Theory of reliability and Life testing

Contents

- References
- Course contents
- Basic results in probability
- Some continuous distributions
- Some discrete distributions
- Functions of random variables
- List of some commonly used distributions
- Exercise

▪

$:S_b\{0,1\}$

Recommended References

References

- Barlow, R. and Proschan, E. (1981). Statistical Theory of Reliability and life Testing: Probability Models. To Begin With.
- Pham , H (2003). Handbook of Reliability Engineering Springer.
- Leemis, L. M. (1995). Reliability: Probabilistic models and Statistical methods. Prentice-Hall, Englewood Cliffs.

.

$:S_b\{0,1\}$

Basic Concepts of probability

- Envision an experiment for which the result is unknown. The collection of all possible outcomes is called the *sample space*. A set of outcomes, or subset of the sample space, is called an *event*.
- A probability space is a three-tuple (Ω ,$\Im$, P) where Ω is a sample space, $\Im$ is a collection of events from the sample space and P is a probability law that assigns a number to each event in $\Im$. For any events A and B, Pr must satisfy:
  - $P(\Omega) = 1$
  - $P(A) \geq 0$, $P(A^C) = 1 - Pr(A)$
  - For any countable mutually disjoint events

$A_i \in \Omega, i = 1,2,\ldots,\infty$, Then. $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} p(A_i$

Basic Concepts

- $\Im$ is a –algebra which is defined as : a nonempty collection of subsets of Ω such that the following hold:
- 1. Ω is in $\Im$.
- 2. If $A \in \Im$ then so is the complement $A^c \in \Im$
- 3. If is a sequence of elements $A_i \in \Im, \mathrm{i}=1,\ldots,\mathrm{n}$ then the union of these elements is in $\Im$ i.e. $.\bigcup_{i=1}^{n} A_i \in \Im$

- One can construct many sigma-algebra such as

$$\Im = \{\phi, \Omega\}$$

Distribution and densityfunctions

- **If *X* is a continuous rv then**

$$\Pr(X \le x) = F(x) \text{ is the distribution of } X.$$

$$\textit{and the density is defined by}$$

$$f(\mathrm{x}) = \frac{d}{dx} F(x), \textit{if it exists}.$$

Thus the density can be of the form

$$F(x) = \int_{-\infty}^{x} f(u)\,du$$

$$= P(-\infty < X < x)$$

$$= P(X < x)$$

Basic probability and Distributions

Topics in Probability:

- **What is a random variable?**
- **Discrete vs. Continuous**
- **Density (mass) function**
- **Probability distribution function**
- **Forms of distributions**
- **Joint distributions**
- **Conditional distributions**
- **Functions of random variables**
- **Moments of random variables**
- **Transforms and generating functions**
- **Family or sequence of random variables**

▪

What is a Random Variable

- A *random variable*, usually written as*X*, is a variable whose possible values are numerical assigned to the outcomes of a random phenomenon or experiment**.**

There are two types of random variables, *discrete* and *continuous*

- Suppose that an airline mandates that all pilots must weigh between 60 and 80 Kgm. The weight of a pilot would be an example of a **continuous variable**; since a pilot weight could take on any value between 60 and 80 Kgm.
- Suppose one flips a coin and count the number of heads. The number of heads could be any **integer value** between 0 and plus infinity. However, it could not be any number between 0 and plus infinity. We could not, for example, get 2.5 heads. Therefore, the number of heads must be a **discrete variable**..

$:S_b\{0,1\}$

What is a Random Variable?

- A random variable is discrete if it has a **finite or countable infinite number** of possible outcomes that can be listed.

- A random variable is continuous if it has an infinite number of possible outcomes represented by an interval on the real line.

- A random variable is a **mixed** of discrete and continuous if it has a **finite or countable infinite number** of possible values at part of its values and it has an infinite number of possible values represented by an interval on the real line on other parts..

Types of Random variables!

A random variable can be:

- **the number of people waiting at the entrance of the Louvre Museum**
- **The number of people waiting at the barber shop**
- **Number of patients waiting at the outpatients clinics or in hospital beds**
- **etc**

Distribution and densityfunctions

- **If *X* is a continuous rv then**

$$\Pr(X \le x) = F(x) \text{ is the distribution of } X.$$

$$\textit{and the density is defined by}$$

$$f(x) = \frac{d}{dx} F(x), \textit{if it exists}.$$

Thus the density can be of the form

$$F(x) = \int_{-\infty}^{x} f(u)\,du$$
$$= P(-\infty < X < x)$$
$$= P(X < x)$$

Properties of Density and distribution Functoions

- **The distribution function F(x):**

It must satisfies the following conditions:

$$i)\ F(-\infty)=0,\ \ F(\infty)=1.$$

$$ii)F(x)is\ nondecrea\text{ing}\ in\ x\ .$$

$$iii)\ \text{F(x) is left continuous}$$

$$iv)F(x)=P(X\ \leq x)$$

Whereas the density f(x) has to satisfy

$$)\ f\ (x)\geq 0.$$

$$i)\ \int_{-\infty}^{\infty} f\ (u)du\ =1$$

$$ii)P(a<X\ ,b)=\int_{a}^{b} f\ (u)du.for\ anyreala\,and\ \ b$$

Functions of Random variables

- **Often one is interested in some combinations of r.v.'s**
  - **Sum of the first *k* interarrival times = time of the *k*th arrival** $Z = X_1 + ... + X_k$

Z is called the convolution of $X_1, ..., X_k$

- Minimum of service times for parallel independent servers = time until next departure

If Z = min(X,Y), where X and Y are independent then

$$F_Z(x) = \mathrm{P}(Z \le x)$$

$$= \mathrm{P}(\min[X, Y\} \le x)$$

$$= 1 - \mathrm{P}(\min[X, Y\} \ge x)$$

$$= 1 - \mathrm{P}(X \ge x \ \ and\, Y \ge x) \qquad P(X \le x) = P(Y \le x)P(Z \le x)$$

$$= 1 - \mathrm{P}(X \ge x)\mathrm{P}(Y \ge x), if \ \ independents$$

$$= 1 - [1 - F_X(x)][1 - F_X(x)]$$

$:S_b\{0,1\}$

Functions of Random variables

- **Maximum of service times for parallel servers = time until next departure**

If *Z* = max(*X*, Y) then

$$F_Z(x) = \mathrm{P}\left(Z \le x\right)$$
$$= \mathrm{P}\left(\max[X, Y\} \le x\right)$$
$$= \mathrm{P}\left(X \le x \; and \; Y \le x\right)$$
$$= \mathrm{P}\left(X \le x\right)\mathrm{P}\left(Y \prec x\right), if \;\; independent$$
$$= F_X(x) F_Y(x)$$

Functions of Random variables

- **Maximum of service times for parallel servers = time until next departure**

- **If one has n independent rv's** $X_1, .., X_n$ **then**

If $Z = Min\{X_1,..,X_k\} then$

$$F_Z(x) = \mathrm{P}\left(Z \le x\right)$$

$$= \mathrm{P}\left(min\{X_1,..,X_n\} \le x\right)$$

$$= 1 - \mathrm{P}\left(X_1 \ge x, .., X_n\} \ge x\right)$$

$$= 1 - \mathrm{P}\left(min\{X_1,..,X_n\} \ge x\right)$$

$$= 1 - \mathrm{P}\left(X_1 \ge x\right)...\mathrm{P}\left(X_n \ge x\right), if \ \ independents$$

$$= 1 - [1 - F_{X_1}(x)]....[1 - F_{X_n \backslash}(x)]$$

$:S_b\{0,1\}$

Funtions of Random variables

- maximum of service times for parallel servers = time until next departure

- If one has n independent rv's $X_1, .., X_n$ then
- If $Z = \max\{X_1,..,X_n\} then$

$$
\begin{aligned}
F_Z(x) &= \mathrm{P}\left(Z \le x\right) \\
&= \mathrm{P}\left(\max\{X_1,..,X_n\} \le x\right) \\
&= \mathrm{P}\left(X_1 \le x ,..,X_n \le x\right) \\
&= \mathrm{P}\left(X_1 \le x\right)...\mathrm{P}\left(X_n \le x\right), if\ \ independent \\
&= F_{X_1}(x)]....\ F_{X_n \backslash}(x)]
\end{aligned}
$$

Some Special Distributions

Distribution arise frequently in this course

- Discrete

- **– Bernoulli - Binomial**
- **– Geometric - Poisson**
- **– Discrete uniform**

- **– Continuous**
- **- Uniform - Exponential**
- **- Gamma - Normal, log normal**
- **- Weibull -**

Bernoulli Distribution

"Single coin flip" p = Pr(success)

N = 1 if success, 0 otherwise

$$\Pr(N=n) = \begin{cases} p, & n=1 \\ 1-p, & n=0 \end{cases}$$

$$E(N) = p$$

$$\mathrm{Var}(N) = p(1-p)$$

$$Cv_N^2 = \frac{1-p}{p}$$

$$M(\theta) = \left(1-p+pe^{\theta}\right)$$

Binomial Distribution

"*n* independent coin flips" p = Pr(success)

N = # of successes

$$\Pr(N=k)=\binom{n}{k}p^k(1-p)^{n-k},\ k=0,1,\ldots,n$$

$$E(N)=np$$

$$\mathrm{Var}(N)=np(1-p)$$

$$Cv_N^2=\frac{1-p}{np}$$

$$M(\theta)=\left(1-p+pe^{\theta}\right)^n$$

Bernoulli Distribution

"Single coin flip" p = Pr(success)

N = 1 if success, 0 otherwise

$$\Pr(N = n) = \begin{cases} p, & n = 1 \\ 1 - p, & n = 0 \end{cases}$$

$$E(N) = p$$

$$\mathrm{Var}(N) = p(1 - p)$$

$$Cv_N^2 = \frac{1 - p}{p}$$

$$M^*(\theta) = \left(1 - p + pe^{\theta}\right)$$

Poisson Distribution

"Occurrence of rare events" λ = average rate of occurrence per period;

N = # of events in an arbitrary period

$$\Pr(N=k)=\frac{\lambda^k e^{-\lambda}}{k!}, k=0,1,2,\ldots$$

$$E(N)=\lambda$$

$$\text{Var}(N)=\lambda$$

$$Cv_N^2=1/\lambda$$

z-Transform for Geometric Distribution

Given $P_n = (1-p)^{n-1}p$, $n = 1, 2, \ldots,$ find

$$G(z) = \sum_{n=0}^{\infty} P_n z^n$$

Then,

$$G(z) = \sum_{n=1}^{\infty} (1-p)^{n-1} p z^n = pz \sum_{n=1}^{\infty} \left((1-p)z\right)^{n-1} = pz \sum_{n=0}^{\infty} \left((1-p)z\right)^n$$

$$= \frac{pz}{1-(1-p)z}, \text{ using geometric series } \sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \text{ for } |a| < 1$$

$$E(N) = \left.\frac{dG(z)}{dz}\right|_{z=1} = \left.\frac{p}{(1-p+pz)^2}\right|_{z=1} = \frac{1}{p}$$

$$E(N^2) - E(N) = \left.\frac{d^2G(z)}{dz^2}\right|_{z=1} = \frac{2(1-p)}{p^2}, \text{ so } E(N^2) = \frac{2-p}{p^2} \text{ and}$$

$$\mathrm{Var}(N) = E(N^2) - (E(N))^2 = \frac{1-p}{p^2}$$

Discrete Uniform Distribution

.

$$E(X)=\int_a^b X\left(\frac{1}{b-a}\right)dx=\left(\frac{1}{b-a}\right)\frac{y^2}{2}\Bigg|_a^b$$

$$=\frac{b^2-a^2}{2(b-a)}=\frac{(b-a)(b+a)}{2(b-a)}=\frac{b+a}{2}$$

$$E\left(Y^2\right)=\int_a^b y^2\left(\frac{1}{b-a}\right)dy=\left(\frac{1}{b-a}\right)\frac{y^3}{3}\Bigg|_a^b$$

$$=\frac{b^3-a^3}{3(b-a)}=\frac{(b-a)(a^2+b^2+ab)}{3(b-a)}$$

$$=\frac{(a^2+b^2+ab)}{3}$$

Discrete Uniform Distribution

.

$$V(Y) = E\left(Y^2\right) - \left[E(Y)\right]^2 = \frac{(a^2+b^2+ab)}{3} - \left[\frac{b+a}{2}\right]^2 =$$

$$= \frac{4(a^2+b^2+ab) - 3(b^2+a^2+2ab)}{12} = \frac{a^2+b^2-2ab}{12} = \frac{(b-a)^2}{12}$$

Thus

$$\sigma = \sqrt{\frac{(b-a)^2}{12}} = \frac{b-a}{\sqrt{12}} \approx 0.2887(b-a)$$

Geometric Distribution

.the geometric distribution caan have the following form

$$p_x = p(X = x)$$

$$= (1-p)^{x-1} p, \quad x = 1, 2, \ldots$$

$$P_x = P(X \le x)^x$$

$$= 1-(1-p)^x$$

$$\mu = EX$$

$$= 1/p$$

$$\sigma^2 = \operatorname{var} X = (1-p)/p^2$$

Continuous distributions
Uniform Distribution

X is equally likely to fall anywhere within interval (*a*,*b*)

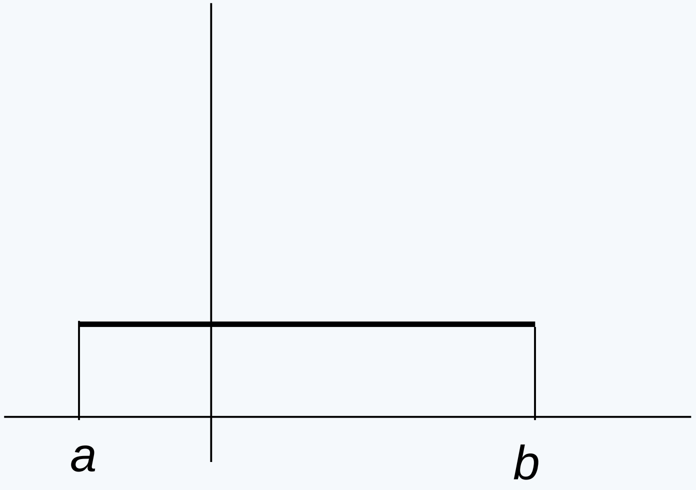
$$f_X(x) = \frac{1}{b-a}, \quad a \le x \le b$$

$$F(x) = \frac{x-b}{b-a}, \quad a \le x \le b$$

$$E(X) = \frac{a+b}{2}$$

$$\mathrm{Var}(X) = \frac{(b-a)^2}{12}$$

$$Cv_X^2 = \frac{(b-a)^2}{3(b+a)^2}$$



The pdf for the exponential is

$$f(y) = \begin{cases} \frac{1}{\theta} e^{-y/\theta} & y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$f(y) = \begin{cases} \frac{1}{\theta} e^{-y/\theta} & y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

Other form is

$$f(x) = \lambda e^{-\lambda x}$$

Exponential distribution

The distribution for the exponential is

$$F(y) = \int_0^y \frac{1}{\theta} e^{-t/\theta} dt$$

$$f(y) = \begin{cases} \frac{1}{\theta} e^{-y/\theta} & y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$= \frac{1}{-1/\theta} \left(\frac{1}{\theta} \right) e^{-t/\theta} \Big|_0^y$$

$$= -e^{-y/\theta} - \left(-e^{-0} \right)$$

$$= 1 - e^{-y/\theta} \quad y > 0$$

Exponential distribution

The Random variable X is said to have memory less property or forgetfulness property if

$$P(X \geq s+t / X \geq s) = P(X \geq t)$$

In other word

$$R(s+t) = R(s)R(t) \ where \ R(.) = 1 - F()$$

<u>Theorem 1</u>. A continuous R.V. X is exponentially distributed if and only if for $s, t \geq 0,$

$$P\{X > s+t \mid X > t\} = P\{X > s\} \qquad (1)$$

or equivalently,

$$P\{X > s+t\} = P\{X > s\}P\{X > t\}.$$

Memoryless property -I

The Random variable X is said to have memoryless property or forgetfulness property if

$$P(X \geq s+t \,/\, X \geq s) = P(X \geq t)$$

In other word

$$R(s+t) = R(s)R(t) \;\; where \;\; R(.) = 1 - F()$$

<u>Theorem 1</u>. A continuous R.V. X is exponentially distributed if and only if for $s, t \geq 0,$

$$P\{X > s+t \mid X > t\} = P\{X > s\} \qquad (1)$$

or equivalently,

$$P\{X > s+t\} = P\{X > s\}P\{X > t\}.$$

Memoryless property -II

A random variable with this property is said to be memoryless.A

Show that the Geometric RV's enjoy the memoryless property.is property.

Gamma Distribution

X is nonnegative, by varying parameter *b* get a variety of shapes

$$f_X(x) = \frac{\lambda^b x^{b-1} e^{-\lambda x}}{\Gamma(b)}, \quad x \geq 0, \text{ where } \Gamma(b) = \int_0^\infty x^{b-1} e^{-x} dx \text{ for } b > 0$$

$$E(X) = \frac{b}{\lambda}$$

$$\mathrm{Var}(X) = \frac{b}{\lambda^2}$$

$$Cv_X^2 = \frac{1}{b}$$

When *b* is an integer, *k say* , this is called the Erlang-*k* distribution, and $\Gamma(k) = (k-1)!$ Erlang-1 is same as exponential.

Normal Distribution

X follows a "bell-shaped" density function

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}, \quad -\infty < x < \infty$$

$$E(X) = \mu$$

$$\mathrm{Var}(X) = \sigma^2$$

From the central limit theorem, the distribution of the sum of independent and identically distributed random variables approaches a normal distribution as the number of summed random variables goes to infinity.

FORMS OF some PROBABILITY DISTRIBUTIONS

Discrete Distributions

Distribution	Probability Function	Mean	Variance	Moment-Generating Function
Binomial	$p(y) = \binom{n}{y} p^y (1-p)^{n-y};$ $y = 0, 1, \ldots, n$	np	$np(1-p)$	$[pe^t + (1-p)]^n$
Geometric	$p(y) = p(1-p)^{y-1};$ $y = 1, 2, \ldots$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$\frac{pe^t}{1-(1-p)e^t}$
Hypergeometric	$p(y) = \frac{\binom{r}{y}\binom{N-r}{n-y}}{\binom{N}{n}};$ $y = 0, 1, \ldots, n$ if $n \le r$, $y = 0, 1, \ldots, r$ if $n > r$	$\frac{nr}{N}$	$n\left(\frac{r}{N}\right)\left(\frac{N-r}{N}\right)\left(\frac{N-n}{N-1}\right)$	
Poisson	$p(y) = \frac{\lambda^y e^{-\lambda}}{y!};$ $y = 0, 1, 2, \ldots$	λ	λ	$\exp[\lambda(e^t - 1)]$
Negative binomial	$p(y) = \binom{y-1}{r-1} p^r (1-p)^{y-r};$ $y = r, r+1, \ldots$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$	$\left[\frac{pe^t}{1-(1-p)e^t}\right]^r$

FORMS OF PROBABILITY DISTRIBUTIONS

Continuous Distributions

Distribution	Probability Function	Mean	Variance	Moment-Generating Function
Uniform	$f(y) = \frac{1}{\theta_2 - \theta_1};\ \theta_1 \le y \le \theta_2$	$\frac{\theta_1 + \theta_2}{2}$	$\frac{(\theta_2 - \theta_1)^2}{12}$	$\frac{e^{t\theta_2} - e^{t\theta_1}}{t(\theta_2 - \theta_1)}$
Normal	$f(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\left(\frac{1}{2\sigma^2}\right)(y-\mu)^2\right]$ $-\infty < y < +\infty$	μ	σ^2	$\exp\left(\mu t + \frac{t^2\sigma^2}{2}\right)$
Exponential	$f(y) = \frac{1}{\beta} e^{-y/\beta};\quad \beta > 0$ $0 < y < \infty$	β	β^2	$(1 - \beta t)^{-1}$
Gamma	$f(y) = \left[\frac{1}{\Gamma(\alpha)\beta^\alpha}\right] y^{\alpha-1} e^{-y/\beta};$ $0 < y < \infty$	$\alpha\beta$	$\alpha\beta^2$	$(1 - \beta t)^{-\alpha}$
Chi-square	$f(y) = \frac{(y)^{(v/2)-1} e^{-y/2}}{2^{v/2}\Gamma(v/2)};$ $y^2 > 0$	v	$2v$	$(1-2t)^{-v/2}$
Beta	$f(y) = \left[\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\right] y^{\alpha-1}(1-y)^{\beta-1};$ $0 < y < 1$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	does not exist in closed form

EXCERCISES-1

2.1 Suppose a family contains two children of different ages, and we are interested in the gender of these children. Let F denote that a child is female and M that the child is male and let a pair such as FM denote that the older child is female and the younger is male. There are four points in the set S of possible observations:

$$S = \{FF, FM, MF, MM\}.$$

Let A denote the subset of possibilities containing no males; B, the subset containing two males; and C, the subset containing at least one male. List the elements of A, B, C, $A \cap B$, $A \cup B$, $A \cap C$, $A \cup C$, $B \cap C$, $B \cup C$, and $C \cap \overline{B}$.

2.2 Suppose that A and B are two events. Write expressions involving unions, intersections, and complements that describe the following:

a Both events occur.

b At least one occurs.

c Neither occurs.

d Exactly one occurs.

2.3 Draw Venn diagrams to verify DeMorgan's laws. That is, for any two sets A and B, $\overline{(A \cup B)} = \overline{A} \cap \overline{B}$ and $\overline{(A \cap B)} = \overline{A} \cup \overline{B}$.

EXCERCISES-2

2.4 If A and B are two sets, draw Venn diagrams to verify the following:

a $A = (A \cap B) \cup (A \cap \overline{B})$.

b If $B \subset A$ then $A = B \cup (A \cap \overline{B})$.

2.5 Refer to Exercise 2.4. Use the identities $A = A \cap S$ and $S = B \cup \overline{B}$ and a distributive law to prove that

a $A = (A \cap B) \cup (A \cap \overline{B})$.

b If $B \subset A$ then $A = B \cup (A \cap \overline{B})$.

c Further, show that $(A \cap B)$ and $(A \cap \overline{B})$ are mutually exclusive and therefore that A is the union of two mutually exclusive sets, $(A \cap B)$ and $(A \cap \overline{B})$.

d Also show that B and $(A \cap \overline{B})$ are mutually exclusive and if $B \subset A$, A is the union of two mutually exclusive sets, B and $(A \cap \overline{B})$.

2.6 From a survey of 60 students attending a university, it was found that 9 were living off campus, 36 were undergraduates, and 3 were undergraduates living off campus. Find the number of these students who were

a undergraduates, were living off campus, or both.

b undergraduates living on campus.

c graduate students living on campus.

2.7 A group of five applicants for a pair of identical jobs consists of three men and two women. The employer is to select two of the five applicants for the jobs. Let S denote the set of all possible outcomes for the employer's selection. Let A denote the subset of outcomes corresponding to the selection of two men and B the subset corresponding to the selection of at least one woman. List the outcomes in A, $\overline{B}$, $A \cup B$, $A \cap B$, and $A \cap \overline{B}$. (Denote the different men and women by M_1, M_2, M_3 and W_1, W_2, respectively.)

EXCERCISES-3

- **2.71** If two events, A and B, are such that $P(A) = .5$, $P(B) = .3$, and $P(A \cap B) = .1$, find the following:

  a $P(A|B)$
  b $P(B|A)$
  c $P(A|A \cup B)$
  d $P(A|A \cap B)$
  e $P(A \cap B|A \cup B)$

2.75 Cards are dealt, one at a time, from a standard 52-card deck.

a If the first 2 cards are both spades, what is the probability that the next 3 cards are also spades?
b If the first 3 cards are all spades, what is the probability that the next 2 cards are also spades?
c If the first 4 cards are all spades, what is the probability that the next card is also a spade?

2.79 If $P(A) > 0$, $P(B) > 0$, and $P(A) < P(A|B)$, show that $P(B) < P(B|A)$.

2.80 Suppose that $A \subset B$ and that $P(A) > 0$ and $P(B) > 0$. Are A and B independent? Prove your answer.

2.81 Suppose that A and B are mutually exclusive events, with $P(A) > 0$ and $P(B) < 1$. Are A and B independent? Prove your answer.

2.82 Suppose that $A \subset B$ and that $P(A) > 0$ and $P(B) > 0$. Show that $P(B|A) = 1$ and $P(A|B) = P(A)/P(B)$.

EXCERCISES-4

3.12 Let Y be a random variable with $p(y)$ given in the accompanying table. Find $E(Y)$, $E(1/Y)$, $E(Y^2 - 1)$, and $V(Y)$.

y	1	2	3	4
$p(y)$	.4	.3	.2	.1

3.22 A single fair die is tossed once. Let Y be the number facing up. Find the expected value and variance of Y.

***3.29** If Y is a discrete random variable that assigns positive probabilities to only the positive integers, show that

$$E(Y) = \sum_{i=1}^{\infty} P(Y \geq k).$$

3.77 If Y has a geometric distribution with success probability p, show that

$$P(Y = \text{ an odd integer }) = \frac{p}{1 - q^2}.$$

EXCERCISES-5

3.121 Let Y denote a random variable that has a Poisson distribution with mean $\lambda = 2$. Find

- **a** $P(Y = 4)$.
- **b** $P(Y \geq 4)$.
- **c** $P(Y < 4)$.
- **d** $P(Y \geq 4 | Y \geq 2)$.

3.123 The random variable Y has a Poisson distribution and is such that $p(0) = p(1)$. What is $p(2)$?

3.138 Let Y have a Poisson distribution with mean λ. Find $E[Y(Y-1)]$ and then use this to show that $V(Y) = \lambda$.

3.145 If Y has a binomial distribution with n trials and probability of success p, show that the moment-generating function for Y is

$$m(t) = (pe^t + q)^n, \qquad \text{where } q = 1 - p.$$

3.146 Differentiate the moment-generating function in Exercise 3.145 to find $E(Y)$ and $E(Y^2)$. Then find $V(Y)$.

3.147 If Y has a geometric distribution with probability of success p, show that the moment-generating function for Y is

$$m(t) = \frac{pe^t}{1 - qe^t}, \qquad \text{where } q = 1 - p.$$

3.148 Differentiate the moment-generating function in Exercise 3.147 to find $E(Y)$ and $E(Y^2)$. Then find $V(Y)$.

3.149 Refer to Exercise 3.145. Use the uniqueness of moment-generating functions to give the distribution of a random variable with moment-generating function $m(t) = (.6e^t + .4)^3$.

EXCERCISES-6

3.158 If Y is a random variable with moment-generating function $m(t)$ and if W is given by $W = aY + b$, show that the moment-generating function of W is $e^{tb}m(at)$.

3.159 Use the result in Exercise 3.158 to prove that, if $W = aY + b$, then $E(W) = aE(Y) + b$ and $V(W) = a^2V(Y)$.

4.11 Suppose that Y possesses the density function

$$f(y) = \begin{cases} cy, & 0 \leq y \leq 2, \\ 0, & \text{elsewhere.} \end{cases}$$

a Find the value of c that makes $f(y)$ a probability density function.

b Find $F(y)$.

c Graph $f(y)$ and $F(y)$.

d Use $F(y)$ to find $P(1 \leq Y \leq 2)$.

e Use $f(y)$ and geometry to find $P(1 \leq Y \leq 2)$.

4.29 The temperature Y at which a thermostatically controlled switch turns on has probability density function given by

$$f(y) = \begin{cases} 1/2, & 59 \leq y \leq 61, \\ 0, & \text{elsewhere.} \end{cases}$$

Find $E(Y)$ and $V(Y)$.

4.41 A random variable Y has a uniform distribution over the interval (θ_1, θ_2). Derive the variance of Y.

EXCERCISES-7

A random variable Y is said to have a *gamma distribution with parameters* $\alpha > 0$ and $\beta > 0$ if and only if the density function of Y is

$$f(y) = \begin{cases} \dfrac{y^{\alpha-1}e^{-y/\beta}}{\beta^{\alpha}\Gamma(\alpha)}, & 0 \le y < \infty, \\ 0, & \text{elsewhere}, \end{cases}$$

where

$$\Gamma(\alpha) = \int_0^{\infty} y^{\alpha-1}e^{-y}\,dy.$$

4.81 **a** If $\alpha > 0$, $\Gamma(\alpha)$ is defined by $\Gamma(\alpha) = \int_0^{\infty} y^{\alpha-1}e^{-y}\,dy$, show that $\Gamma(1) = 1$.

***b** If $\alpha > 1$, integrate by parts to prove that $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$.

4.82 Use the results obtained in Exercise 4.81 to prove that if n is a positive integer, then $\Gamma(n) = (n-1)!$. What are the numerical values of $\Gamma(2)$, $\Gamma(4)$, and $\Gamma(7)$?

Theory of Reliability and Life Testing

Lectures 040506

Abdulrahman M. Abouammoh
Department of Statistics & OR
King Saud University

Contents

- System and components
- Binary system and components
- Series structures
- Parallel structures
- k-out-of-n Structures
- Some theorems and basic results

.

$:S_b\{0,1\}$

Reliability Theory of Binary systems and components -I

System

– a collection of interacting or interdependent components, organized to provide a function or functions

Components

– can be unique

– can be redundant

Types of systems and components

- **Engineering**
- **Biological**

$:S_b\{0,1\}$

Reliability of systems and components -I

Reliability

- the ability of a system or component to perform its required functions under stated conditions for a specified period of time

System reliability is a function of:

- the **reliability** of the components
- the **interdependence** of the components
- the **topology** of the components

.

.

Binary systems and components -I

Consider a system comprised of *n* components, where each component is either functioning or has failed. Define

$$x_i = \begin{cases} 1, & \text{if the } i^{\text{th}} \text{ component is functioning} \\ 0, & \text{if the } i^{\text{th}} \text{ component has failed} \end{cases}$$

The vector ***x*** = $\{x_1, \ldots, x_n\}$ is called the *state vector*.

Or

.

$$\underline{x} = (x_1, x_2, \ldots, x_n)$$

$:S_b\{0,1\}$

Reliability Theory of Binary systems and components -I

Similarly one can define the binary function to represent the state of the system

Assume that whether the system as a whole is functioning is completely determined by the state vector ***x***. Define

$$\phi(\boldsymbol{x}) = \begin{cases} 1, & \text{if the system is functioning when the state vector is } \boldsymbol{x} \\ 0, & \text{if the system has failed when the state vector is } \boldsymbol{x} \end{cases}$$

The function $\phi(\boldsymbol{x})$ is called the *structure function* of the system.

$: S_b \{0,1\}$

Reliability Theory of Binary systems and components -I

x_i is a binary variable

Binary means it takes either of two values, here 0 and 1 .

ϕ is a binary function

Binary means it takes either of two values, here 0 and 1 .

It is clear that ϕ is a function of all the components of the system.

$: S_b \{0,1\}$

Structural reliability of Binary Systems

State vector; is the vector that represents the state of all components $\boldsymbol{x} = (x_1, \ldots, x_n)$ in the system.

- The state of the system is a function of the states of the components as

$$\phi(\boldsymbol{x}) = \phi(x_1, \ldots, x_n)$$

Definition: ϕ is called **the structure function** of the system..

Since the knowledge of the system gives us knowledge of the structure function ϕ and vice-versa

We sometimes refer to this function by the structure ϕ of the system.

$: S_b \{0,1\}$

Reliability Theory of Binary systems and components -I

Definition: the number of the components in the system are known by the order of the system.

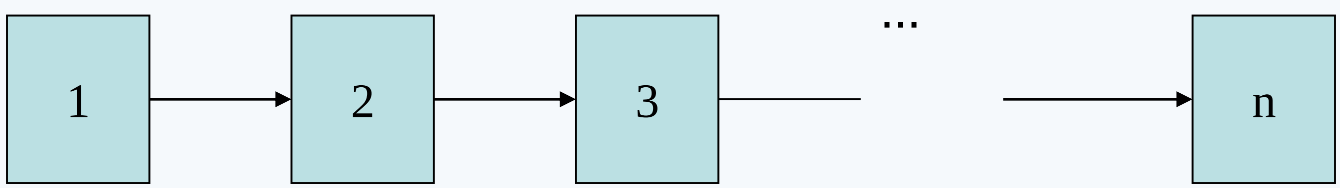
Thus a system of n components is called a system of order n.

Block or Venn diagrams can be used to visualize systems of components.

Types of systems

1. Series systems

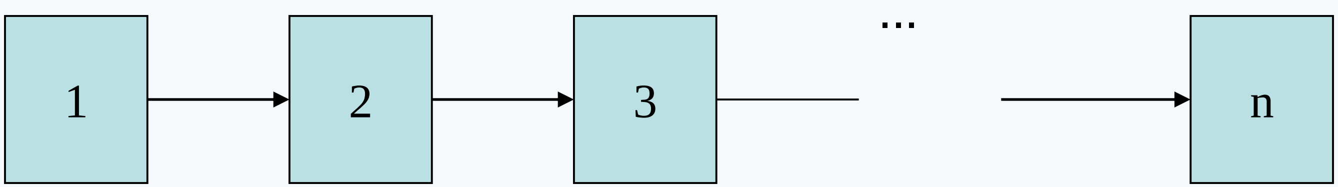
- The block diagram corresponding to a series system is



- The block diagram represents the logical relationship of the operation of the components and the system, it does not represent their physical layout.
- The idea is that if a path can be traced from left to right through the system, then the system operates.

The Series Structure

A series system functions if and only if all of its *n* components are functioning:



Its structure function is given by

$$\phi(\boldsymbol{x}) = \min\{x_1, \ldots, x_n\}$$

$$= x_1 . x_2 \ldots x_n$$

$$= \prod_{i=1}^{n} x_i .$$

The Parallel Structure

2. Parallel system

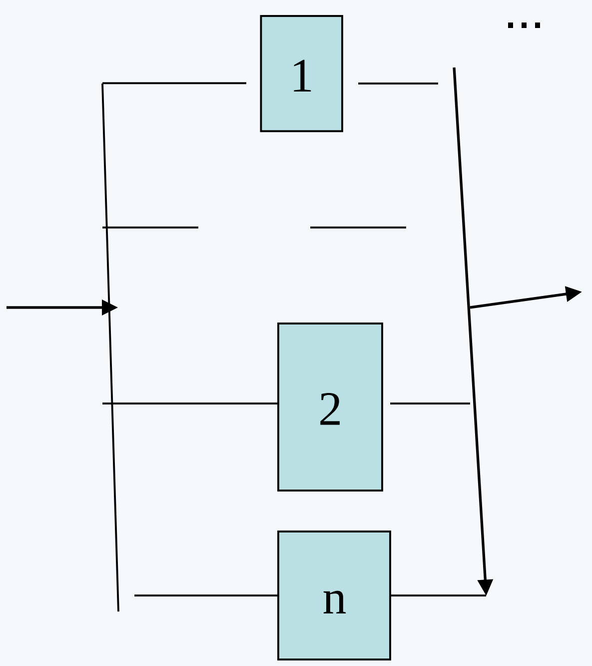
A parallel system functions if and only if at least one of its *n* components iss functioning:

Its structure function is given by

$$\phi(\boldsymbol{x}) = \max_{i=1,\ldots,n} x_i$$

$$= x_1 \vee x_2 \vee \ldots \vee x_n$$

$$= \coprod_{i=1}^{n} x_i$$



The Parallel Structure

Notation: for any binary {0,1) variable

$$\coprod_{i=1}^{n} x_i = 1 - \prod_{i=1}^{n} (1 - x_i)$$

For a system of order 2, on, we has

$$\coprod_{i=1}^{2} x_i = 1 - (1 - x_1)(1 - x_2)$$

$:S_b\{0,1\}$

The k-out-of-n Structure

3. A *k*-out-of-*n* system

Examples:

- Tri-Star aero plane that can function with at least two functioning engines
- Citroen: that can move with at least three working wheels

***A k*-out-of-*n* system:** functions if and only if at least *k* of its *n* components are functioning:

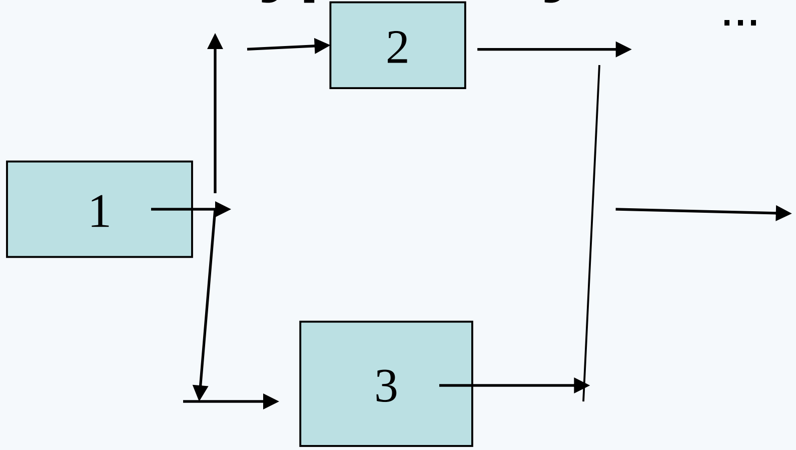
Its structure function is given by

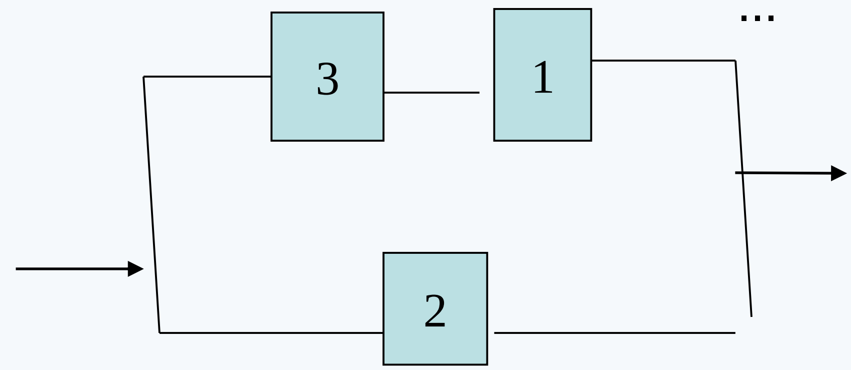
$$\phi(\boldsymbol{x}) = \begin{cases} 1, & \text{if } \sum_{i=1}^{n} x_i \geq k \\ 0, & \text{if } \sum_{i=1}^{n} x_i < k. \end{cases}$$

$: S_b\{0,1\}$

The k-out-of-n Structure

3. Other type of systems





A system of three components can be series parallel, 2-out-of-3, or either one of the above systems

Structural reliability of Binary Systems

Note that: for k-out-of-n system one has for k=n

or equivalently,

$$\phi(\mathbf{x}) = \prod_{1}^{n} x_i \qquad \text{for} \quad k = n,$$

and for k=1

$$\phi(\mathbf{x}) = \coprod_{i=1}^{n} x_i = \max(x_1, \ldots, x_n)$$

Hence a k-out-of-n system can be expressed as

$$\phi(\mathbf{x}) = (x_1 \cdots x_k) \amalg (x_1 \cdots x_{k-1} x_{k+1}) \amalg \cdots \amalg (x_{n-k+1} \cdots x_n)$$

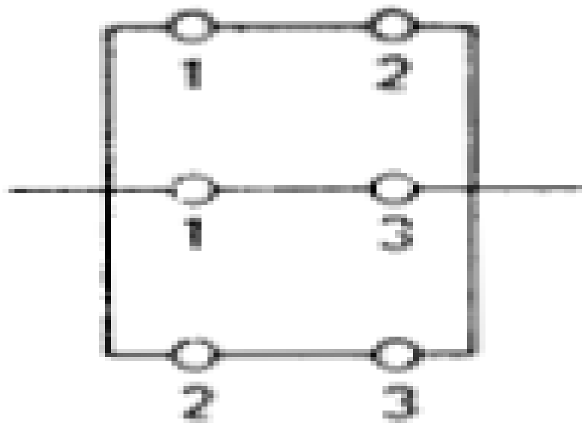
$$\equiv \max\{(x_1 \cdots x_k), (x_1 \cdots x_{k-1} x_{k+1}), \ldots, (x_{n-k+1} \cdots x_n)\}$$

for $1 \leq k \leq n$, where every choice of k out of the n x's appears once exactly.

$:S_b\{0,1\}$

Structural reliability of Binary Systems

The block diagram, for this system is



Note that: i) a series structure is n-out-of-n system

ii) a parallel structure is 1-out-of-n system

Binary Systems

Example: the structure function for a 2-out-of-3 system

$$\phi(\mathbf{x}) = x_1x_2 \amalg x_1x_3 \amalg x_2x_3$$

$$\equiv x_1x_2x_3 + x_1x_2(1 - x_3) + x_1(1 - x_2)x_3 + (1 - x_1)x_2x_3.$$

Note that we have replicated each component for purposes of analysis; physically each component appears once only.

1.5. Example. A more elaborate example is a stereo hi-fi system with the following components:

(a) FM tuner.
(b) Record changer.
(c) Amplifier.
(d) Speaker A.
(e) Speaker B.

$:S_b\{0,1\}$

Structural reliability of Binary Systems

Note that:

We consider the system functioning if we can obtain music (monaural or stereo) through FM or records. The system diagram is illustrated in Figure 1.1.4.

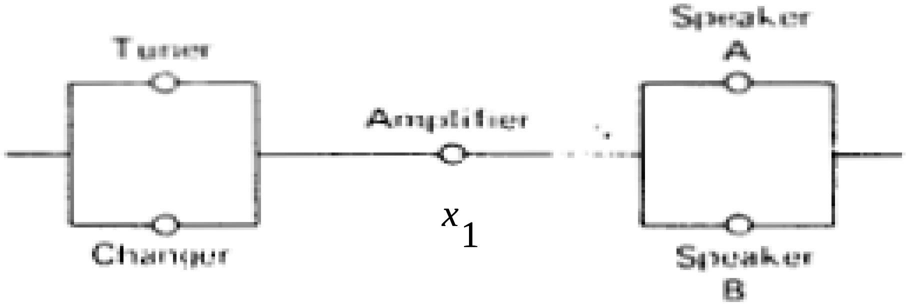


Figure 1.1.4.
Hi-fi system.

Thus the corresponding structure functions

$$\phi(\underline{x}) = (x_1 \vee x_2) x_3 (x_4 \vee x_5)$$

:S_b {0,1}

Structural reliability of Binary Systems

Note that:

1.6. Definition. The ith component is *irrelevant* to the structure ϕ if ϕ is constant in x_i; that is, $\phi(1_i, \mathbf{x}) = \phi(0_i, \mathbf{x})$ for all $(\cdot_i, \mathbf{x})$.[1] Otherwise the ith component is *relevant* to the structure.

[1] *Notation.*

$$(1_i, \mathbf{x}) \equiv (x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n),$$
$$(0_i, \mathbf{x}) \equiv (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n),$$
$$(\cdot_i, \mathbf{x}) \equiv (x_1, \ldots, x_{i-1}, \cdot, x_{i+1}, \ldots, x_n).$$

The irrelevant component

For example, component 2 is irrelevant to the structure pictured in Figure 1.1.5.

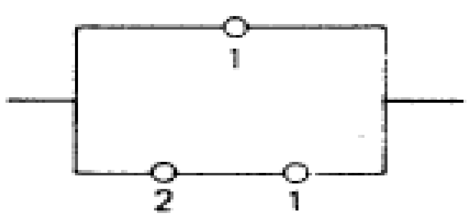


Figure 1.1.5.
Example of irrelevant component.

$: S_b \{0,1\}$

The k-out-of-n Structure

Equivalently k-out-of-n:

or equivalently,

$$\phi(\mathbf{x}) = \prod_1^n x_i \qquad \text{for} \quad k = n,$$

while

$$\begin{aligned}\phi(\mathbf{x}) &= (x_1 \cdots x_k) \amalg (x_1 \cdots x_{k-1} x_{k+1}) \amalg \cdots \amalg (x_{n-k+1} \cdots x_n) \\ &\equiv \max\{(x_1 \cdots x_k), (x_1 \cdots x_{k-1} x_{k+1}), \ldots, (x_{n-k+1} \cdots x_n)\}\end{aligned}$$

for $1 \le k \le n$, where every choice of k out of the n x's appears once exactly.

To illustrate the k-out-of-n structure further, consider the special case of the 2-out-of-3 structure, with structure function given by

$$\begin{aligned}\phi(\mathbf{x}) &= x_1 x_2 \amalg x_1 x_3 \amalg x_2 x_3 \\ &\equiv x_1 x_2 x_3 + x_1 x_2 (1 - x_3) + x_1 (1 - x_2) x_3 + (1 - x_1) x_2 x_3.\end{aligned}$$

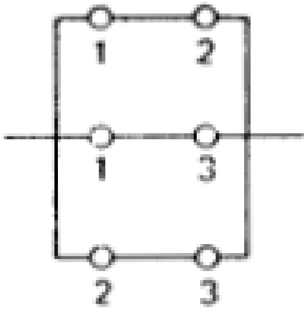
Note that we have replicated each component for purposes of analysis; physically each component appears once only.

$:S_b\{0,1\}$

The k-out-of-n Structure

Examples:

1. An airplane which is capable of functioning if and only if at least two of its three engines are functioning is an example of a 2-out-of-3 system.



2.Prove the following relation for any binary variable

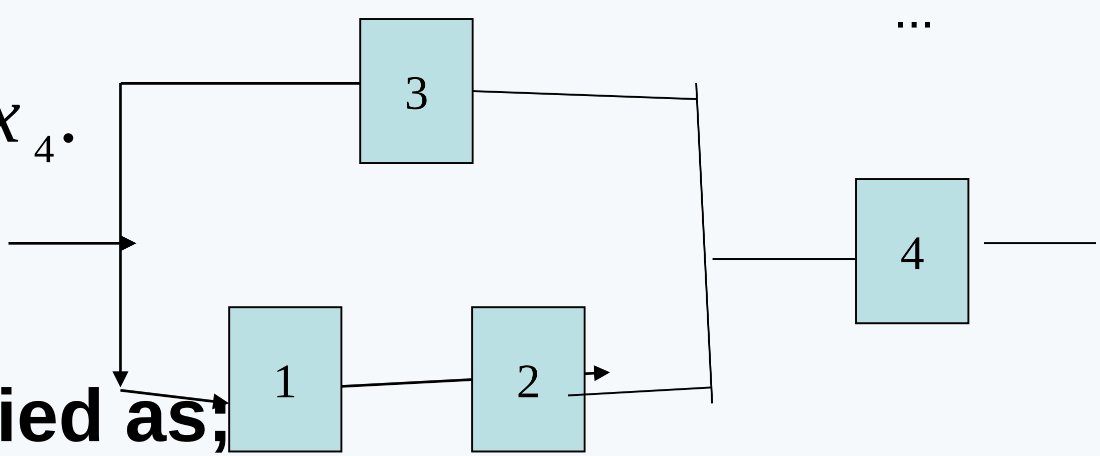
$$\coprod_{i=1}^{n} x_i \equiv 1 - \prod_{i=1}^{n} (1 - x_i),$$

$$x_1 \amalg x_2 = 1 - (1 - x_1)(1 - x_2).$$

Structural reliability of Binary Systems

3. Find the structure function for the following diagrams

$$\phi(\boldsymbol{x}) = [(x_1 x_2) \vee x_3] x_4.$$



This can be simplified as;

$$\begin{aligned}\phi(\boldsymbol{x}) &= [(x_1 x_2) \vee x_3] x_4 \\ &= [1 - (1 - x_1 x_2)(1 - x_3)] x_4 \\ &= (x_1 x_2 + x_3 - x_1 x_2 x_3) x_4 \\ &= x_1 x_2 x_4 + x_3 x_4 - x_1 x_2 x_3 x_4\end{aligned}$$

:S_b {0,1}

Remarks on Structures

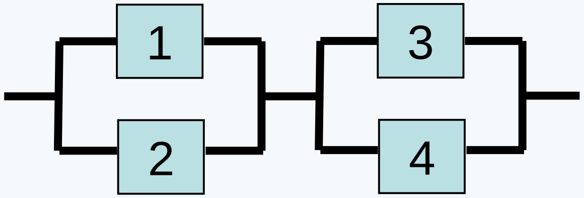
4. Note that for a parallel system

$$\phi(X) = \max\{X_1,\ldots,X_n\},$$
$$= 1 - \prod_{i=1}^{n}(1 - X_i).$$

5. For 2-out-of- 3 system

$$\phi(X) = 1 - (1 - X_1X_2)(1 - X_2X_3)(1 - X_1X_3).$$

6. Given the block diagram



Then the structure function is

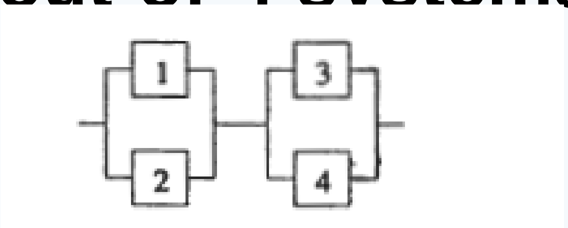
$$\phi(X) = (1 - (1 - X_1)(1 - X_2))(1 - (1 - X_3)(1 - X_4)).$$

Structural reliability of Binary Systems

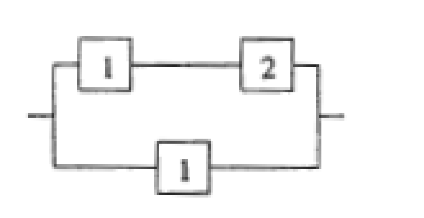
Additional exercises:

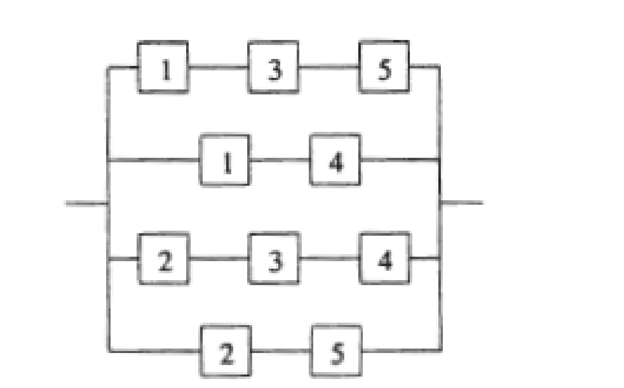
1. Obtain the structure functions of the following systems:

   a) a 2-out-of-4 systems

   b) 

x_1

   c) 

   d) 

:S_b {0,1}

Binary Systems

Additional exercises

2. Draw the block diagrams for the coherent systems having structure functions

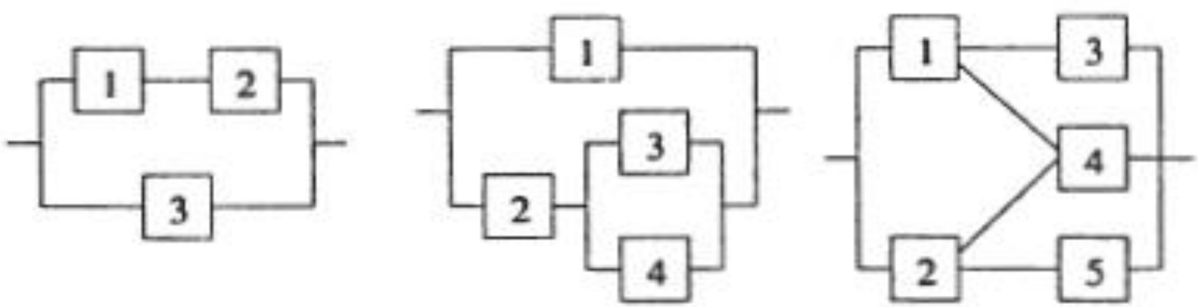
(a) $\phi(\mathbf{x}) = x_1x_2(1-(1-x_3)(1-x_4))$.

(b) $\phi(\mathbf{x}) = (1-(1-x_1)(1-x_2x_3)(1-x_4))x_5$.

(c) $\phi(\mathbf{x}) = x_1x_2 + x_1x_3 + x_2x_3 - x_1^2x_2x_3 - x_1x_2^2x_3 - x_1x_2x_3^2 + (x_1x_2x_3)^2$.

x_1

3. Find the structure functions of the following three a b and c systems



System A System B System C

$:S_b\{0,1\}$

Some Notations and Definitions

Relevant component: The i *th* component is **irrelevant** to the structure ϕ if ϕ is constant in xi

$$\phi(1_i, x) = \phi(0_i, x)$$

$$Note: \quad \phi(1_i, X) = (x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)$$

$$\phi(0_i, X) = (x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)$$

Pivotal decompositions

$$\phi(\mathbf{x}) = x_i \phi(1_i, \mathbf{x}) + (1 - x_i)\phi(0_i, \mathbf{x}) \qquad \text{for all} \quad \mathbf{x} \; (i = 1, \ldots, n). \quad (1.1)$$

The n-Pivotal decompositions:

$$\phi(\mathbf{x}) = \sum_{\mathbf{y}} \prod_{j=1}^{n} x_j^{y_j}(1 - x_j)^{1-y_j}\phi(\mathbf{y}),$$

$: S_b\{0,1\}$

Structural reliability of Binary Systems

Order and Monotonicity:

A partial order is defined on the set of state vectors as follows. Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be two state vectors. We define

$$\boldsymbol{x} \le \boldsymbol{y} \text{ if } x_i \le y_i,\ i = 1, \ldots, n.$$

Furthermore,

x_1

$$\boldsymbol{x} < \boldsymbol{y} \text{ if } \boldsymbol{x} \le \boldsymbol{y} \text{ and } x_i < y_i \text{ for some } i.$$

We assume that if $\boldsymbol{x} \le \boldsymbol{y}$ then $\phi(\boldsymbol{x}) \le \phi(\boldsymbol{y})$. In this case we say that the system is *monotone*.

$:S_b\{0,1\}$

Some Notations and Definitions

Dual structure;

1.9. Definition. Given a structure ϕ, we define its *dual* ϕ^D by

$$\phi^D(\mathbf{x}) = 1 - \phi(\mathbf{1} - \mathbf{x}),$$

where $\mathbf{1} - \mathbf{x} = (1 - x_1, \ldots, 1 - x_n)$.

Or $$\phi^D(\mathbf{x}) = 1 - \phi(\mathbf{1} - \mathbf{x})$$

Coherent systems:

A system of component is coherent if:

1. Its structure function is increasing;
2. Each component is relevant

$: S_b \{0,1\}$

Teoremson binary systems

Theorem 1: Let $\varphi(x)$ be a coherent structure of n components. Then

$$\min\{x_1,\ldots,x_n\} < \phi(x) < \max\{x_1,\ldots,x_n\}.$$

Proof::

Theorem 2:

1.The dual of a series (parallel) system of n component is a parallel (series) system of n component.

2. The dual of a k-out-of – n structure is an (n-k+1) – out-of – n structure

Proof:.

Structural reliability of Binary Systems

Minimal Path Sets:

- A state vector $\boldsymbol{x}$ is call a <u>*path vector*</u> if

$$\phi(\boldsymbol{x}) = 1.$$

- If $\phi(\boldsymbol{y}) = 0$ for all $\boldsymbol{y} < \boldsymbol{x}$, then $\boldsymbol{x}$ is a **minimal path vector.**

x_1

- If $\boldsymbol{x}$ is a minimal path vector, then the set $A = \{i: x_i = 1\}$ is a **minimal path set**.

Theory of Reliability and Life Testing

Lectures 070809

Abdulrahman M. Abouammoh

Department of Statistics & OR

King Saud University

Structural reliability of Binary Systems

Dual Structures

Definition. Given a structure ϕ, we define its *dual* ϕ^D by

$$\phi^D(\mathbf{x}) = 1 - \phi(\mathbf{1} - \mathbf{x}),$$

where $\mathbf{1} - \mathbf{x} = (1 - x_1, \ldots, 1 - x_n)$.

Examples;

Find the dual structures of the following systems;

1) **Parallel system of order n**
2) **Series structure of order n**
3) **k-out-of-n system**
4) **The radio system of five components**

:S_b {0,1}

Structural reliability of Binary Systems

Dual Structures

Definition. Given a structure ϕ, we define its *dual* ϕ^D by

$$\phi^D(\mathbf{x}) = 1 - \phi(\mathbf{1} - \mathbf{x}),$$

where $\mathbf{1} - \mathbf{x} = (1 - x_1, \ldots, 1 - x_n)$.

<u>Examples;</u>

Find the dual structures of the following systems;

1) **Parallel system of order n**
2) **Series structure of order n**

Structural reliability of Binary Systems

Dual Structures

Solution:

1. Let the structure $\{\phi, C\}$ represents a parallel structure , then

$$\phi(\underline{x}) = \coprod_{i=1}^{n} x_i \quad = 1 - \prod_{i=1}^{n} (1 - x_i)$$

$Thus$

$$\phi(\underline{1} - \underline{x}) = \coprod_{i=1}^{n} (1 - x_i) \quad = 1 - \prod_{i=1}^{n} x_i$$

$Therfore,$

$:S_b\{0,1\}$

Structural reliability of Binary Systems

Dual Structures

$$\phi^D(\underline{x}) = 1 - \phi(\underline{1} - \underline{x})$$

$$= 1 - [1 - \prod_{i=1}^{n} x_i]$$

$$= \prod_{i=1}^{n} x_i$$

This means that the dsual is a series structure of order n.

Structural reliability of Binary Systems

Dual Structures

2. Let the structure $\{\phi, C\}$ **represents a series structure , then**

$$\phi(\underline{x}) = \prod_{i=1}^{n} x_i$$

$$Thus$$

$$\phi(\underline{1} - \underline{x}) = \prod_{i=1}^{n} (1 - x_i)$$

$$Therfore,$$

$:S_b\{0,1\}$

Structural reliability of Binary Systems

Dual Structures

$$\phi^D = 1 - \phi(1 - x) = 1 - \prod_{i=1}^{n}(1 - x_i)$$

$$= \coprod_{i=1}^{n} x_i$$

Thus the dual structure of a series system of order n is a parallel structure of order n

Structural reliability of Binary Systems

Dual Structures

3. Let the structure $\{\phi, C\}$ represents a k-out-of-n structure , then show that $\{\phi^D, C\}$ a (n-k+1)-out-of-n system

Left as an exercise

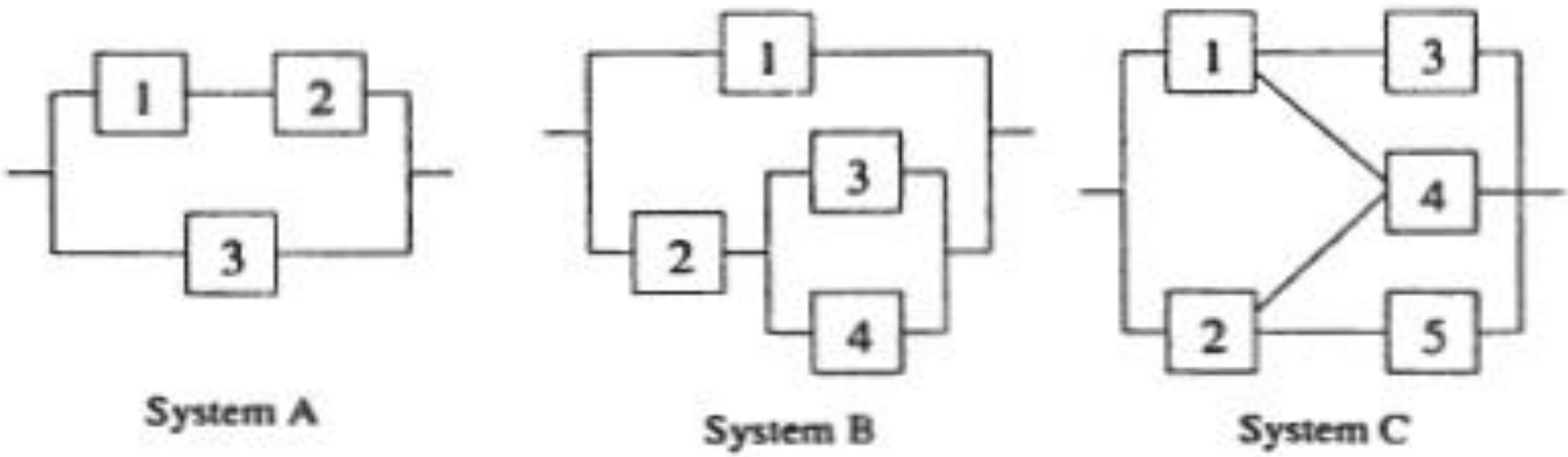
Structural reliability of Binary Systems

Dual Structures

Exercises: Find the sual structure of the following systems:

1) **k-out-of-n system**
2) **The radio system of five components**

3) **Find the dual structure for the following systems**



Structural reliability of Binary Systems

Minimal Path Sets

Let $A_1, \ldots, A_s$ be the minimal path sets of a system. A system will function if and only if all the components of at least one minimal path set are functioning, so that

$$\phi(\boldsymbol{x}) = \max_j \prod_{i \in A_j} x_i .$$

This expresses the system as a parallel arrangement of series systems.

$:S_b \{0,1\}$

Structural reliability of Binary Systems

Minimal Cut Sets

- A state vector ***x*** is call a ***cut vector*** if

$$\phi(\boldsymbol{x}) = 0.$$

- If $\phi(\boldsymbol{y}) = 1$ for all $\boldsymbol{y} > \boldsymbol{x}$, x_1 then ***x*** is a ***minimal cut vector*.**
- If ***x*** is a minimal cut vector, then the set $C = \{i: x_i = 0\}$ is a ***minimal cut set*.**

Structural reliability of Binary Systems

Examples

- The Series System of order n
  - There are *n* minimal cut sets, namely, the sets consisting of all but one component.
- The Parallel System of order n
  - There is one minimal cut set, namely, the empty set.
- The *k*-out-of-*n* System
  - There are $\cdot \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle$ minimal path sets, namely all of
  - the sets consisting of exactly k components.

$:S_b\{0,1\}$

Structural reliability of Binary Systems

Minimal cut sets

Let $C_1, \ldots, C_k$ be the minimal cut sets of a system. A system will not function if and only if all the components of at least one minimal cut set are not functioning, so that

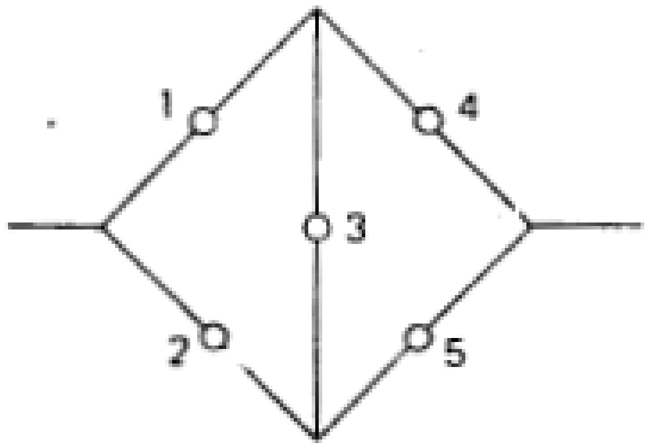
$$\phi(\boldsymbol{x}) = \prod_{j=1}^{k} \max_{i \in C_j} x_i .$$

This expresses the system as a series arrangement of parallel systems.

:

Structural reliability of Binary Systems

Example: the Bridge System



The system whose structure is shown below is called the bridge system. Its minimal path sets are:

{1, 4}, {1, 3, 5}, {2, 5}, {2, 3, 4}.

For example, the system will work if only 1 and 4 are working, but will not work if only 1 is working.

Structural reliability of Binary Systems

The bridge system:-

Its structure function is given by

$$\begin{aligned}\phi(\boldsymbol{x}) &= \max\{x_1x_4, x_1x_3x_5, x_2x_5, x_2x_3x_4\} \\ &= 1-(1-x_1x_4)(1-x_1x_3x_5)(1-x_2x_5)(1-x_2x_3x_4).\end{aligned}$$

Also, one may notice that

The system whose structure is shown below is called the bridge system. Its minimal cut sets are:

{1, 2}, {1, 3, 5}, {4, 5}, {2, 3, 4}.

For example, the system will work if 1 and 2 are not working, but it can work if either 1 or 2 are working.

Structural reliability of Binary Systems

The bridge system: -

Its structure function is given by

$$\varphi(\boldsymbol{x}) = \max\{x_1, x_2\}.\max\{x_1, x_3, x_5\}.\max\{x_4, x_5\}$$

$$.\max\{x_2, x_3, x_4\}.$$

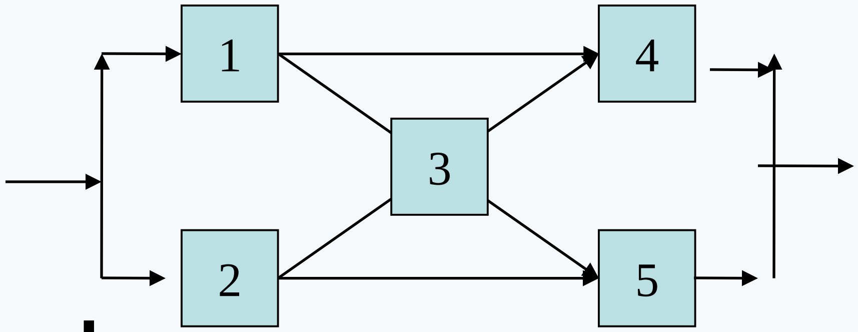
Thus if there are p minimal path sets $\{P_1, P_2, \ldots, P_p\}$ with structure functions $\{\rho_1, \rho_2, \ldots, \rho_p\}$

and k minimal cut set $\{\kappa_1, \kappa_2, \ldots, \kappa_k\}$

Then one has $\{\kappa_1, \kappa_2, \ldots, \kappa_k\}$

Structural reliability of Binary Systems

Here again the bridge system of five components



Here for any path set one has

$$\rho_j(\mathbf{x}) = \prod_{i \in P_j} x_i.$$

Hence one may write the corresponding structure function as

$$\phi(\mathbf{x}) \equiv \coprod_{j=1}^{p} \rho_j(\mathbf{x}) \equiv 1 - \prod_{j=1}^{p} [1 - \rho_j(\mathbf{x})],$$

Structural reliability of Binary Systems

Also for any cut set one has

$$\kappa_j(x) = \coprod_{i \in K_j} x_i,$$

Hence one may write the corresponding structure function as

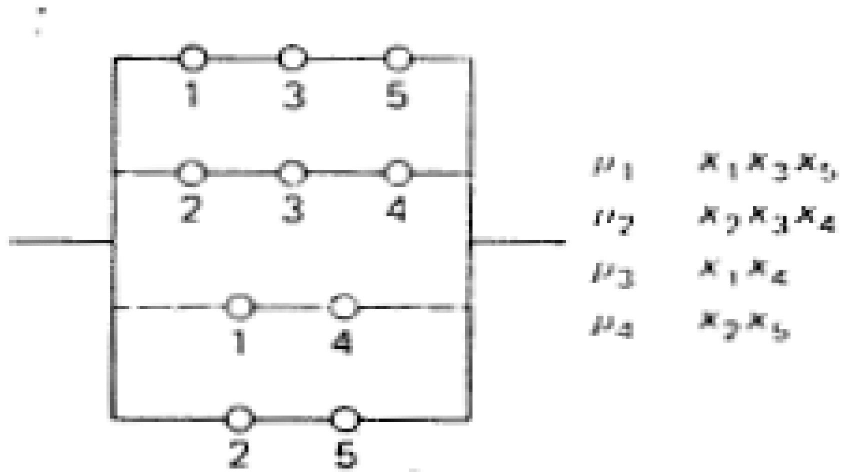
$$\phi(x) = \prod_{j=1}^{k} \kappa_j(x),$$

:S_b {0,1}

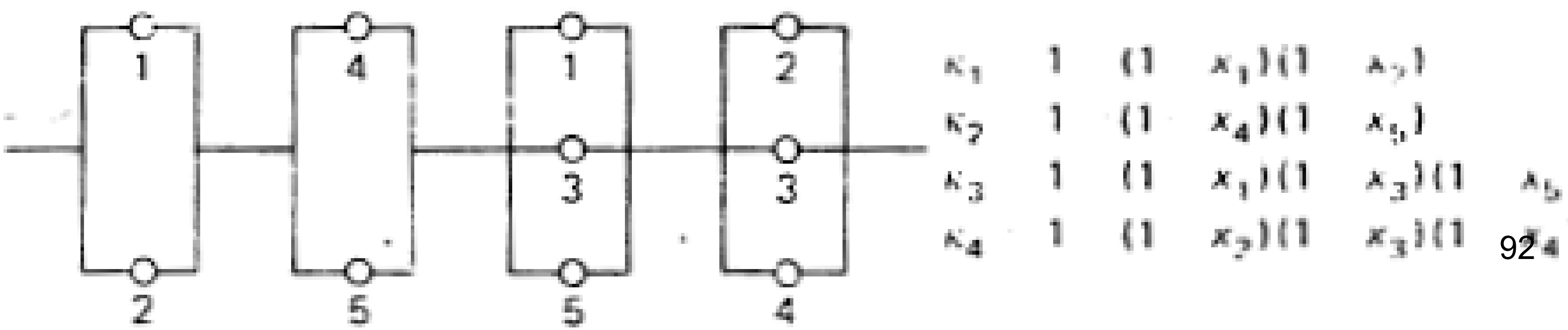
Structural reliability of Binary Systems

Representation of the bridge system:

a) Minimal path representation



b) Minimal cut representation



$:S_b\{0,1\}$

Structural reliability of Binary Systems

These representations can be written as:

$$\phi(\mathbf{x}) \equiv \max_{1 \le j \le p} \rho_j(\mathbf{x}) \equiv \max_{1 \le j \le p} \min_{i \in P_j} x_i,$$

$$\phi(\mathbf{x}) \equiv \min_{1 \le j \le k} \kappa_j(\mathbf{x}) \equiv \min_{1 \le j \le k} \max_{i \in K_j} x_i.$$

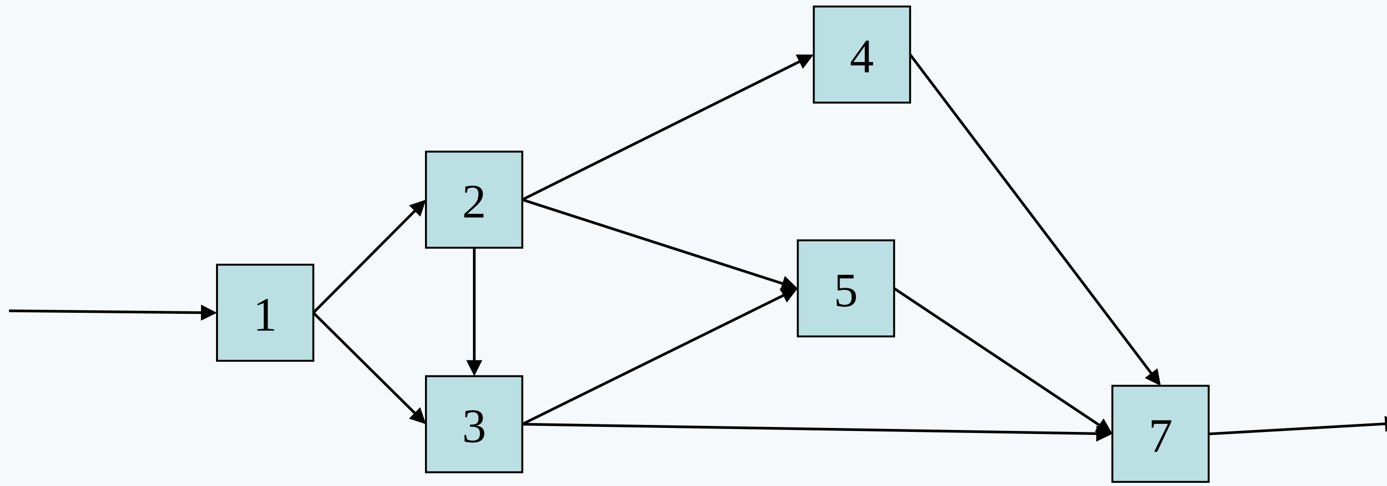
Now let t_i be the life length of the i^{th} components where i=1, 2, …, n, then the life of the system is $\tau_\phi(t)$ hence one has

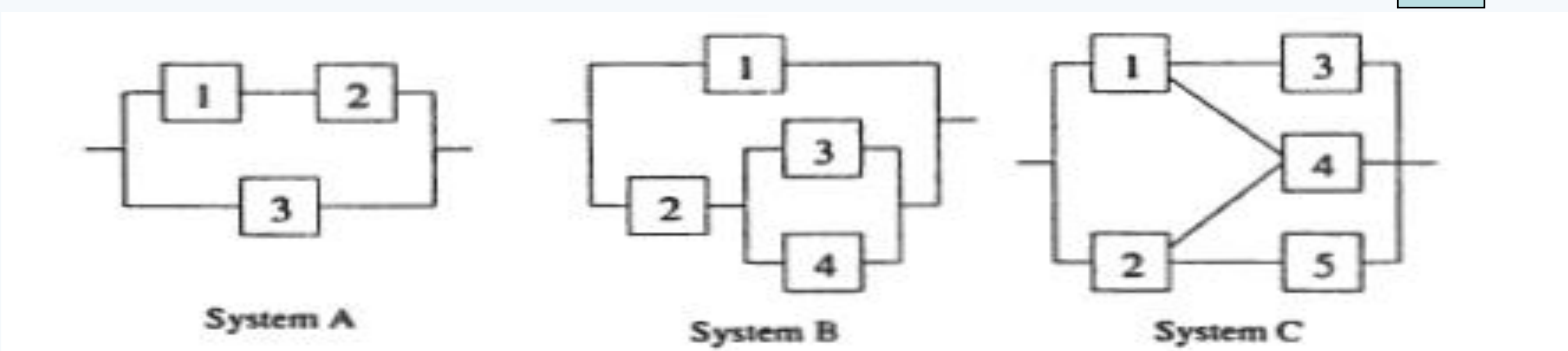
and minimal cut sets $K_1, \ldots, K_k$, then

$$\max_{1 \le j \le p} \min_{i \in P_j} t_i \equiv \tau_\phi(\mathbf{t}) \equiv \min_{1 \le j \le k} \max_{i \in K_j} t_i.$$

Structural reliability of Binary Systems

Example : What are the minimal path sets and the minimal cut sets for the following four systems?





$: S_b \{0,1\}$

Structural reliability of Binary Systems

Components relevance aand Coherent structures

Definition: the component I is irrelevant to the structure if is constant in x_i or

Otherwise the component is relevant..

Remark: If component i is relevant then one can find at least one vector of states such that: $(\cdot_i, \underline{x})$ **such that**

$$\phi(0_i, \underline{x}) \neq \phi(1_i, \underline{x})$$

or in other word

$$\phi(0_i, \underline{x}) < \phi(1_i, \underline{x})$$

: S_b {0,1}

Structural reliability of Binary Systems

Coherent structures

Definition: A system of components $\{\phi, C\}$ is called **coherent** if

i) its structure function is increasing and

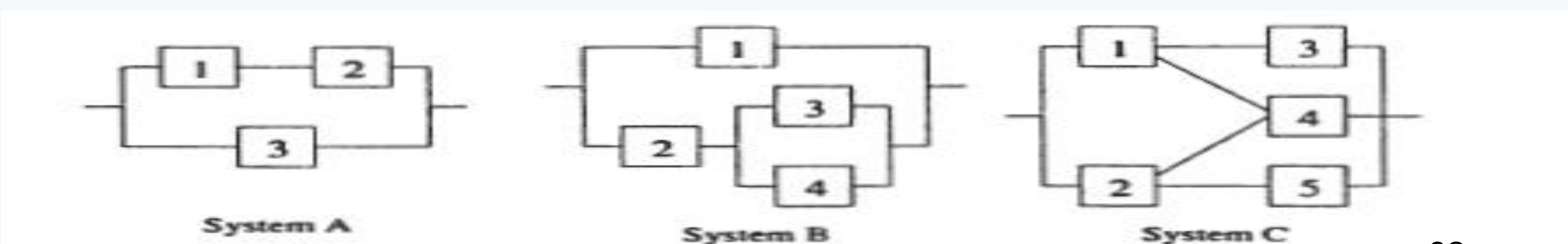
ii) every component is relevant.

Exercises: Show that every components of the following systems is relevant

1) Series system of order n.

2) Parallel system of order n. iii) 2-out-of n system

3)



Structural reliability of Binary Systems

Coherent structures

<u>Theorem</u> : The **dual** structure $\{\phi^D, C\}$ of a coherent system $\{\phi, C\}$ is coherent

Proof: Assume that the structure $\{\phi, C\}$ is coherent, then one has

i) its structure function ϕ is increasing and

ii) every component is relevant.

Therefore if ϕ is increasing, means that for any two vectors $\underline{x} < \underline{y}$ leads to $\phi(\underline{x}) \leq \phi(\underline{y})$

Further, note that

$$\underline{1} - \underline{x} > \underline{1} - \underline{y}$$

Structural reliability of Binary Systems

Coherent structures

Proof:- Now note that

$$\phi(\underline{1}-\underline{x}) \geq \phi(\underline{1}-\underline{y})$$

This leads to

$$1-\phi(1_i,\underline{1}-\underline{x}) \leq 1-\phi(0_i,\underline{1}-\underline{x})$$

or

$$\phi^D(\underline{x}) \leq \phi^D(\underline{y})$$

This means that ϕ^D is increasing.

To complete the proof let unit i be relevant to the structure ϕ, then there exist a vector $(\cdot_i,\underline{x})$ such that $\phi(1_i,\underline{x}) > \phi(0_i,\underline{x})$

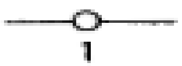
:S_b {0,1}

Structural reliability of Binary Systems

Coherent structures

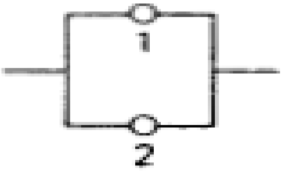
Samples of coherent structures of order 1, 2 and 3

Order 1. $\phi(x)$ x_1

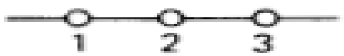


Order 2. $\phi(x)$ $x_1 x_2$

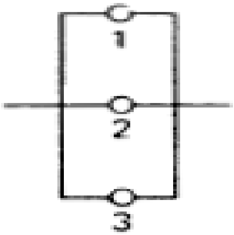
$\phi(x)$ $x_1 \amalg x_2$



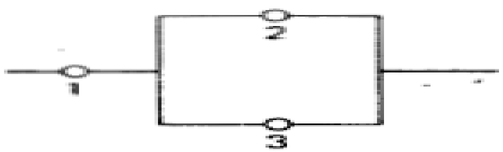
Order 3. $\phi(x)$ $x_1 x_2 x_3$



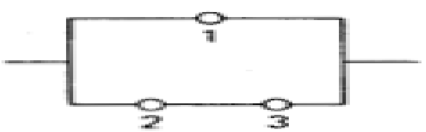
$\phi(x)$ $x_1 \amalg x_2 \amalg x_3$



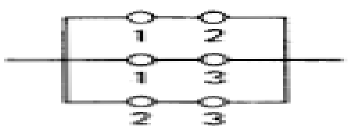
$\phi(x) - x_1 (x_2 \amalg x_3)$



$\phi(x) - x_1 \amalg x_2 x_3$



$\phi(x) - x_1 x_2 \amalg x_1 x_3 \amalg x_2 x_3$



Structural reliability of Binary Systems

Theorems

<u>Theorem</u>: For any coherent structure , one has

i) $\phi(\underline{x} \vee \underline{y}) \geq \phi(\underline{x}) \vee \phi(\underline{y})$ equality sign hold for all vectors $\underline{x}$ and $\underline{y}$ vectors iff the system is parallel

ii) $\phi(\underline{x} . \underline{y}) \leq \phi(\underline{x}) . \phi(\underline{y})$ equality sign hold for all vectors $\underline{x}$ and $\underline{y}$ vectors iff the system is series.

PROOF. $x_i \amalg y_i \geq x_i \ \forall i$, so that $\phi(\mathbf{x} \amalg \mathbf{y}) \geq \phi(\mathbf{x})$ since ϕ is increasing. Similarly, $x_i \amalg y_i \geq y_i \ \forall i$, so that $\phi(\mathbf{x} \amalg \mathbf{y}) \geq \phi(\mathbf{y})$. It follows that

$$\phi(\mathbf{x} \amalg \mathbf{y}) \geq \max[\phi(\mathbf{x}), \phi(\mathbf{y})] \equiv \phi(\mathbf{x}) \amalg \phi(\mathbf{y}).$$

A similar argument proves (b).

If the system is series, $\phi(\mathbf{x} \cdot \mathbf{y}) = \prod_{i=1}^{n} x_i y_i = \prod_{i=1}^{n} x_i \prod_{i=1}^{n} y_i = \phi(\mathbf{x})\phi(\mathbf{y})$.

Contents

- **System and components**
- **Structural reliability of Binary Systems**
- **Structural importance**
- **Reliability importance**
- **Standardized importance**
- **Bivariate importance**

.

Structural reliability of Binary Systems

Relative Importance

Definition: **Relative Importance of Components.** For a given coherent system, some components are more important than others in determining whether the system functions or not. For example, if a component is in series with the rest of the system, then it would seem to be at least as important as any other component in the system. It is clearly of value to the designer and reliability analyst to have a quantitative measure of the importance of the individual components in the system.

How important is component i in determining whether the system functions or not? First, suppose we are given the state of each of the remaining components, $(\cdot_i, \mathbf{x})$. Then if $\phi(1_i, \mathbf{x}) = 1$ while $\phi(0_i, \mathbf{x}) = 0$, that is, if

$$\phi(1_i, \mathbf{x}) - \phi(0_i, \mathbf{x}) = 1,$$

Structural reliability of Binary Systems

Relative Importance

Definition:

we would consider component i more important than if $\phi(1_i, \mathbf{x}) = 1 = \phi(0_i, \mathbf{x})$ or $\phi(1_i, \mathbf{x}) = 0 = \phi(0_i, \mathbf{x})$. In the first case, (3.8), the state of component i determines whether the system functions or not, whereas in the alternative cases [(3.8) not true], the state of component i is of no consequence. When (3.8) holds, we call $(1_i, \mathbf{x})$ a *cr...... path vector for* i and $C_1(1_i, \mathbf{x})$ the corresponding *critical path set for* i. We let

The definition of relative importance of a component i is based on the number of all critical path vector for component i compared to all possible state vectors.

Structural reliability of Binary Systems
Relative Importance

$$n_\phi(i) = \sum_{\{\mathbf{x} \mid x_i = 1\}} [\phi(1_i, \mathbf{x}) - \phi(0_i, \mathbf{x})]$$

denote the total number of critical path vectors for i (or equivalently, the total number of critical path sets for i).

This suggests the following plausible measure of the *structural importance of component* i:

Now the relative or structural importance of a component with structure function and a system of order n is given by

$$I_\phi(i) = \frac{1}{2^{n-1}} n_\phi(i) = \frac{1}{2^{n-1}} \sum_{\{\mathbf{x} \mid x_i = 1\}} [\phi(1_i, \mathbf{x}) - \phi(0_i, \mathbf{x})],$$

the proportion of the 2^{n-1} outcomes having $x_i = 1$ which are critical path vectors for i.

Thus for any given structure ϕ, we may order the components as to structural importance by ordering the values $I_\phi(1), \ldots, I_\phi(n)$.

Structural reliability of Binary Systems

Relative Importance

The structural, or relative, importance of unit i in the coherent system $\{C, \phi\}$ is given by

$$SI_\phi(i) = strucural\ \ impor\ tance\ \ for$$

$$component\ \ i\ \ under\ \ the\ \ structure\ \ \phi$$

$$= n_\phi(\mathrm{i}) / 2^{n-1}$$

Where, $n_\phi(\mathrm{i})$ is the number of all possible critical vectors for unit i that is found by

$$n_\phi(\mathrm{i}) = \sum_{\underline{x}} [\phi(1_i, \underline{x}) - \phi(0_i, \underline{x})]$$

Structural reliability of Binary Systems

Relative Importance

Examples. (a) Let ϕ be a 2-out-of-3 structure. Then $I_\phi(1) = 1/2^2 \cdot 2 = 1/2$, since among the four outcomes 100, 101, 110, 111, there are two critical path vectors for component 1 (101 and 110).

By symmetry, $I_\phi(2) = I_\phi(3) = 1/2$ also.

(b) Let $\phi(\mathbf{x}) = x_1(x_2 \amalg x_3)$. Then $I_\phi(1) = 1/2^2 \cdot 3 = 3/4$, since among the four outcomes 100, 101, 110, 111, there are three critical path vectors for component 1 (101, 110, and 111).

However, $I_\phi(2) = 1/2^2 \cdot 1 = 1/4$, since among the four outcomes 010, 011, 110, 111, there is only one critical path vector for component 2—namely, 110. By symmetry, $I_\phi(3) = 1/4$ also.

Note that component 1 is distinctly more important than component 2 or 3. This is to be expected, since component 1 is in series with the rest of the system.

Structural reliability of Binary Systems

Relative Importance

Exercise; Find the structural importance of every component in the following systems

(a) The 2-out-of-3 system;

(b) The 3-out-of-4 system;

(c) The series system of three components;

(d) The parallel system of three components.

Exercise; Find the structural importance of every component in the following systems

i) the radio or relay system.

ii) the bridge system

.

Structural reliability of Binary Systems

Relative Importance

Exercise; Find the structural importance of every component in the following systems.

i) Series system of order n

ii) Parallel system of order n

Solution:

i) For the series system: note that the only critical vector for component i is $(1,1,\ldots,\cdot_i,1,\ldots,1)$

$$SI_\phi(i) = n_\phi(\mathrm{i}) / 2^{n-1}$$

$$= 1/2^{n-1}$$

$: S_b \{0,1\}$

Structural reliability of Binary Systems

Relative Importance

Solution:

i) For the parallel system: note that the only critical vector for component i is $(0, 0, \ldots, \cdot_i, 0, \ldots, 0)$

Thus

$$SI_\phi(i) = n_\phi(\text{i}) / 2^{n-1}$$

$$= 1 / 2^{n-1}$$

Try to find the relative importance of k-out-of-n-system

Structural reliability of Binary Systems

Relative Importance

Remarks

I) The sum of the structural or relative importance of all units in a coherent system need not to be one

II) Relative or structural importance is measure for the importance of a component compared to other components in the same system and it is not an absolute value by itself.

$:S_b \{0,1\}$

Structural reliability of Binary Systems

Standardized Relative Importance

Definition: The standardized structural or relative importance of any component i in a cohernt system of order n is

$$SSI_{\phi}(i) = SSI_{\phi}(i)$$

$$= SI_{\phi}(i) / \sum_{j=1}^{n} SI_{\phi}(j) s$$

The above definition leads to the following

$$\sum_{j=1}^{n} SSI_{\phi}(i) = 1$$

Theory of Reliability and Life Testing

Lectures 131415

System Reliability

Abdulrahman M. Abouammoh
Department of Statistics & OR
King Saud University

$: S_b \{0,1\}$

System Reliability

Observing components and system over time one can note that

Component Reliability:

$$p_i = P\{X_i = 1\}$$

Further

$$EX_i = \sum x_i P\{X_i = x_i\}$$

$$: \quad = 1.P\{X_i = 1\} + 0.P\{X_i = 0\}$$

$$= P\{X_i = 1_i\} = p_i$$

$:S_b\{0,1\}$

System Reliability

System Reliability:

$$h = E\,\phi(X)$$

$$= \sum_{\phi(x)} \phi(\mathbf{x}) P\{\phi(\mathbf{X}) = \phi(\mathbf{x})\}$$

$$= 1.P\{\phi(\mathbf{X}) = 1\} + 0.P\{\phi(\mathbf{X}) = 0\}$$

$$= P\{\phi(\mathbf{X}) = 1\}$$

$$= \mathrm{h}(\mathbf{p})$$

When the components are independent then one can represent the system in term of the components that is

:

:

System Reliability

When the components are independent, then *h* can be expressed as a function of the component reliabilities:

$$h = h(\mathbf{p}), \text{ where } \mathbf{p} = (p_1, \ldots, p_n).$$

The function $h(\mathbf{p})$ is called the ***reliability function***.

System Reliability

- The system reliability, h, is defined by

$$h = P(\varphi(X) = 1).$$

- h is a function of p, the reliability vector $(p_1, \ldots, p_n)$, so we write h=h(p).
- There are several techniques for calculating h(p).
  - Applying the definition.
  - Using the expectation operator.
  - Using path vectors.
  - Using cut vectors.- Decomposition. -Monte Carlo Simulation in spreadsheets.

System Reliability

$$h = h(\mathbf{p}),\ where\ \ p = \ (p_1,\ \ldots, p_n).$$

The function $h(\mathbf{p})$ is called the **reliability function** and if all the components have the same reliability then

$$h = h(\mathbf{p}) = h(p),\ where\ \ p = \ (p,\ \ldots, p).$$

Examples:-

1. The series system

$$h(\mathbf{p}) = E(\phi(\boldsymbol{X})) = P\{\phi(\boldsymbol{X}) = 1\}$$

$$= E\prod_{i=1}^{n} X_i . = \prod_{i=1}^{n} EX_i .$$

$$= \prod_{i=1}^{n} p_i .$$

:
:

$:S_b\{0,1\}$

System Reliability

2. The parallel system

$$h(\mathbf{p}) = P\{\phi(\boldsymbol{X}) = 1\}$$

$$= E\,\phi(\boldsymbol{X})$$

$$= \mathrm{E}\coprod_{i-1}^{n} X_i$$

$$= E\,[1 - \prod_{i=1}^{n}(1 - X_i).]$$

$$= [1 - \prod_{i=1}^{n}(1 - EX_i).]$$

$$= 1 - \prod_{i=1}^{n}(1 - p_i).$$

:

System Reliability

Using Expectation

- Be careful when using the expectation method.
- What is the system reliability of a 2 out of 3 system?
  - Notice that $E[X_i^2]=p_i$, not p_i^2, so

System Reliability

Using Expectation

$$
\begin{aligned}
h(p) &= E[\phi(X)] \\
&= E[1-(1-X_1X_2)(1-X_2X_3)(1-X_1X_3)] \\
&= E[X_1X_2 + X_2X_3 + X_1X_3 \\
&\quad -X_1^2X_2X_3 - X_1X_2^2X_3 - X_1X_2X_3^2 \\
&\quad +X_1^2X_2^2X_3^2] \\
&= p_1p_2 + p_2p_3 + p_1p_3 \\
&\quad -p_1p_2p_3 - p_1p_2p_3 - p_1p_2p_3 \\
&\quad +p_1p_2p_3 \\
&= p_1p_2 + p_2p_3 + p_1p_3 - 2p_1p_2p_3.
\end{aligned}
$$

$: S_b \{0,1\}$

System Reliability

3. k-out-of-n system where

$$p_i = p, \quad \forall\, i = 1, 2, \ldots, n$$

Then

$$h(\mathbf{p}) = P\{\phi(\mathbf{X}) = 1\}$$

$$= P\{\sum_{i=1}^{n} X_i \geq k\}$$

:

$$= \sum_{i=k}^{n} \binom{n}{i} p^i (1-p)^{n-i}.$$

System Reliability

<u>Theorem</u>. If h(**p**) is the reliability function of a coherent system of independent components, then *h*(**p**) is an increasing function in **p**.

Proof: Since $\{c, \phi\}$ is a coherent structure then $\phi(x)$ is increasing function in every argument.

Thus
$$h(\mathbf{p}) = P\{\phi(\mathbf{X}) = 1\}$$
$$= E[\phi(\mathbf{X})]$$
$$= \sum_i i \; p(\phi(\mathbf{X}) = i)$$

System Reliability

Theorem.: Let $h(p)$ and be $h(p^{/})$ the reliability of coherent system of order n with units' reliabilities. $0 < p_i < 1,\ and\ 0 < p^{/}_i < 1,\ i = 1, 2, \ldots, n$

Then

i) $h(p \vee p^{/}) \geq h(p) \vee h(p^{/})$

ii) $\bar{\phi}(p \,.\, \bar{p}^{/}) \leq \bar{h}(p).h\,\bar{(}p^{/})$

Equality hold for all values for (i) if the system is parallel and for (ii) if the system is series.

.

System Reliability

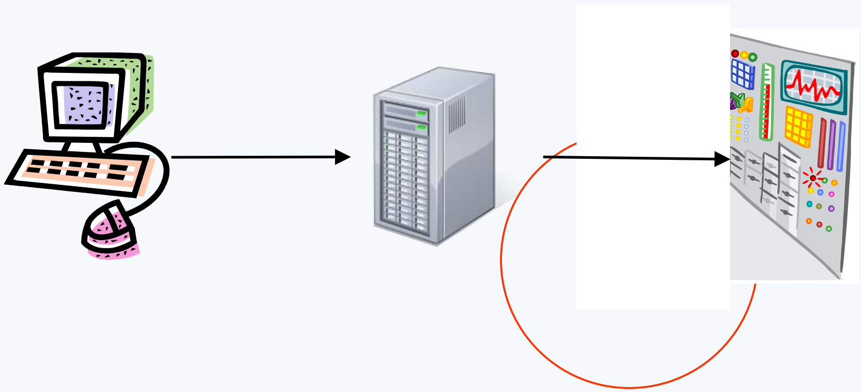
Example - Communications System

Suppose we are concerned about the reliability of the controller, server and transformer. Assume that these components are independent, and that

$p_{controller} = .95$

$p_{server} = .96$

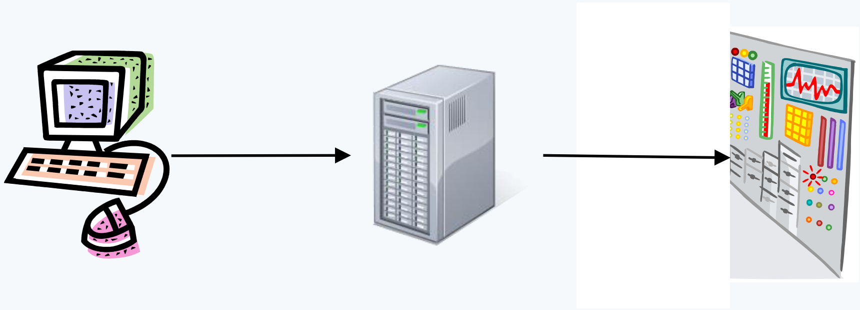
$p_{transformer} = .99$



System Reliability

Example - Communications System

Since these three components connect in series, the system A consisting of these components has reliability

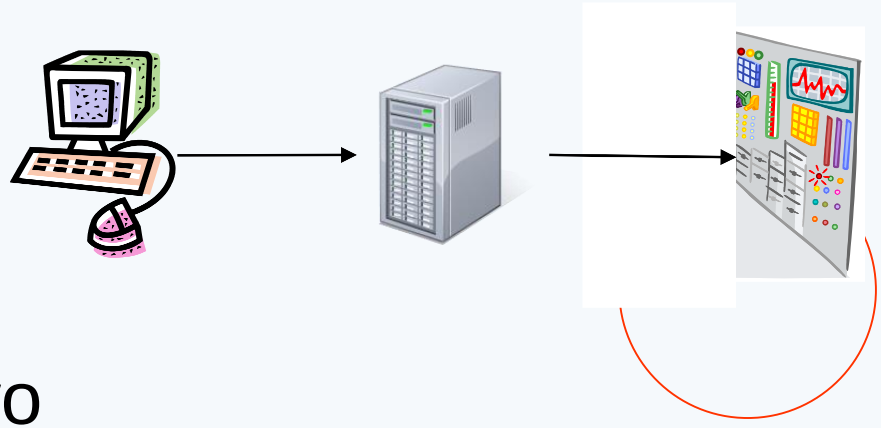


$$h_{system\_A}$$
$$= p_{controller} \cdot p_{server} \cdot p_{transformer}$$
$$= .90$$

System Reliability

Example - Communications System

Suppose that we want to increase the reliability of system A. What are our options?

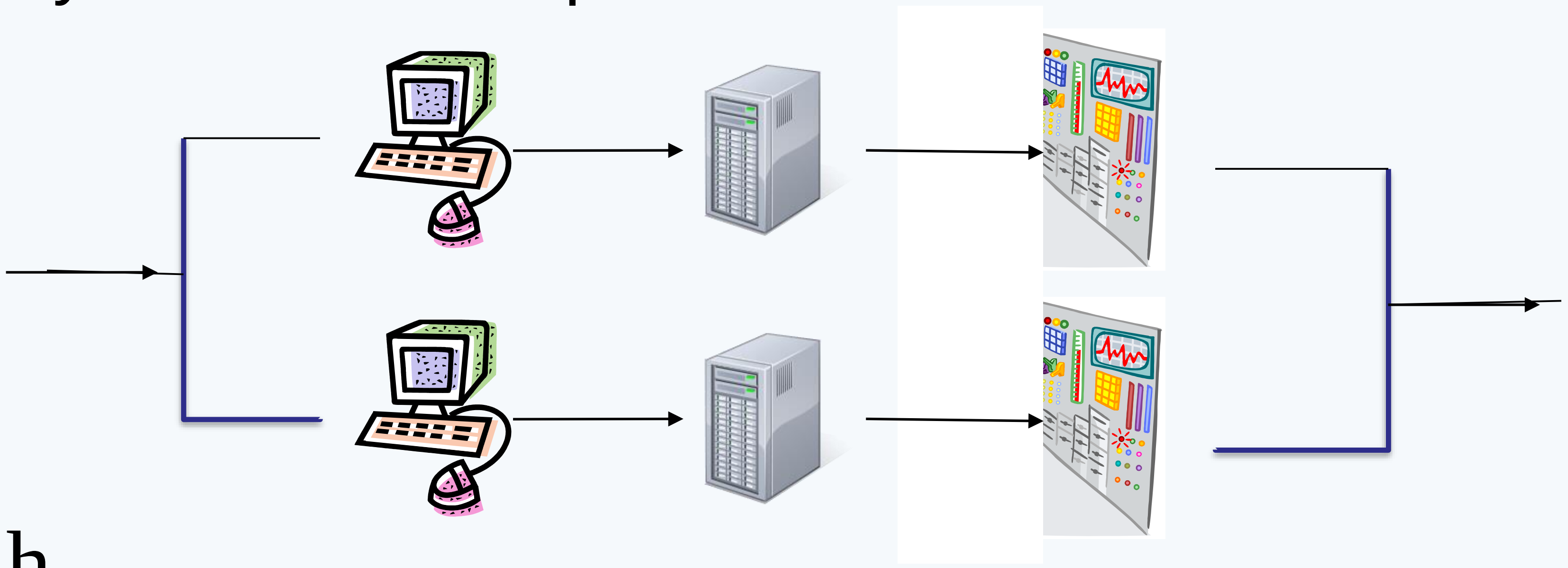


Suppose that we have two controllers, two servers, and two transformers.

System Reliability

Example - Communications System

Option 1: Duplicate the entire system A, creating a new system B made up of two "A"s

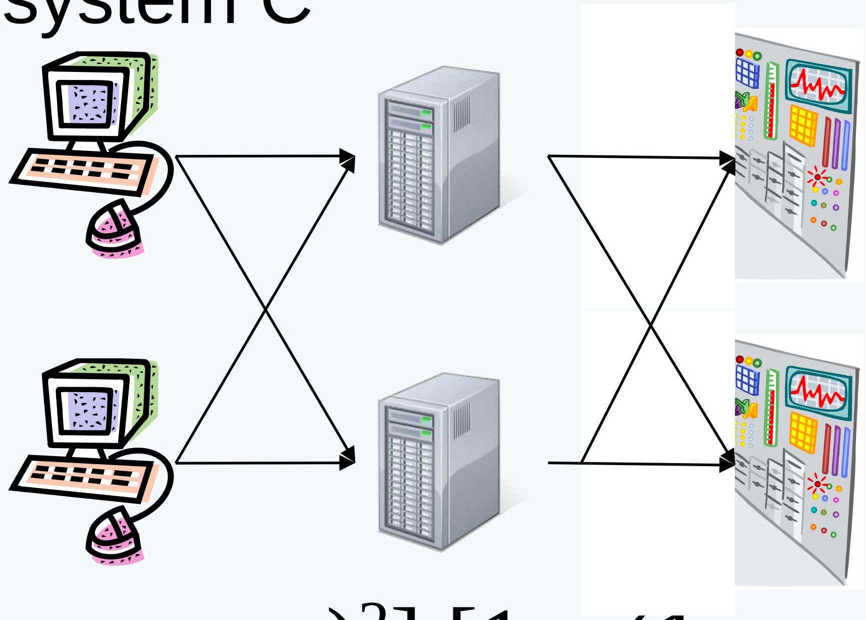


$$h_{system\_B} = 1 - (1 - h_{system\_A})^2 = .991$$

System Reliability

Example - Communications System

Option 2: Replicate components within system A, creating a new system C



h_{system_C}

$= [1 – (1 – p_{controller})^2] [1 – (1 – p_{server})^2] [1 – (1 – p_{transformer})^2]$

$= .996$

System Reliability

Example - Communications System

Find the value for the following reliability for the earlier system

1. $\phi(\underline{p} \,.\, \underline{p}^{/})$

2. $h(\underline{p}).h(\underline{p}^{/})$

and compare the results.

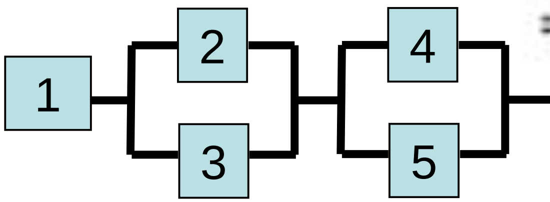
:S_b {0,1}

System Reliability

Example. If we take the structure of Example 4.2 of Chapter 1, then

$$\phi(\mathbf{x}) = x_1(x_2 \amalg x_3)(x_4 \amalg x_5)$$

$$= x_1(x_2 + x_3 - x_2x_3)(x_4 + x_5 - x_4x_5)$$



$$= x_1x_2x_4 + x_1x_2x_5 - x_1x_2x_4x_5$$

$$+ x_1x_3x_4 + x_1x_3x_5 - x_1x_3x_4x_5$$

$$- x_1x_2x_3x_4 - x_1x_2x_3x_5 + x_1x_2x_3x_4x_5.$$

Hence

$$h(\mathbf{p}) = E\phi(\mathbf{X}) = p_1p_2p_4 + p_1p_2p_5 - p_1p_2p_4p_5 + p_1p_3p_4 + p_1p_3p_5$$

$$- p_1p_3p_4p_5 - p_1p_2p_3p_4 - p_1p_2p_3p_5 + p_1p_2p_3p_4p_5.$$

Theory of Reliability and Life Testing

Lectures 161718

System Reliability

Associationa and Reliability Bounds

Abdulrahman M. Abouammoh
Department of Statistics & OR
King Saud University

Modeling Failures Over Time

- So far we have only seen indicators of failure over a given period.
- This does not give us information about when items fail during the period.
- We may be interested in changing the mission length.
- So how do we incorporate time information into the analysis.

Modeling Failures Over Time

- Firstly, we must define the random variables.
- Let T_i = time to failure of the i-th component.
- In the next class we will see different ways to characterize the distribution of T_i.

Survival Probability

- The survival probability of the i-th item at time t is $S_i(t)=P(T_i>t)$.
- In standard probability modeling, we use the cumulative distribution function $F_i(t)= P(T_i \leq t)$.
- Of course, $S_i(t)=1- F_i(t)$..

Survival Probability

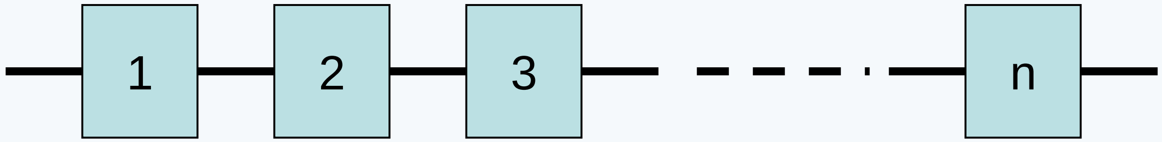
- We use the survival probability as it sounds more optimistic.

- We will discuss this at greater length next week.

- However, it is useful to use this representation of the distribution for system lifetime (T) calculations.

Series Systems

- A series system functions when all of its components function.



  – Thus $\phi(X) = 1$ if all the X_i take the value 1 and 0 otherwise.

$$\phi(T) = \min\{T_1, \ldots, T_n\}.$$

Series Systems

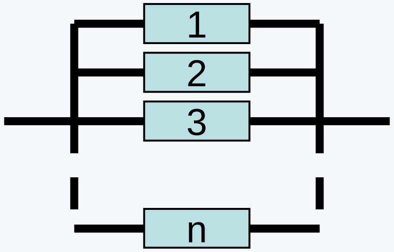
– This can be written in several ways

– So

$$
\begin{aligned}
P(T > t) &= P(\min\{T_1, \ldots, \mathrm{T_n}\} > \mathrm{t}) \\
&= \mathrm{P}(T_1 > t, \ldots, \mathrm{T_n} > \mathrm{t}) \\
&= \prod_{i=1}^{n} S_i(t), \text{ if } \mathrm{T_i} \text{ are independent.}
\end{aligned}
$$

Parallel Systems

- A parallel system functions when any of its components function.



  - Thus $\phi(X) = 1$ if all the X_i take the value 1 and 0 otherwise.

$$\phi(X) = \max\{X_1, \ldots, X_n\}.$$

Parallel Systems

- A This can be written in several ways
  - So

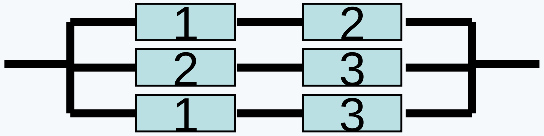
$$
\begin{aligned}
P(T > t) &= P(\max\{T_1,\ldots,\mathrm{T_n}\} > \mathrm{t}) \\
&= 1 - P(\max\{T_1,\ldots,\mathrm{T_n}\} \le \mathrm{t}) \\
&= 1 - \mathrm{P}(T_1 \le t,\ldots,\mathrm{T_n} \le \mathrm{t}) \\
&= 1 - \prod_{i=1}^{n}(1 - S_i(t)), \text{ if } \mathrm{T_i} \text{ are independent.}
\end{aligned}
$$

Examples

- Suppose

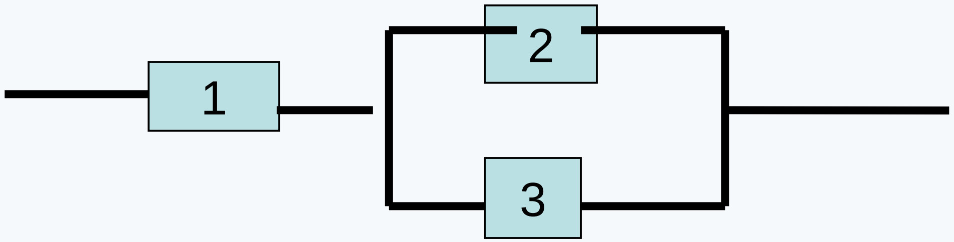
$$S_i(t) = e^{-\lambda_i t}$$

- What is the S(t) for:
  - A series system of n such components?
  - A parallel system of n such components?
  - A 2 out of 3 system?



Examples

– Find the survival function for the system



If the dlife distribution for each components is

$$f_i(t) = \lambda_i e^{-\lambda_i t}, \quad i = 1, 2, 3$$

Theory of Reliability and Life Testing

Lectures 192021

Families of life distributions

Abdulrahman M. Abouammoh
Department of Statistics & OR
King Saud University

Reliability properties

Failure rate revisited and the notion of aging

The reliability (or survival probability) of a fresh unit corresponding to a mission of duration x is, by definition, $\bar{F}(x) \equiv 1 - F(x)$, where F is the life distribution of the unit. The corresponding *conditional reliability of a unit of age t* is

$$\bar{F}(x \mid t) = \frac{\bar{F}(t+x)}{\bar{F}(t)} \qquad \text{if} \quad \bar{F}(t) > 0.$$

Similarly, the conditional probability of failure during the next interval of duration x of a unit of age t is

$$F(x \mid t) = \frac{F(t+x) - F(t)}{\bar{F}(t)} = 1 - \bar{F}(x \mid t).$$

Finally, we may obtain a conditional *failure rate* $r(t)$ at time t:

$$r(t) = \lim_{x \to 0} \frac{1}{x} \frac{F(t+x) - F(t)}{\bar{F}(t)},$$

Reliability properties

Failure rate revisited and the notion of aging

so that

$$r(t) = \frac{f(t)}{\overline{F}(t)}, \tag{1.1}$$

when $f(t)$ exists and $\overline{F}(t) > 0$. [Alternate names for $r(t)$ defined in (1.1) are hazard rate, force of mortality, and intensity rate.]

Useful identities are readily obtained by integrating both sides of (1.1),

$$\int_0^x r(t)\,dt = -\log \overline{F}(x), \tag{1.2}$$

and then exponentiating,

$$\overline{F}(x) = \exp\left[-\int_0^x r(t)\,dt\right]. \tag{1.3}$$

Reliability properties

Another approach to define failure rate

The hazard function can be derived using conditional probability. First, consider the probability of failure between t and $t + \Delta t$:

$$P[t \le T \le t + \Delta t] = \int_{t}^{t+\Delta t} f(\tau)d\tau = S(t) - S(t + \Delta t).$$

Conditioning on the event that the item is working at time t yields

$$P[t \le T \le t + \Delta t \mid T \ge t] = \frac{P[t \le T \le t + \Delta t]}{P[T \ge t]} = \frac{S(t) - S(t + \Delta t)}{S(t)}.$$

:S_b {0,1}

Reliability properties

Another approach to define failure rate

If this conditional probability is averaged over the interval $[t, t + \Delta t]$ by dividing by Δt, an average rate of failure is obtained:

$$\frac{S(t) - S(t + \Delta t)}{S(t)\Delta t}.$$

As $\Delta t \to 0$, this becomes the instantaneous failure rate, which is the hazard function

$$\begin{aligned} h(t) &= \lim_{\Delta t \to 0} \frac{S(t) - S(t + \Delta t)}{S(t)\Delta t} \\ &= \frac{-S'(t)}{S(t)} \\ &= \frac{f(t)}{S(t)} \qquad t \geq 0. \end{aligned}$$

Reliability properties

Another approach to define failure rate

Thus, the hazard function is the ratio of the probability density function to the survivor function. Using the previous derivation, a probabilistic interpretation of the hazard function is

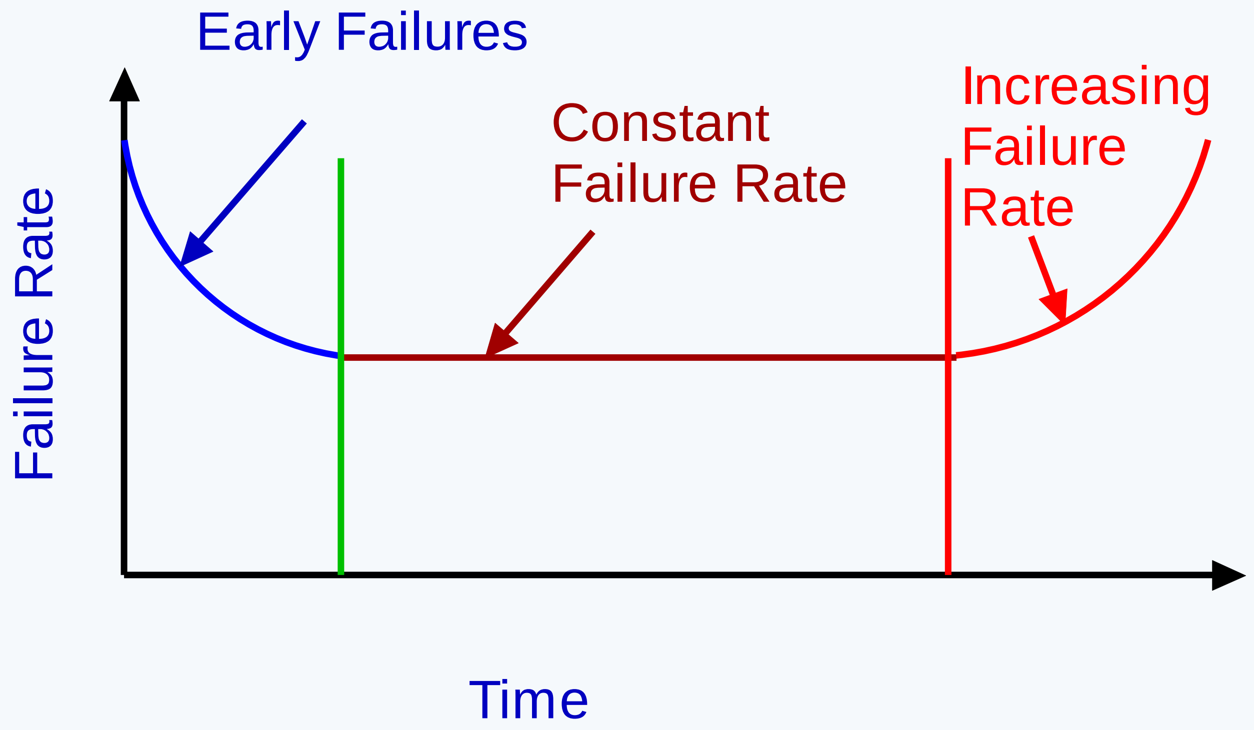
$$h(t)\Delta t = P[t \le T \le t + \Delta t \mid T \ge t]$$

for small Δt values, which is a conditional version of the interpretation for the probability density function. All hazard functions must satisfy two conditions:

$$\int_0^\infty h(t)\,dt = \infty \qquad h(t) \ge 0 \text{ for all } t \ge 0.$$

Some Initial Thoughts Warranty

- Will you buy additional warranty?
- Burn in and removal of early failures. (Lemon Law).



Reliability Definitions

Reliability is a time dependent characteristic.

- It can only be determined after an elapsed time but can be predicted at any time.

- It is the probability that a product or service will operate properly for a specified period of time (design life) under the design operating conditions without failure.

Other Measures of Reliability

Availability is used for repairable systems

- It is the probability that the system is operational at any random time t.

- It can also be specified as a proportion of time that the system is available for use in a given interval (0,T).

Other Measures of Reliability

Mean Time To Failure (MTTF): *It is the average time that elapses until a failure occurs.*

It does not provide information about the distribution of the TTF, hence we need to estimate the variance of the TTF.

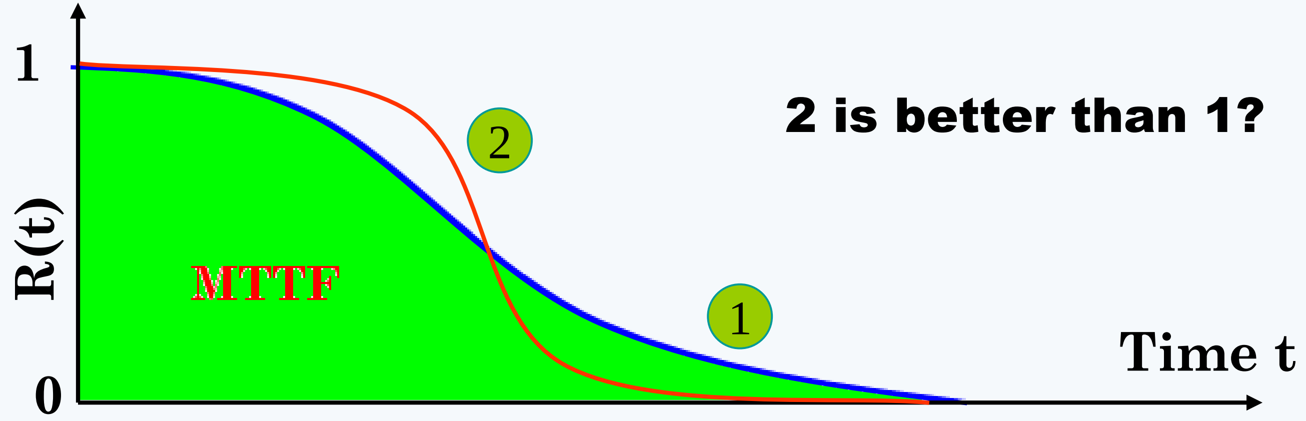
Mean Time Between Failure (MTBF): It is the average time between successive failures.

It is used for repairable systems.

Mean Time to Failure: MTTF

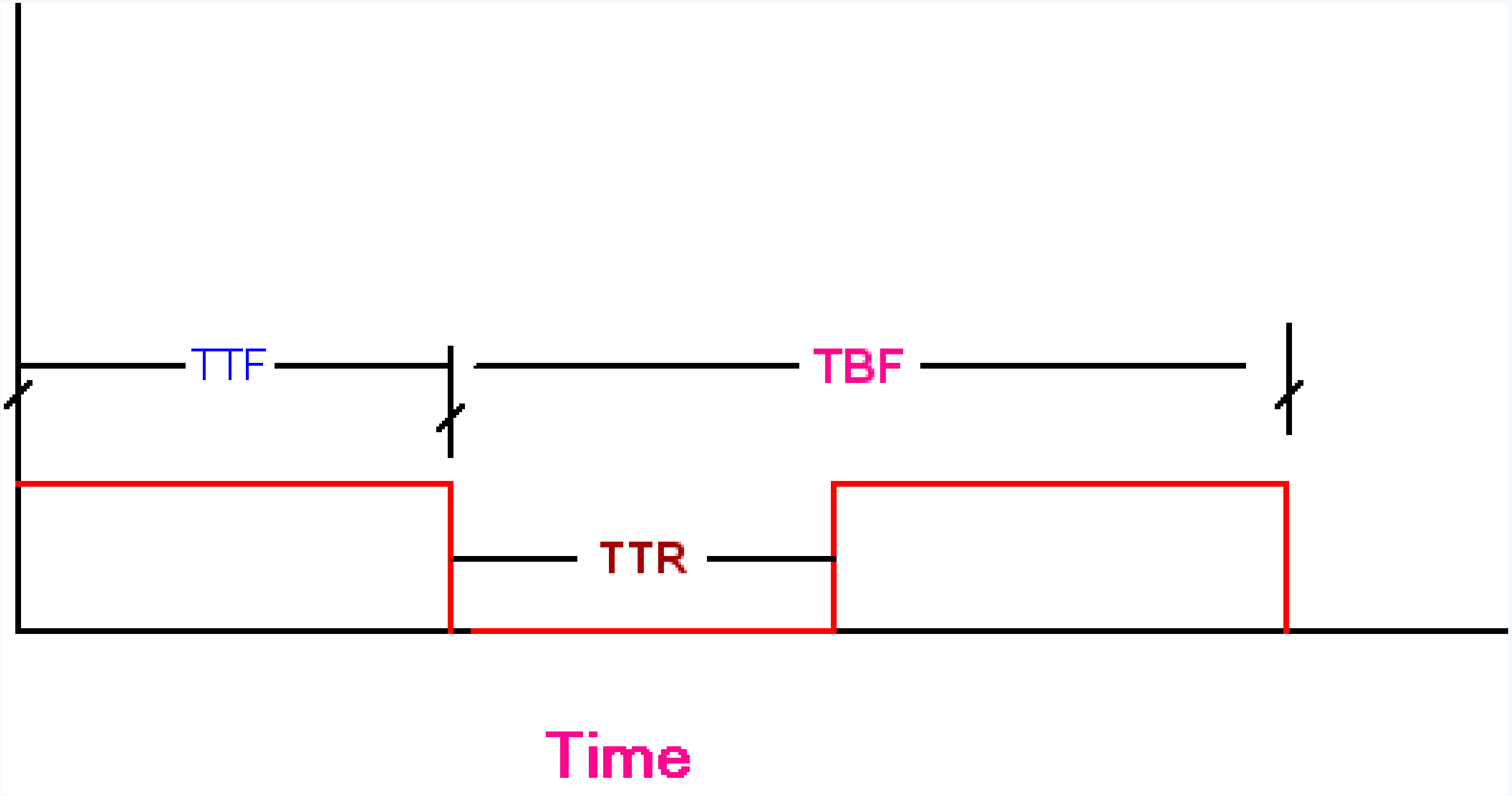
$$MTTF = \int_0^{\infty} t f(t) dt = \int_0^{\infty} R(t) dt$$

$$MTTF = \frac{1}{n} \sum_{i=1}^{n} t_i$$



2 is better than 1?

Mean Time Between Failure: MTBF



Other Measures of Reliability

Mean Residual *Life (MRL):* It is the expected remaining life, T-t, given that the product, component, or a system has survived to time t.

$$L(t) = E[T - t \mid T \geq t] = \frac{1}{R(t)} \int_t^{\infty} \tau f(\tau) d\tau - t$$

Failure Rate *(FITs failures in 10^9 hours):* The failure rate in a time interval [$t_1 - t_2$] is the probability that a failure per unit time occurs in the interval given that no failure has occurred prior to the beginning of the interval.

Hazard Function: It is the limit of the failure rate as the length of the interval approaches zero.

Basic Calculations

Suppose n_0 identical units are subjected to a test. During the interval (t, t+Δt), we observed $n_f(t)$ failed components. Let $n_s(t)$ be the surviving components at time t, then the MTTF, failure density, hazard rate, and reliability at time t are:

$$MTTF = \frac{\sum_{i=1}^{n_0} t_i}{n_0}, \qquad \hat{f}(t) = \frac{n_f(t)}{n_0 \Delta t}$$

$$\hat{\lambda}(t) = \frac{n_f(t)}{n_s(t)\Delta t}, \qquad \hat{R}(t) = P_r(T > t) = \frac{n_s(t)}{n_0}$$

Basic Definitions :-

The unreliability *F*(*t*) is

$$F(t) = 1 - R(t)$$

Example: 200 light bulbs were tested and the failures in 1000-hour intervals are

Time Interval (Hours)	Failures in the interval
0-1000	100
1001-2000	40
2001-3000	20
3001-4000	15
4001-5000	10
5001-6000	8
6001-7000	7
Total	200

$$MTTF = \frac{\sum_{i=1}^{n_0} t_i}{n_0}, \qquad \hat{f}(t) = \frac{n_f(t)}{n_0 \Delta t}$$

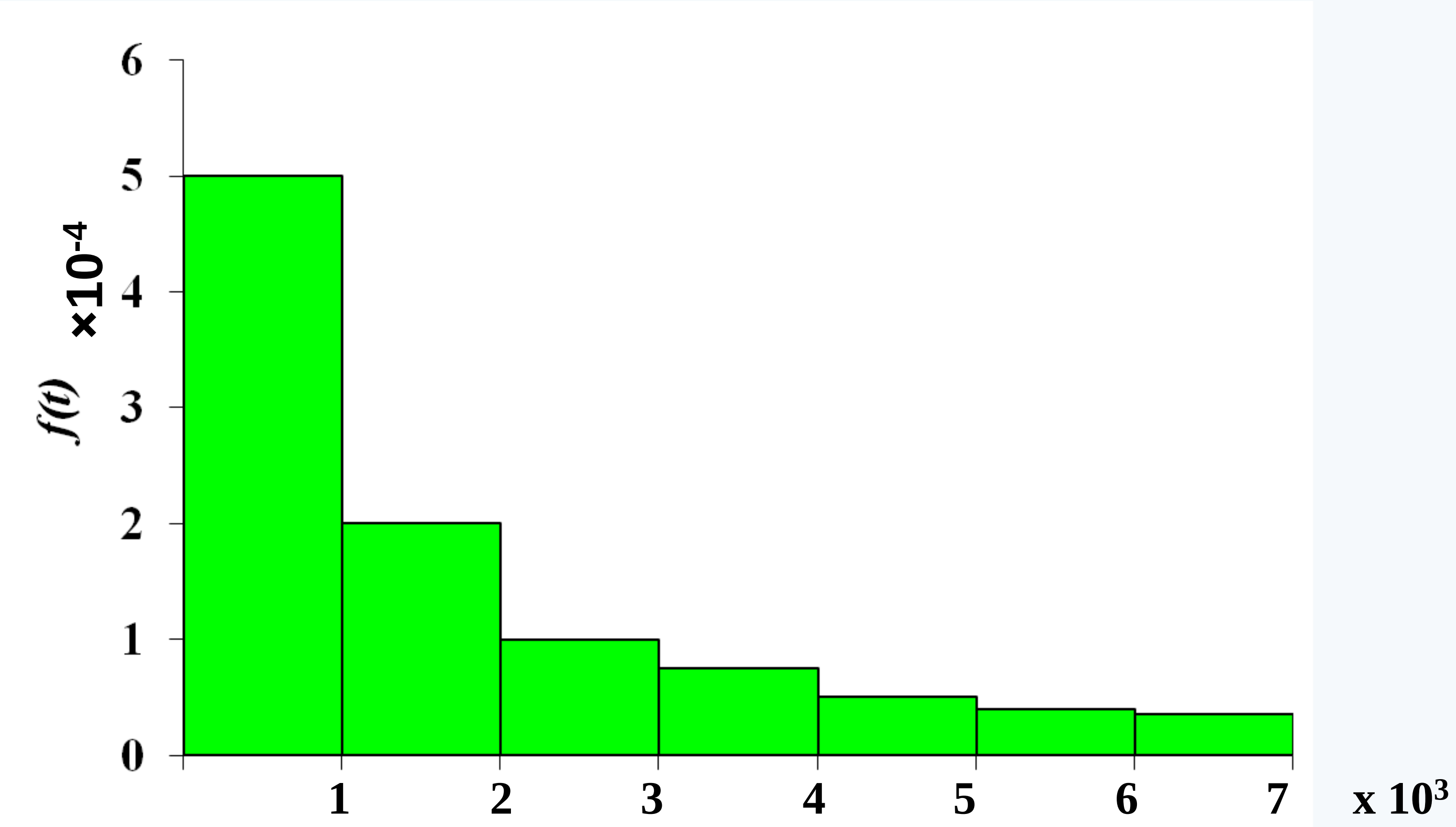
$$\hat{\lambda}(t) = \frac{n_f(t)}{n_s(t)\Delta t}, \qquad \hat{R}(t) = P_r(T > t) = \frac{n_s(t)}{n_0}$$

Calculations

Time Interval (Hours)	Failures in the interval
0-1000	100
1001-2000	40
2001-3000	20
3001-4000	15
4001-5000	10
5001-6000	8
6001-7000	7
Total	200

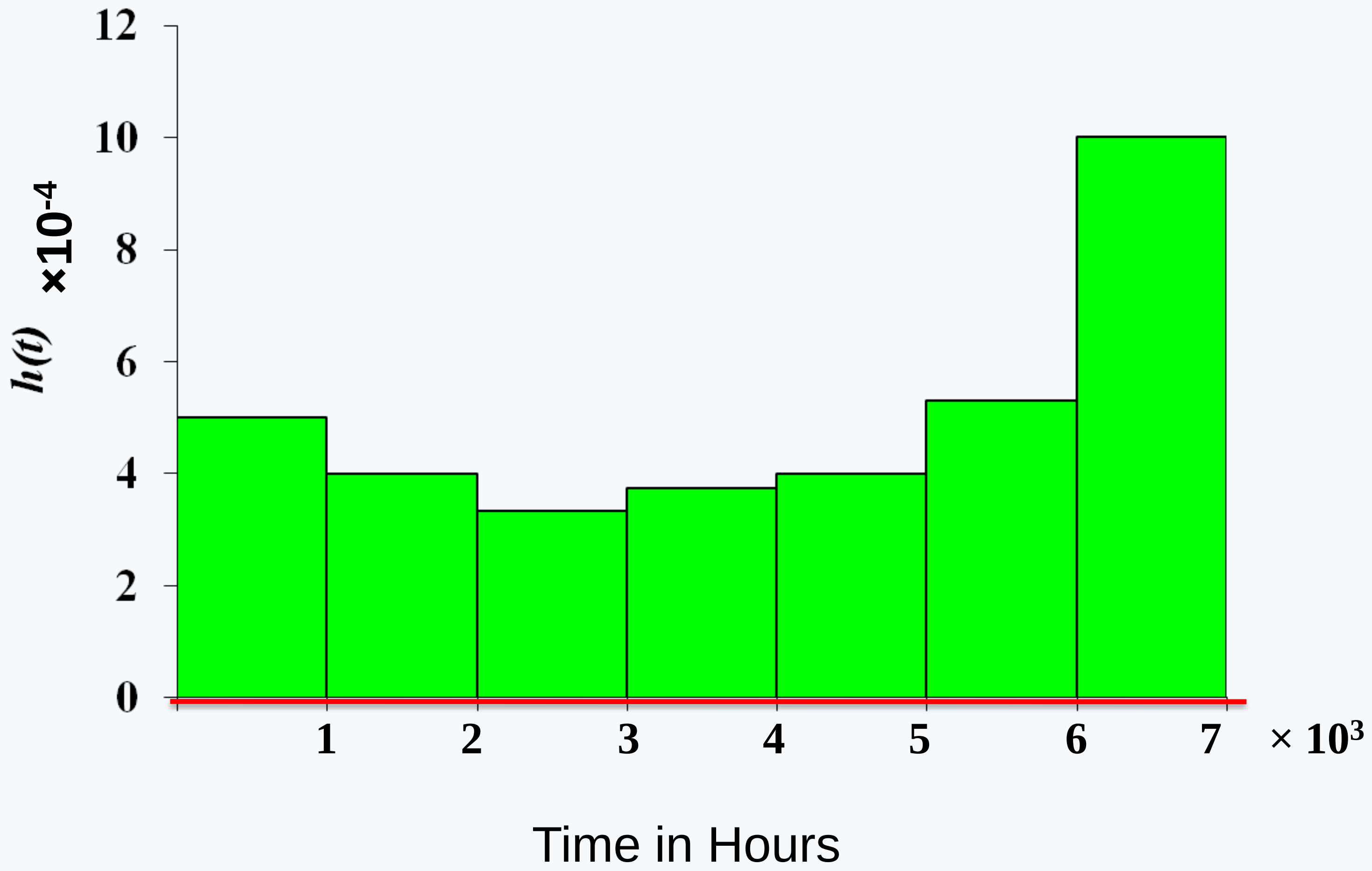
Time Interval	Failure Density $f(t)$ x 10^{-4}	Hazard rate $h(t)$ x 10^{-4}
0–1000	$\frac{100}{200 \times 10^3} = 5.0$	$\frac{100}{200 \times 10^3} = 5.0$
1001–2000	$\frac{40}{200 \times 10^3} = 2.0$	$\frac{40}{100 \times 10^3} = 4.0$
2001–3000	$\frac{20}{200 \times 10^3} = 1.0$	$\frac{20}{60 \times 10^3} = 3.33$
……	……..	……
6001–7000	$\frac{7}{200 \times 10^3} = 0.35$	$\frac{7}{7 \times 10^3} = 10$

Failure Density vs. Time



Time in hours

Hazard Rate vs. Time



$$MTTF = \frac{\sum_{i=1}^{n_0} t_i}{n_0}, \qquad \hat{f}(t) = \frac{n_f(t)}{n_0 \Delta t}$$

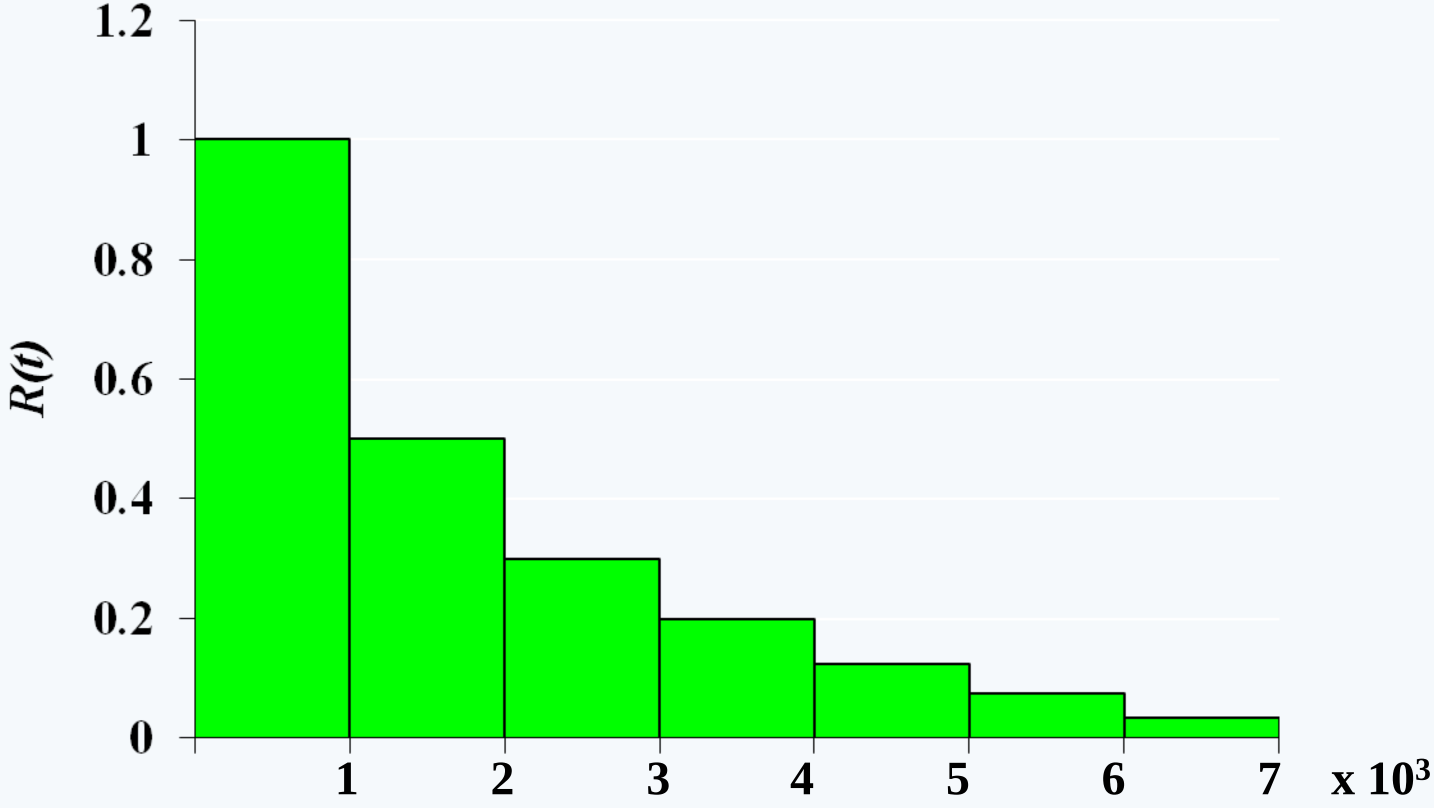
$$\hat{\lambda}(t) = \frac{n_f(t)}{n_s(t)\Delta t}, \qquad \hat{R}(t) = P_r(T > t) = \frac{n_s(t)}{n_0}$$

Calculations

Time Interval (Hours)	Failures in the interval
0-1000	100
1001-2000	40
2001-3000	20
3001-4000	15
4001-5000	10
5001-6000	8
6001-7000	7
Total	200

Time Interval	Reliability $R(t)$
0–1000	200/200=1.0
1001–2000	100/200=0.5
2001–3000	60/200=0.33
……	……
6001–7000	0.35/10=.035

Reliability vs. Time



Time in hours

Reliability properties

Example:

Consider the Weibull distribution defined by the survivor function

$$S(t) = e^{-(\lambda t)^\kappa} \qquad t \geq 0$$

with positive scale parameter λ and positive shape parameter κ. Find the hazard function.

By differentiating the survivor function with respect to t and negating, the probability density function is

$$f(t) = \lambda\kappa(\lambda t)^{\kappa-1} e^{-(\lambda t)^\kappa} \qquad t \geq 0,$$

and the hazard function is

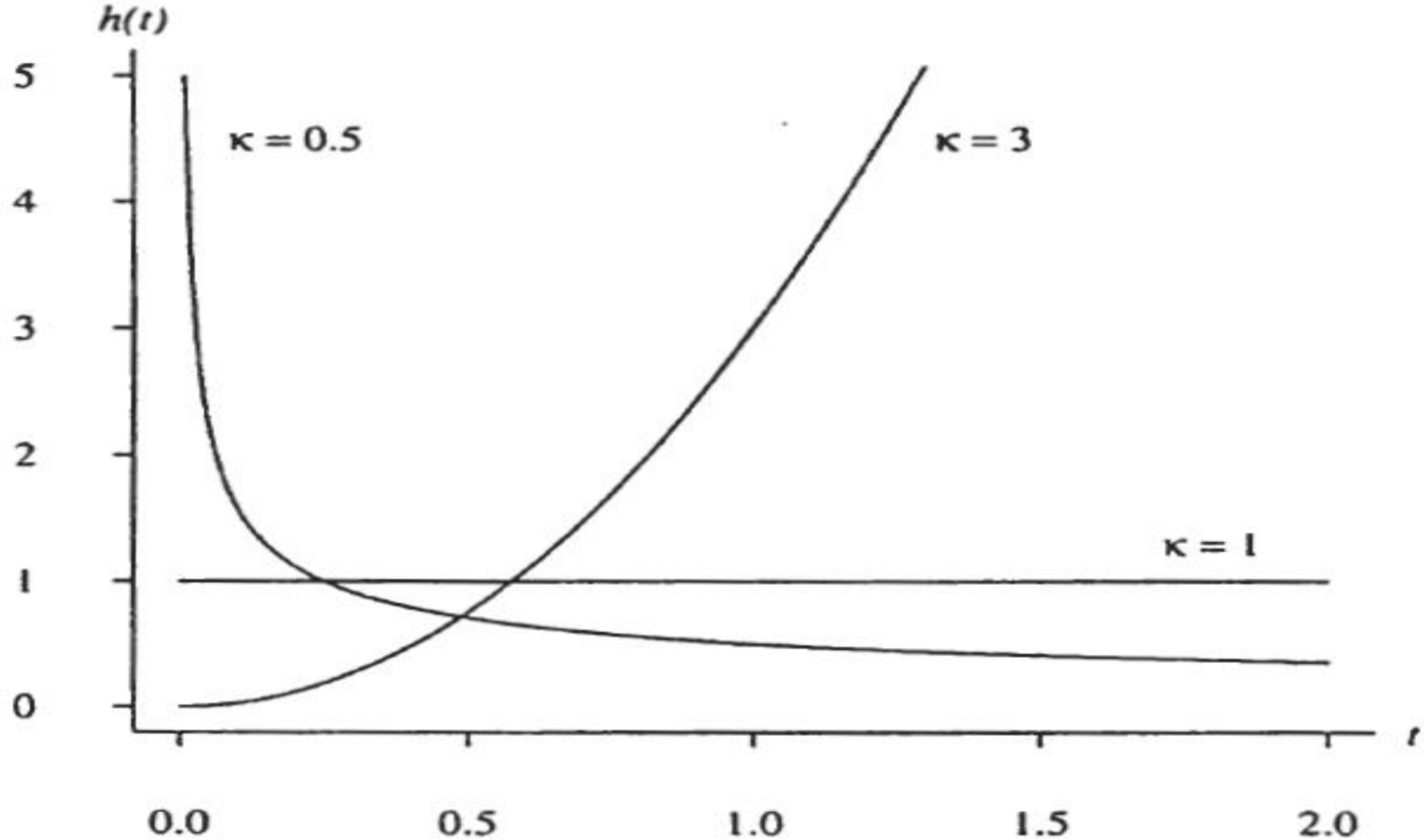
$$h(t) = \frac{f(t)}{S(t)} = \lambda\kappa(\lambda t)^{\kappa-1} \qquad t \geq 0.$$

Figure 3.4 shows that the hazard function for $\lambda = 1$ and several κ values is constant if $\kappa = 1$, increasing if $\kappa > 1$ and decreasing if $\kappa < 1$.

:S_b {0,1}

Reliability properties

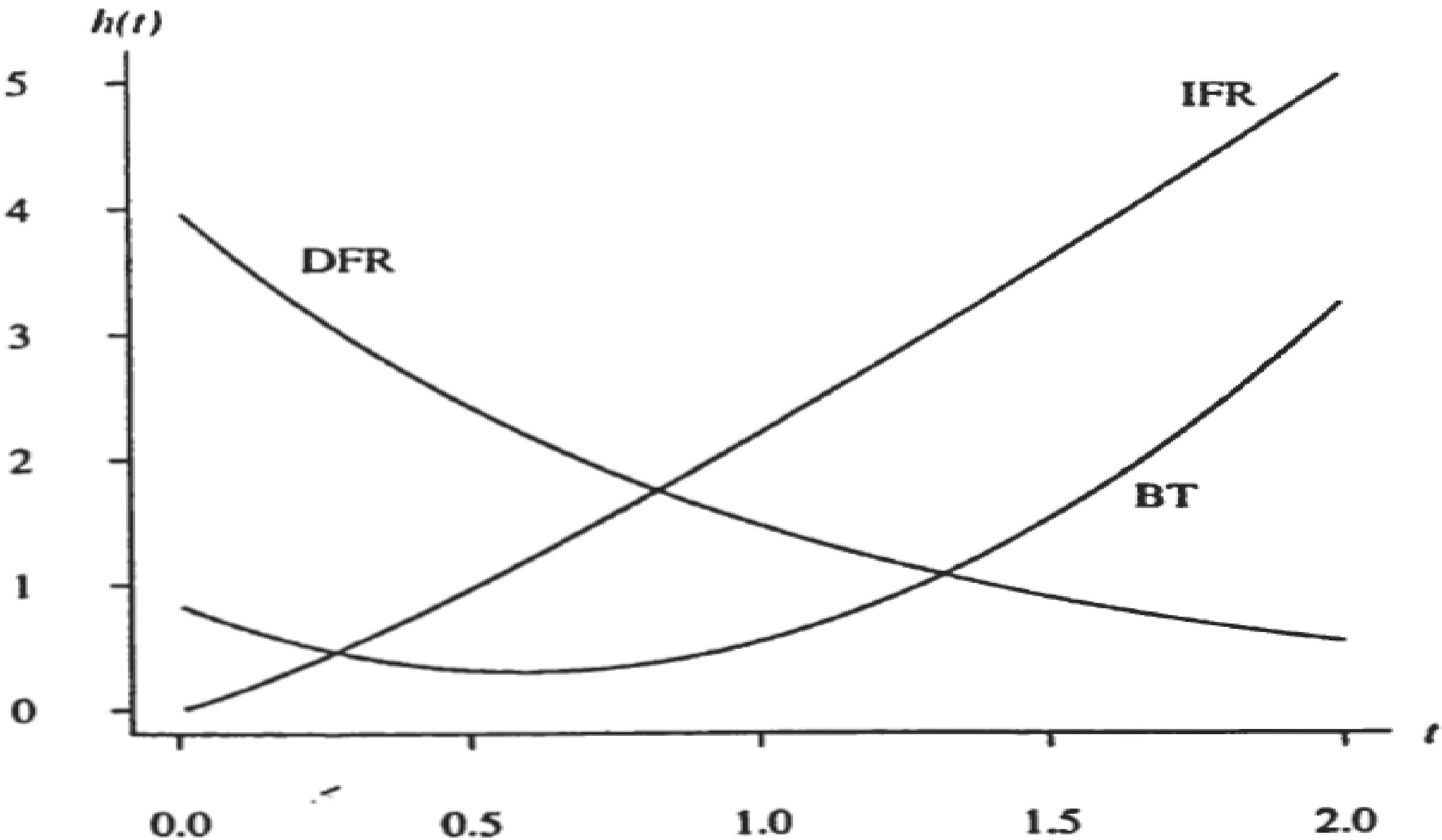
Example: Failur or hazard rate for Weibull with scal parameter one and different shape parameter



:S_b {0,1}

Reliability properties

Shapes for IFR, DFR and bath tub failure rates



:S_b {0,1}

Reliability properties

The mean remaining life

Mean Residual Life Function

The last lifetime distribution representation, the *mean residual life* function, $L(t)$, is defined by

$$L(t) = E[T - t \mid T \geq t] \qquad t \geq 0.$$

The mean residual life function is the expected *remaining life*, $T - t$, given that the item has survived to time t. The unconditional mean of the distribution, $E[T]$, is a special case given by $L(0)$. To determine a formula for this expectation, the conditional probability density function is needed

$$f_{T \mid T \geq t}(\tau) = \frac{f(\tau)}{S(t)} \qquad \tau \geq t.$$

This conditional probability density function is actually a family of probability density functions (one for each value of t), each of which has an associated mean

$$E[T \mid T \geq t] = \int_t^{\infty} \tau f_{T \mid T \geq t}(\tau) d\tau = \int_t^{\infty} \tau \frac{f(\tau)}{S(t)} d\tau.$$

Reliability properties

The mean remaining life

Since the mean residual life function is the expected *remaining* life, t must be subtracted, yielding

$$L(t) = E[T - t \mid T \geq t] = \int_t^\infty (\tau - t)\,\frac{f(\tau)}{S(t)}\,d\tau = \int_t^\infty \tau\,\frac{f(\tau)}{S(t)}\,d\tau - t = \frac{1}{S(t)}\int_t^\infty \tau f(\tau)d\tau - t.$$

All mean residual life functions associated with distributions having a finite mean must satisfy three conditions:

$$L(t) \geq 0 \qquad L'(t) \geq -1 \qquad \int_0^\infty \frac{dt}{L(t)} = \infty.$$

Example 3.2

Consider the exponential distribution defined by the survivor function

$$S(t) = e^{-\lambda t} \qquad t \geq 0$$

with positive scale parameter λ. Find the mean residual life function.

Reliability properties

The mean remaining life

By differentiating the survivor function with respect to t and negating, the probability density function is

$$f(t) = \lambda e^{-\lambda t} \qquad t \geq 0.$$

Using the survivor function and the probability density function, the mean residual life function is

$$L(t) = \frac{1}{S(t)} \int_t^{\infty} \tau f(\tau) d\tau - t = e^{\lambda t} \int_t^{\infty} \tau \lambda e^{-\lambda \tau} d\tau - t = \frac{1}{\lambda} \qquad t \geq 0$$

by using integration by parts. This indicates that regardless of the age of the item the mean remaining lifetime is always $1/\lambda$.

$:S_b\{0,1\}$

Reliability properties

Llife time distribution representation relationships

	$f(t)$	$S(t)$	$h(t)$	$H(t)$	$L(t)$
$f(t)$	•	$\int_t^\infty f(\tau)d\tau$	$\frac{f(t)}{\int_t^\infty f(\tau)d\tau}$	$-\log\left[\int_t^\infty f(\tau)d\tau\right]$	$\frac{\int_t^\infty \tau f(\tau)d\tau}{\int_t^\infty f(\tau)d\tau}-t$
$S(t)$	$-S'(t)$	•	$\frac{-S'(t)}{S(t)}$	$-\log S(t)$	$\frac{1}{S(t)}\int_t^\infty S(\tau)d\tau$
$h(t)$	$h(t)e^{-\int_0^t h(\tau)d\tau}$	$e^{-\int_0^t h(\tau)d\tau}$	•	$\int_0^t h(\tau)d\tau$	$\frac{\int_t^\infty e^{-\int_0^\tau h(y)dy}d\tau}{e^{-\int_0^t h(\tau)d\tau}}$
$H(t)$	$H'(t)e^{-H(t)}$	$e^{-H(t)}$	$H'(t)$	•	$e^{H(t)}\int_t^\infty e^{-H(\tau)}d\tau$
$L(t)$	$\frac{1+L'(t)}{L(t)}e^{-\int_0^t \frac{1+L'(\tau)}{L(\tau)}d\tau}$	$e^{-\int_0^t \frac{1+L'(\tau)}{L(\tau)}d\tau}$	$\frac{1+L'(t)}{L(t)}$	$\int_0^t \frac{1+L'(\tau)}{L(\tau)}d\tau$	•

Reliability properties

Example:

Given $h(t) = 18t$, find $f(t)$.

Using the $(h(t), f(t))$ element of the matrix,

$$f(t) = h(t)\, e^{-\int_0^t h(\tau)\,d\tau} = 18t\, e^{-\int_0^t 18\tau\,d\tau} = 18t\, e^{-9t^2} \qquad t \geq 0,$$

which is a special case of the Weibull distribution with $\lambda = 3$ and $\kappa = 2$.

Reliability properties

Failure rate revisited and the notion of aging

The cumulative failure rate, $R(x) = \int_0^x r(t)\,dt$, is referred to as the *hazard* function, or simply the hazard. Equation (1.3) gives a useful theoretical representation of reliability as a function of failure rate. An alternate representation,

$$\bar{F}(x) = e^{-R(x)}, \tag{1.4}$$

gives reliability in terms of hazard.

Now consider a device which does not age stochastically; that is, its survival probability over an additional period of duration x is the same regardless of its present age t. Symbolically,

$$\bar{F}(x \mid t) = \bar{F}(x) \qquad \text{for all } x, \quad t \geq 0.$$

Equivalently,

$$\bar{F}(t + x) = \bar{F}(t)\bar{F}(x) \qquad \text{for all } x, \quad t \geq 0. \tag{1.5}$$

Equation (1.5) is classical; taking into account the requirement that $0 \leq \bar{F}(x) \leq 1$, its solution is of the form

$$\bar{F}(x) = e^{-\lambda x}, \qquad \lambda > 0, \quad x \geq 0, \tag{1.6}$$

an exponential survival probability, as shown in Theorem 2.2 below. In Section 2 we derive a number of useful properties of the exponential distribution.