $Z^{\pi}_t = \frac{\bar{Z}^{\pi}_t}{F'(F^{-1}(\bar{Y}^{\pi}_t))}$. The pair (Y^{π}, Z^{π}) is an approximation of the unique solution (Y, Z) of QBSDE (1). Then, (Y^{π}, Z^{π}) satisfies the following recursive quadratic BSDEs

$$\begin{cases} Y^{\pi}_{t_n} &= \xi^{\pi}, \\ Y^{\pi}_t &= \varphi\Big(\int_{t_i}^{t_{i+1}} F'(Y^{\pi}_r) Z^{\pi}_r \mathrm{d}r\Big) + \int_t^{t_{i+1}} f(Y^{\pi}_r)|Z^{\pi}_r|^2 \mathrm{d}r - \int_t^{t_{i+1}} Z^{\pi}_r \mathrm{d}W_r, \end{cases} \tag{45}$$

where $t \in [t_i, t_{i+1})$, $i = n-1,\ n-2, \ldots,\ 0$ and

$$\begin{aligned} & \varphi\Big(\int_{t_i}^{t_{i+1}} F'(Y^{\pi}_r) Z^{\pi}_r \mathrm{d}r\Big) \\ = \; & F^{-1}\Big(F\Big(Y^{\pi}_{t_{i+1}}\Big) + F'(Y^{\pi}_{t_{i+1}}) h\Big(t_{i+1}, Y^{\pi}_{t_{i+1}}\Big) \triangle_i + h_1(t_{i+1}) \int_{t_i}^{t_{i+1}} Z^{\pi}_r F'(Y^{\pi}_r) \mathrm{d}r\Big). \end{aligned}$$

In fact, Equation (45) can be written in the form:

$$Y_t = F^{-1}\Big(\bar{\xi} + g\Big(\int_u^v F'(Y_r) Z_r \mathrm{d}r\Big)\Big) + \int_t^v f(Y_r)|Z_r|^2 \mathrm{d}r - \int_t^v Z_r \mathrm{d}W_r. \tag{46}$$

for $t \in [u,\ v]$ and $0 \le u < v \le T$, $\bar{\xi}$ is $\mathcal{F}_v$-measurable and $g : (\Omega \times \mathbb{R}, \mathcal{F}_v \otimes \mathcal{B}) \longrightarrow (\mathbb{R}, \mathcal{B})$ is a given function.

In the following theorem we will give an existence and uniqueness result for this new type of QBSDE.

Theorem 4. *Let $0 \le u < v \le T$ and $p \ge 2$. Let g be a Lipschitz function such that $g(0) \in L^p(\omega)$ and $\bar{\xi} \in L^p(\omega)$, then Equation (46) admits a unique solution*

$$(Y, Z) \in \mathcal{S}^p_{\mathcal{F}}([u,\ v]) \times \mathcal{H}^p_{\mathcal{F}}([u,\ v]).$$

Proof. Itô's formula applied to $\bar{Y}_t = F(Y_t)$, taking into account that $\bar{Z}_t = F'(Y_t) Z_t$, yields

$$\begin{cases} \mathrm{d}\bar{Y}_t &= F'(Y_t)\Big(-f(Y_t)|Z_t|^2 \mathrm{d}t + Z_t \mathrm{d}W_t\Big) + \frac{1}{2} F''(Y_t)|Z_t|^2 \mathrm{d}t = \bar{Z}_t \mathrm{d}W_t \\ \bar{Y}_v &= \bar{\xi} + g\Big(\int_u^v \bar{Z}_r \mathrm{d}r\Big). \end{cases}$$

Or equivalently in its integral form

$$\bar{Y}_t = \bar{\xi} + g\Big(\int_u^v \bar{Z}_r \mathrm{d}r\Big) - \int_t^v \bar{Z}_r \mathrm{d}W_r, \quad t \in [u,\ v]. \tag{47}$$

Notice that g and $\bar{\xi}$ satisfy all the conditions of Theorem 4.1 in [18] and therefore the BSDE (47) has a unique solution $(\bar{Y},\bar{Z}) \in \mathcal{S}^p_{\mathcal{F}}([u,\, v]) \times \mathcal{H}^p_{\mathcal{F}}([u,\, v])$.
Now, Itô's formula applied to $F^{-1}(\bar{Y}_t)$ shows that

$$\mathrm{d}F^{-1}(\bar{Y}_t) = (F^{-1})'(\bar{Y}_t)\mathrm{d}\bar{Y}_t + \frac{1}{2}(F^{-1})''(\bar{Y}_t)\mathrm{d}\langle \bar{Y}_\cdot\rangle_t.$$

By invoking notations (26) and (30), we have

$$\mathrm{d}F^{-1}(\bar{Y}_t) = \frac{\bar{Z}_t}{F'(F^{-1}(\bar{Y}_t))}\mathrm{d}W_t - \frac{1}{2}\frac{F''(F^{-1}(\bar{Y}_t))}{(F'(F^{-1}(\bar{Y}_t)))^3}|\bar{Z}_t|^2\mathrm{d}t,$$

which will be written in its integral form as

$$Y_t = Y_v + \int_t^v f(Y_r)|Z_r|^2\mathrm{d}r - \int_t^v Z_r\mathrm{d}W_r. \tag{48}$$

□

Theorem 5. *For any $p \geq 2$, let $(Y^\pi, Z^\pi) \in \mathcal{S}^p_{\mathcal{F}}([0,\, T]) \times \mathcal{H}^p_{\mathcal{F}}([0,\, T])$ be a solution of the Equation (48). Let Assumption 3 and (2.i), (2.iii) and (2.iv) in Assumption 2 be satisfied. Then, there exist two positive constants δ and K which are independent from π, such that, if $|\pi| < \delta$, the rate of convergence of the implicit scheme (45) of QBSDE (1) is of this type:*

$$\mathbb{E}\sup_{0\leq t\leq T}|Y_t - Y_t^\pi|^p \leq K\Big(|\pi|^{\frac{p}{2}} + \mathbb{E}|\xi - \xi^\pi|^p\Big),$$

and

$$\mathbb{E}\bigg(\int_0^T |Z_t - Z_t^\pi|^2\mathrm{d}t\bigg)^{\frac{p}{2}} \leq K\Big(|\pi|^{\frac{p}{2}} + \max\Big(\mathbb{E}|\xi - \xi^\pi|^p,\ |\pi|^{-\frac{p}{2}}\mathbb{E}|\xi - \xi^\pi|^{2p}\Big)\Big).$$

Proof. Thanks to assertion (2.iv) in Assumption 2, and considering that h_1 is a constant function and the boundedness of F', we can easily check that $\bar{h}$ satisfies condition (40). Assuming that ξ^π is p-integrable, it can be easily verified that $\bar{\xi}^\pi \in L^p(\omega)$, for any $p \geq 2$. Keeping in mind that $\bar{\xi}$ and $\bar{h}$ satisfy Assumption 1, Proposition 2 shows that there are two positive constants K and δ, independent of the partition π, such that, for $|\pi| < \delta$, we have

$$\mathbb{E}\sup_{0\leq t\leq T}\big|\bar{Y}_t - \bar{Y}_t^\pi\big|^p + \mathbb{E}\bigg(\int_0^T |\bar{Z}_t - \bar{Z}_t^\pi|^2\mathrm{d}t\bigg)^{\frac{p}{2}} \leq K\Big(|\pi|^{\frac{p}{2}} + \mathbb{E}|\xi - \xi^\pi|^p\Big). \tag{49}$$

Since F and F^{-1} are Lipschitz functions and by using (49) we have

$$\begin{aligned}\mathbb{E}\sup_{0\leq t\leq T}|Y_t - Y_t^\pi|^p &\leq K\mathbb{E}\sup_{0\leq t\leq T}\big|\bar{Y}_t - \bar{Y}_t^\pi\big|^p\\ &\leq K\Big(|\pi|^{\frac{p}{2}} + \mathbb{E}|\xi - \xi^\pi|^p\Big).\end{aligned}$$

Now, let us show that:

$$\mathbb{E}\bigg(\int_0^T |Z_t - Z_t^\pi|^2\mathrm{d}t\bigg)^{\frac{p}{2}} \leq K\Big(|\pi|^{\frac{p}{2}} + \max\Big(\mathbb{E}|\xi - \xi^\pi|^p,\ |\pi|^{-\frac{p}{2}}\mathbb{E}|\xi - \xi^\pi|^{2p}\Big)\Big).$$

We have for all $q \geq 1$, $(|x| + |y|)^q \leq 2^{q-1}(|x|^q + |y|^q)$,

$$\mathbb{E}\bigg(\int_0^T |Z_t - Z_t^\pi|^2\mathrm{d}t\bigg)^{\frac{p}{2}} \leq K\Bigg(\mathbb{E}\bigg(\int_0^T |\bar{Z}_t - \bar{Z}_t^\pi|^2\mathrm{d}t\bigg)^{\frac{p}{2}} + \mathbb{E}\bigg(\int_0^T |\bar{Z}_t|^2|\bar{Y}_t - \bar{Y}_t^\pi|^2\mathrm{d}t\bigg)^{\frac{p}{2}}\Bigg)$$

so, by the Cauchy–Schwartz inequality, we obtain

$$\begin{aligned}\mathbb{E}\left(\int_0^T |\bar{Z}_t|^2 \big|\bar{Y}_t - \bar{Y}_t^\pi\big|^2 \mathrm{d}t\right)^{\frac{p}{2}} &\leq \mathbb{E}\sup_{0\leq t\leq T}\big|\bar{Y}_t - \bar{Y}_t^\pi\big|^p \left(\int_0^T |\bar{Z}_t|^2 \mathrm{d}t\right)^{\frac{p}{2}} \\ &\leq \left(\mathbb{E}\sup_{0\leq t\leq T}\big|\bar{Y}_t - \bar{Y}_t^\pi\big|^{2p}\right)^{\frac{1}{2}} \left(\mathbb{E}\left(\int_0^T |\bar{Z}_t|^2 \mathrm{d}t\right)^p\right)^{\frac{1}{2}}.\end{aligned}$$

Now, for all $x > 0$, $y \geq 0$ and $0 < q < 1$, we have $(x+y)^q \leq x^q + qx^{q-1}y$ and by using (49), we obtain

$$\begin{aligned}\left(\mathbb{E}\sup_{0\leq t\leq T}\big|\bar{Y}_t - \bar{Y}_t^\pi\big|^{2p}\right)^{\frac{1}{2}} &\leq K\Big(|\pi|^p + \mathbb{E}|\xi - \xi^\pi|^{2p}\Big)^{\frac{1}{2}} \\ &\leq K\Big(|\pi|^{\frac{p}{2}} + |\pi|^{-\frac{p}{2}}\mathbb{E}|\xi - \xi^\pi|^{2p}\Big).\end{aligned}$$

Moreover, one has

$$\mathbb{E}\left(\int_0^T |\bar{Z}_t|^2 \mathrm{d}t\right)^p \leq T^p \mathbb{E}\sup_{0\leq t\leq T}|\bar{Z}_t|^{2p} < +\infty.$$

and therefore

$$\mathbb{E}\left(\int_0^T |\bar{Z}_t|^2 \big|\bar{Y}_t - \bar{Y}_t^\pi\big|^2 \mathrm{d}t\right)^{\frac{p}{2}} \leq K\Big(|\pi|^{\frac{p}{2}} + |\pi|^{-\frac{p}{2}}\mathbb{E}|\xi - \xi^\pi|^{2p}\Big).$$

Finally,

$$\begin{aligned}\mathbb{E}\left(\int_0^T |Z_t - Z_t^\pi|^2 \mathrm{d}t\right)^{\frac{p}{2}} &\leq K\Big(|\pi|^{\frac{p}{2}} + \mathbb{E}|\xi - \xi^\pi|^p\Big) \\ &\quad + K\Big(|\pi|^{\frac{p}{2}} + |\pi|^{-\frac{p}{2}}\mathbb{E}|\xi - \xi^\pi|^{2p}\Big) \\ &\leq K\Big(|\pi|^{\frac{p}{2}} + \max\Big(\mathbb{E}|\xi - \xi^\pi|^p,\ |\pi|^{-\frac{p}{2}}\mathbb{E}|\xi - \xi^\pi|^{2p}\Big)\Big).\end{aligned}$$

This ends the proof of theorem. □

Remark 4.

(i) *Implicit and explicit schemes give the same results if $\bar{h}$ does not depend on $\bar{Z}$.*

(ii) *For both explicit and implicit numerical schemes considered in this section, the problem is how to evaluate the processes $\{Z_t^\pi\}_{0\leq t\leq T}$ and $\{\bar{Z}_t^\pi\}_{0\leq t\leq T}$, in order to implement the scheme on computers.*

4.3. A Fully Discrete Scheme for QBSDE

In this part, we consider the following QBSDE

$$Y_t = \xi + \int_t^T \Big(\alpha(s) + \beta(s)Z_s + f(Y_s)|Z_s|^2\Big)\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s \tag{50}$$

under the following Assumptions:

Assumption 4. *(A1) Assume that α and β are deterministic and bounded functions, moreover, there exists a constant $L > 0$, such that for all $t_1, t_2 \in [0,\ T]$*

$$|\alpha(t_2) - \alpha(t_1)| + |\beta(t_2) - \beta(t_1)| \leq L|t_2 - t_1|^{\frac{1}{2}}.$$

(A2) $\mathbb{E}[\sup_{0\le\theta\le T}|D_\theta\xi|^r]$ *is finite for all* $r\ge 1$.
(A3) There exists a constant $M>0$ *such that,* $\xi\le M$ $\mathbb{P}$-*a.s.*

4.3.1. QBSDE $(\xi, \beta(s)z+f(y)|y|^2)$

This paragraph is devoted to the study of a particular case of QBSDE (50) when $\alpha\equiv 0$. By using Zvonkin's transformation and the notations (26), the equation QBSDE (50) will be transformed formally a.s. to the following equation

$$\bar{Y}_t=\bar{\xi}+\int_t^T\beta(s)\bar{Z}_s\mathrm{d}s-\int_t^T\bar{Z}_s\mathrm{d}W_s. \tag{51}$$

The fully discrete numerical scheme of (51) will be defined similarly to the one of (22) with $\bar{h}(t_{i+1},\bar{Y}^\pi_{t_{i+1}},\bar{Z}^\pi_{t_{i+1}})=\beta(t_{i+1})\bar{Z}^\pi_{t_{i+1}}$,

$$\begin{cases}\bar{Y}^\pi_{t_n} &=& \bar{\xi}, \quad \bar{Z}^\pi_{t_n}=D_T\bar{\xi},\\ \bar{Y}^\pi_{t_i} &=& \mathbb{E}\Big[\bar{Y}^\pi_{t_{i+1}}+\beta(t_{i+1})\bar{Z}^\pi_{t_{i+1}}\Delta_i\mid\mathcal{F}_{t_i}\Big],\\ \bar{Z}^\pi_{t_i} &=& \mathbb{E}\Big[\rho^\pi_{t_{i+1},t_n}D_{t_i}\bar{\xi}\mid\mathcal{F}_{t_i}\Big],\end{cases} \tag{52}$$

where $\rho^\pi_{t_i,t_i}=1$, $i=0,1,\dots,n$ and for $0\le i<j\le n$,

$$\rho^\pi_{t_i,t_j}=\exp\left\{\sum_{k=i}^{j-1}\int_{t_k}^{t_{k+1}}\beta(s)\mathrm{d}W_s-\frac{1}{2}\sum_{k=i}^{j-1}\int_{t_k}^{t_{k+1}}\beta^2(s)\mathrm{d}s\right\}.$$

Theorem 6. *For any* $p\ge 2$, *let Assumptions (A1) and (A2) be satisfied and* ξ *satisfy assertion (1.i) in Assumption 1. Then, the rate of convergence of the fully discrete scheme associated with QBSDE (50) is given by*

$$\mathbb{E}\max_{0\le i\le n}\left\{\left|Y_{t_i}-Y^\pi_{t_i}\right|^p+\left|Z_{t_i}-Z^\pi_{t_i}\right|^p\right\}\ \le\ C|\pi|^{\frac{p}{2}-\frac{p}{\left(2\ln\frac{1}{|\pi|}\right)}}\left(\ln\frac{1}{|\pi|}\right)^{\frac{p}{2}}.$$

Proof. Considering that $\bar{\xi}$ satisfies (1.i) in Assumption 1, and Assumptions (A1) and (A2) are satisfied, then thanks to Proposition 3, the following inequality gives the rate of convergence of the fully discrete scheme (52) related to BSDE (51),

$$\mathbb{E}\max_{0\le i\le n}\left\{\left|\bar{Y}_{t_i}-\bar{Y}^\pi_{t_i}\right|^p+\left|\bar{Z}_{t_i}-\bar{Z}^\pi_{t_i}\right|^p\right\}\ \le\ C|\pi|^{\frac{p}{2}-\frac{p}{\left(2\ln\frac{1}{|\pi|}\right)}}\left(\ln\frac{1}{|\pi|}\right)^{\frac{p}{2}}. \tag{53}$$

Let (Y^π,Z^π) such that $Y^\pi_t=F^{-1}(\bar{Y}^\pi_t)$ and $Z^\pi_t=\frac{\bar{Z}^\pi_t}{F'(F^{-1}(\bar{Y}^\pi_t))}$ be an approximation of the unique solution pair (Y,Z) of QBSDE (50). Firstly, since F and F^{-1} are Lipschitz functions, we deduce from (53) that

$$\begin{aligned}\mathbb{E}\max_{0\le i\le n}\left|Y_{t_i}-Y^\pi_{t_i}\right|^p &\le M\mathbb{E}\max_{0\le i\le n}\left|\bar{Y}_{t_i}-\bar{Y}^\pi_{t_i}\right|^p\\ &\le C|\pi|^{\frac{p}{2}-\frac{p}{\left(2\ln\frac{1}{|\pi|}\right)}}\left(\ln\frac{1}{|\pi|}\right)^{\frac{p}{2}}.\end{aligned}$$

where M is the Lipschitz constant of F.

Now, by (53), Cauchy–Schwartz inequality and $\sup_{0\le t\le T}\mathbb{E}[|\bar{Z}_t|^{2p}] < +\infty$, we have

$$\begin{aligned}\mathbb{E}\max_{0\le i\le n}|\bar{Z}_{t_i}|^p\left|\bar{Y}_{t_i}-\bar{Y}^\pi_{t_i}\right|^p &\le \left(\mathbb{E}\max_{0\le i\le n}|\bar{Z}_{t_i}|^{2p}\right)^{\frac{1}{2}}\left(\mathbb{E}\max_{0\le i\le n}\left|\bar{Y}_{t_i}-\bar{Y}^\pi_{t_i}\right|^{2p}\right)^{\frac{1}{2}} \\ &\le C|\pi|^{\frac{p}{2}-\frac{p}{\left(2\ln\frac{1}{|\pi|}\right)}}\left(\ln\frac{1}{|\pi|}\right)^{\frac{p}{2}}. \end{aligned} \tag{54}$$

Then, we have

$$\begin{aligned}\mathbb{E}\max_{0\le i\le n}\left|Z_{t_i}-Z^\pi_{t_i}\right|^p &\le C\left(\mathbb{E}\max_{0\le i\le n}\left|\bar{Z}_{t_i}-\bar{Z}^\pi_{t_i}\right|^p+\mathbb{E}\max_{0\le i\le n}|\bar{Z}_{t_i}|^p\left|\bar{Y}_{t_i}-\bar{Y}^\pi_{t_i}\right|^p\right) \\ &\le C|\pi|^{\frac{p}{2}-\frac{p}{\left(2\ln\frac{1}{|\pi|}\right)}}\left(\ln\frac{1}{|\pi|}\right)^{\frac{p}{2}}.\end{aligned}$$

□

4.3.2. QBSDE $(\xi, \alpha(s)+\beta(s)z+\frac{1}{2}|z|^2)$

In this paragraph, we consider the case where $f\equiv\frac{1}{2}$, so that the generator of QBSDE (50) takes the following form $\alpha(s)+\beta(s)z+\frac{1}{2}|z|^2$. The exponential change of variable $\bar{Y}_t=\exp(Y_t)$ transforms formally Equation (50) a.s.: $\forall\, t\in[0,\ T]$,

$$\begin{aligned}\bar{Y}_t &= \exp(\xi)-\int_t^T Z_s\exp(Y_s)\mathrm{d}W_s \\ &\quad +\int_t^T(\alpha(s)\exp(Y_s)+\beta(s)\exp(Y_s)Z_s)\mathrm{d}s \\ &= \exp(\xi)+\int_t^T(\alpha(s)\bar{Y}_s+\beta(s)\bar{Z}_s)\mathrm{d}s-\int_t^T\bar{Z}_s\mathrm{d}W_s.\end{aligned} \tag{55}$$

The latter BSDE being linear with α is a positive function, and one can define

$$\forall\, t\in[0,\ T],\ \ Y_t=\ln(\bar{Y}_t),\quad Z_t=\frac{\bar{Z}_t}{\bar{Y}_t}.$$

We define the fully discrete numerical scheme of (55) similarly to (22) with

$$\bar{h}(t_{i+1},\bar{Y}^\pi_{t_{i+1}},\bar{Z}^\pi_{t_{i+1}})=\alpha(t_{i+1})\bar{Y}^\pi_{t_{i+1}}+\beta(t_{i+1})\bar{Z}^\pi_{t_{i+1}},$$

$$\left\{\begin{aligned}\bar{Y}^\pi_{t_n} &= \exp(\xi),\quad \bar{Z}^\pi_{t_n}=\exp(\xi)D_T\xi, \\ \bar{Y}^\pi_{t_i} &= \mathbb{E}\left[\bar{Y}^\pi_{t_{i+1}}+\left(\alpha(t_{i+1})\bar{Y}^\pi_{t_{i+1}}+\beta(t_{i+1})\bar{Z}^\pi_{t_{i+1}}\right)\Delta_i\mid\mathcal{F}_{t_i}\right], \\ \bar{Z}^\pi_{t_i} &= \mathbb{E}\left[\rho^\pi_{t_{i+1},t_n}\exp(\xi)D_{t_i}\xi\mid\mathcal{F}_{t_i}\right],\end{aligned}\right. \tag{56}$$

where $\rho^\pi_{t_i,t_i}=1$, $i=0,1,\dots,n$ and for $0\le i<j\le n$,

$$\rho^\pi_{t_i,t_j}=\exp\left\{\sum_{k=i}^{j-1}\int_{t_k}^{t_{k+1}}\beta(s)\mathrm{d}W_s-\sum_{k=i}^{j-1}\int_{t_k}^{t_{k+1}}\left(\alpha(s)-\frac{1}{2}\beta(s)^2\right)\mathrm{d}s\right\}.$$

Let ξ satisfy condition (1.i) in Assumption 1, under Assumptions (A1), (A2) and (A3) and thanks to Proposition 3, there are positive constants C and δ independent of the partition π, such that, when $|\pi|<\delta$ we have, for any $p\ge 2$,

$$\mathbb{E}\max_{0\le i\le n}\left\{\left|\bar{Y}_{t_i}-\bar{Y}^\pi_{t_i}\right|^p+\left|\bar{Z}_{t_i}-\bar{Z}^\pi_{t_i}\right|^p\right\}\le C|\pi|^{\frac{p}{2}-\frac{p}{\left(2\ln\frac{1}{|\pi|}\right)}}\left(\ln\frac{1}{|\pi|}\right)^{\frac{p}{2}}. \tag{57}$$

Theorem 7. *If ξ satisfies (1.i) in Assumption 1, and Assumptions (A1), (A2) and (A3) hold true, then there are positive constants C and δ independent of the partition π, such that, when $|\pi| < \delta$, we have the following rate of convergence*

$$\mathbb{E}\left[\max_{0\le i\le n}\left\{\left|Y_{t_i}-Y^{\pi}_{t_i}\right|^p+\left|Z_{t_i}-Z^{\pi}_{t_i}\right|^p\right\}\right] \le C|\pi|^{\frac{p}{2}-\frac{p}{\left(2\ln\frac{1}{|\pi|}\right)}}\left(\ln\frac{1}{|\pi|}\right)^{\frac{p}{2}}.$$

Proof. We define the approximation of the couple (Y,Z) solution of QBSDE (50), as follows $Y^{\pi}_t=\ln(\bar{Y}^{\pi}_t)$ and $Z^{\pi}_t=\frac{\bar{Z}^{\pi}_t}{\bar{Y}^{\pi}_t}$. Noting that the linear BSDE (55) has a unique solution $(\bar{Y},\bar{Z})$ where $\bar{Y}$ is given explicitly by $\bar{Y}_t=\mathbb{E}[\exp(\xi)\Gamma_{t,T}|\mathcal{F}_t]$, such that $(\Gamma_{t,s})_{s\ge t}$ is the solution of the following SDE

$$\begin{cases} d\Gamma_{t,s} &= \Gamma_{t,s}(\alpha(s)ds+\beta(s)dW_s),\\ \Gamma_{t,t} &= 1.\end{cases}$$

Define $P^*=L_TP$, where

$$L_t=\exp\left(\int_0^t\beta(s)dW_s-\frac{1}{2}\int_0^t|\beta(s)|^2ds\right),$$

then

$$\bar{Y}_t=\mathbb{E}^*\left[\exp(\xi)\exp\left(\int_t^T\alpha(s)ds\right)\mid\mathcal{F}_t\right],$$

where $\mathbb{E}^*$ stands for the mathematical expectation under P^*. Obviously, if $\xi\ge 0$, $\alpha(s)\ge 0$, we have

$$\bar{Y}_t\ge\exp\left(\int_t^T\alpha(s)ds\right)\ge 1.$$

Due the fact that $\ln(\cdot)$ is a Lipschitz function in the interval $[1,+\infty[$ and by using (57), one can obtain

$$\begin{aligned}\mathbb{E}\max_{0\le i\le n}\left|Y_{t_i}-Y^{\pi}_{t_i}\right|^p &= \mathbb{E}\max_{0\le i\le n}\left|\ln(\bar{Y}_{t_i})-\ln\left(\bar{Y}^{\pi}_{t_i}\right)\right|^p \\ &\le C\mathbb{E}\max_{0\le i\le n}\left|\bar{Y}_{t_i}-\bar{Y}^{\pi}_{t_i}\right|^p \\ &\le C|\pi|^{\frac{p}{2}-\frac{p}{\left(2\ln\frac{1}{|\pi|}\right)}}\left(\ln\frac{1}{|\pi|}\right)^{\frac{p}{2}}.\end{aligned} \tag{58}$$

Let $(\bar{Y},\bar{Z})$ be the unique solution of BSDE (55). Since $\exp(\xi)$ and the functions α, β are bounded, it was shown by Proposition 2. d (i) in [20] that $\bar{Y}$ and $\bar{Z}$ are bounded. Moreover, thanks to scheme (56), it is easy to verify that $\bar{Y}^{\pi}$ is also bounded. Elementary calculation shows that

$$\begin{aligned}\mathbb{E}\max_{0\le i\le n}\left|Z_{t_i}-Z^{\pi}_{t_i}\right|^p &= \mathbb{E}\max_{0\le i\le n}\left|\frac{\bar{Z}_{t_i}}{\bar{Y}_{t_i}}-\frac{\bar{Z}^{\pi}_{t_i}}{\bar{Y}^{\pi}_{t_i}}\right|^p \\ &\le \mathbb{E}\left[\max_{0\le i\le n}\left(\left|\frac{1}{\bar{Y}^{\pi}_{t_i}}\left(\bar{Z}_{t_i}-\bar{Z}^{\pi}_{t_i}\right)\right|^p+\left|\frac{\bar{Z}_{t_i}}{\bar{Y}_{t_i}\bar{Y}^{\pi}_{t_i}}\right|^p\left|\bar{Y}_{t_i}-\bar{Y}^{\pi}_{t_i}\right|^p\right)\right] \\ &\le C\left(\mathbb{E}\max_{0\le i\le n}\left|\bar{Z}_{t_i}-\bar{Z}^{\pi}_{t_i}\right|^p+\mathbb{E}\max_{0\le i\le n}\left|\bar{Y}_{t_i}-\bar{Y}^{\pi}_{t_i}\right|^p\right).\end{aligned}$$

Finally, by invoking relation (57), we obtain

$$\mathbb{E}\max_{0\le i\le n}\left|Z_{t_i}-Z^{\pi}_{t_i}\right|^p\le C|\pi|^{\frac{p}{2}-\frac{p}{\left(2\ln\frac{1}{|\pi|}\right)}}\left(\ln\frac{1}{|\pi|}\right)^{\frac{p}{2}}.$$

□

5. Conclusions

In this paper, we addressed questions of existence, uniqueness and regularity of solutions in the Malliavin sense as well as the analysis of explicit and implicit numerical schemes for one-dimensional BSDEs whose generator can be written in the form

$$h(r,y) + h_1(r)z + f(y)z^2.$$

However, general cases such as when h is assumed to be Lipschitz in y and z and/or when $h_1(r)z$ is replaced by a function r, y and z are not solved yet.

It is worth pointing out that we were not able to provide a fully discrete explicit or implicit numerical scheme due to the presence of the integral of Z over some intervals. Therefore, to find fully explicit or implicit schemes, we restricted ourselves to the case where the generator has the following two forms,

$$\beta(s)z + f(y)|z|^2 \text{ and } \alpha(s) + \beta(s)z + \frac{1}{2}|z|^2,$$

the functions α and β are assumed to be deterministic, bounded and Hölder continuous.

Notice that fully explicit or implicit schemes for the case when the generator has the form

$$\alpha(s) + \beta(s)z + f(y)|z|^2,$$

or satisfies the Lipschitz property in both y and z plus the quadratic term $f(y)|z|^2$ are not solved yet. We will focus on the analysis of numerical schemes for these last cases in future research projects and we hope that we can handle all the aforementioned still open problems.

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