Chapter 4

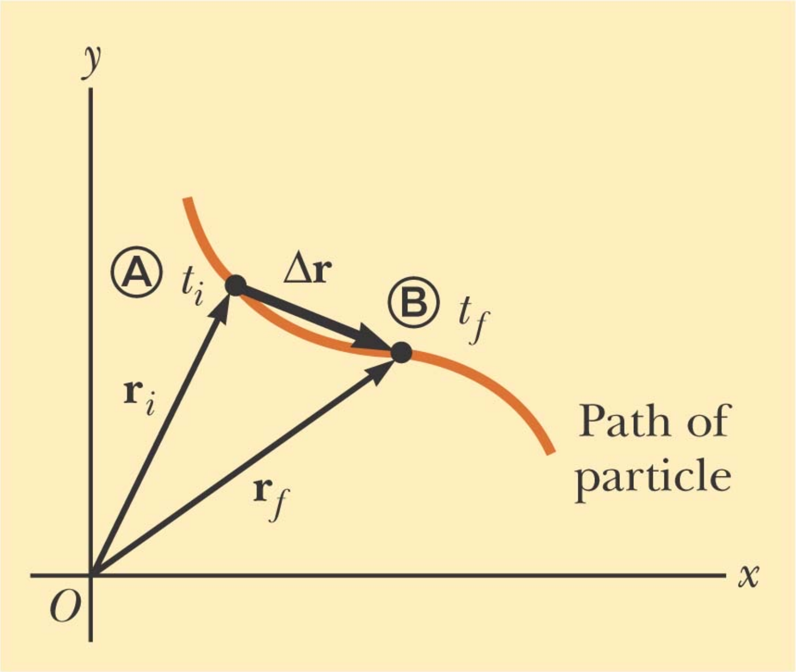
Motion in Two Dimensions

Motion in Two Dimensions

- Using + or – signs is not always sufficient to fully describe motion in more than one dimension
  - Vectors can be used to more fully describe motion
- Still interested in displacement, velocity, and acceleration
- Will serve as the basis of multiple types of motion in future chapters

Position and Displacement

- The position of an object is described by its position vector, **r**
- The **displacement** of the object is defined as the ***change in its position***
  - $\Delta\mathbf{r} = \mathbf{r}_f - \mathbf{r}_i$



General Motion Ideas

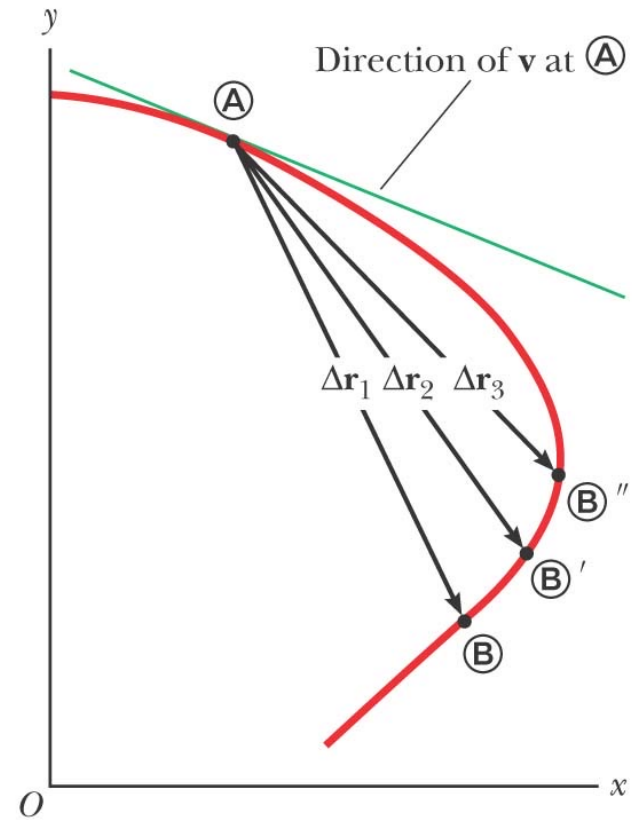
- In two- or three-dimensional kinematics, everything is the same as as in one-dimensional motion except that we must now use full vector notation
  - Positive and negative signs are no longer sufficient to determine the direction

Average Velocity

- The average velocity is the ratio of the displacement to the time interval for the displacement

$$\overline{\mathbf{v}} = \frac{\Delta \mathbf{r}}{\Delta t}$$

  - The direction of the average velocity is the direction of the displacement vector, $\Delta \mathbf{r}$



Average Velocity, cont

- The average velocity between points is *independent of the path* taken
  - This is because it is dependent on the displacement, also independent of the path

Instantaneous Velocity

- The instantaneous velocity is the limit of the average velocity as Δt approaches zero
  - The direction of the instantaneous velocity is along a line that is tangent to the path of the particle's direction of motion

$$\mathbf{v} \equiv \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt}$$

Instantaneous Velocity, cont

- The direction of the instantaneous velocity vector at any point in a particle's path is along a line tangent to the path at that point and in the direction of motion
- The magnitude of the instantaneous velocity vector is the speed
  - The speed is a scalar quantity

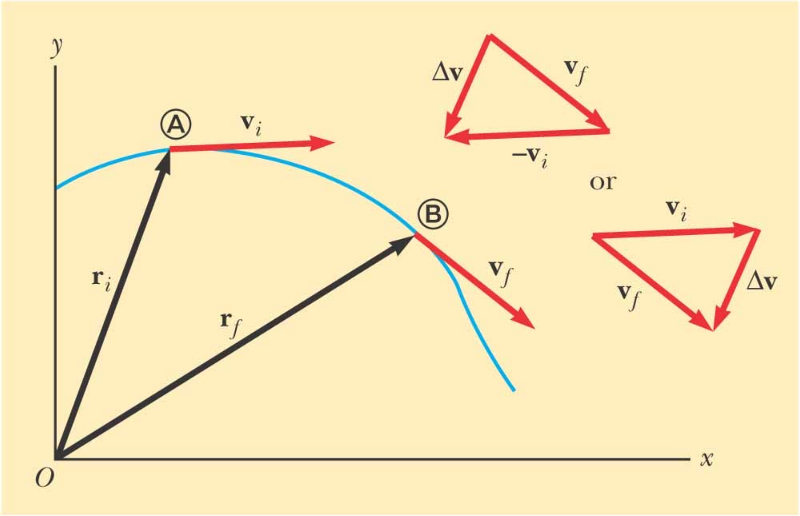
Average Acceleration

- The average acceleration of a particle as it moves is defined as the change in the instantaneous velocity vector divided by the time interval during which that change occurs.

$$\bar{\mathbf{a}} = \frac{\mathbf{v}_{\mathrm{f}} - \mathbf{v}_i}{t_f - t_i} = \frac{\Delta \mathbf{v}}{\Delta t}$$

Average Acceleration, cont

- As a particle moves, $\Delta\mathbf{v}$ can be found in different ways
- The average acceleration is a vector quantity directed along $\Delta\mathbf{v}$



Instantaneous Acceleration

- The instantaneous acceleration is the limit of the average acceleration as $\Delta\mathbf{v}/\Delta t$ approaches zero

$$\mathbf{a} \equiv \lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t} = \frac{d\mathbf{v}}{dt}$$

Producing An Acceleration

- Various changes in a particle's motion may produce an acceleration
  - The magnitude of the velocity vector may change
  - The direction of the velocity vector may change
    - Even if the magnitude remains constant
  - Both may change simultaneously

Kinematic Equations for Two-Dimensional Motion

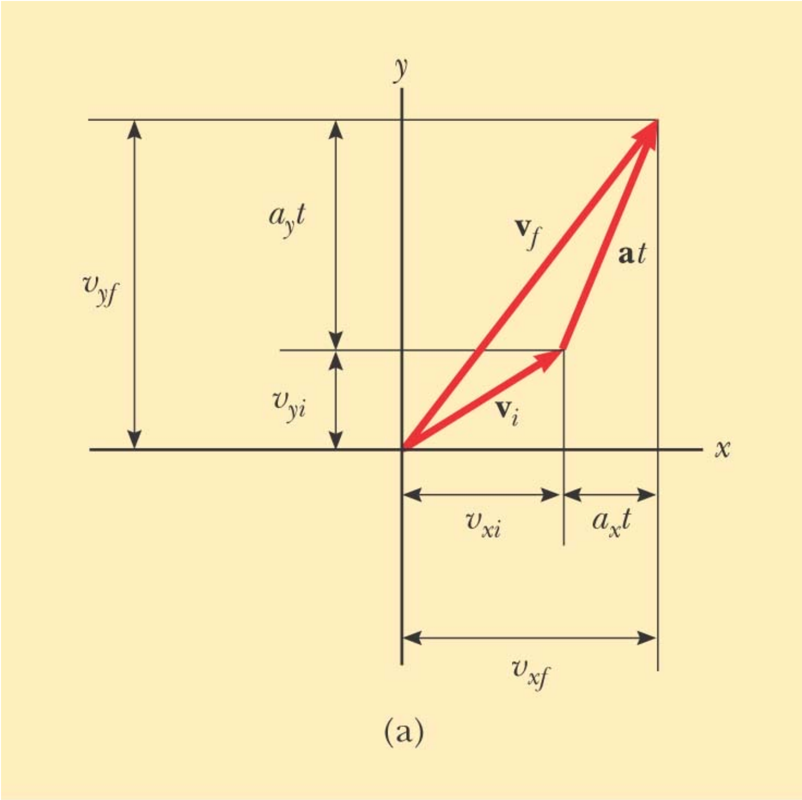
- When the two-dimensional motion has a constant acceleration, a series of equations can be developed that describe the motion
- These equations will be similar to those of one-dimensional kinematics

Kinematic Equations, 2

- Position vector $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$
- Velocity $\mathbf{v} = \frac{d\mathbf{r}}{dt} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}}$
  - Since acceleration is constant, we can also find an expression for the velocity as a function of time: $\mathbf{v}_f = \mathbf{v}_i + \mathbf{a}t$

Kinematic Equations, 3

- The velocity vector can be represented by its components
- $\mathbf{v}_f$ is generally not along the direction of either $\mathbf{v}_i$ or $\mathbf{a}t$

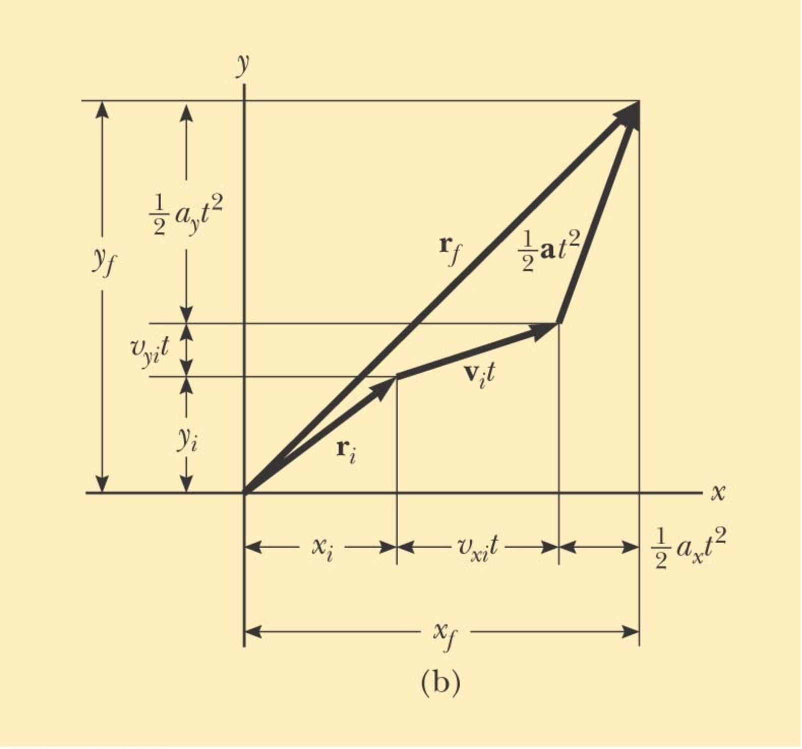


Kinematic Equations, 4

- The position vector can also be expressed as a function of time:
  - $\mathbf{r}_f = \mathbf{r}_i + \mathbf{v}_i t + \frac{1}{2}\mathbf{a}t^2$
  - This indicates that the position vector is the sum of three other vectors:
    - The initial position vector
    - The displacement resulting from $\mathbf{v}_i t$
    - The displacement resulting from $\frac{1}{2}\mathbf{a}t^2$

Kinematic Equations, 5

- The vector representation of the position vector
- $\mathbf{r}_f$ is generally not in the same direction as $\mathbf{v}_i$ or as $\mathbf{a}_i$
- $\mathbf{r}_f$ and $\mathbf{v}_f$ are generally not in the same direction



Kinematic Equations, Components

- The equations for final velocity and final position are vector equations, therefore they may also be written in component form
- This shows that two-dimensional motion at constant acceleration is equivalent to two independent motions
  - One motion in the *x*-direction and the other in the *y*-direction

Kinematic Equations, Component Equations

- $\mathbf{v}_f = \mathbf{v}_i + \mathbf{a}t$ becomes
  - $v_{xf} = v_{xi} + a_x t$ and
  - $v_{yf} = v_{yi} + a_y t$
- $\mathbf{r}_f = \mathbf{r}_i + \mathbf{v}_i t + \frac{1}{2} \mathbf{a}t^2$ becomes
  - $x_f = x_i + v_{xi} t + \frac{1}{2} a_x t^2$ and
  - $y_f = y_i + v_{yi} t + \frac{1}{2} a_y t^2$

Projectile Motion

- An object may move in both the *x* and *y* directions simultaneously
- The form of two-dimensional motion we will deal with is called **projectile motion**

Assumptions of Projectile Motion

- The free-fall acceleration **g** is constant over the range of motion
  - And is directed downward
- The effect of air friction is negligible
- With these assumptions, an object in projectile motion will follow a parabolic path
  - This path is called the ***trajectory***

Verifying the Parabolic Trajectory

- Reference frame chosen
  - y is vertical with upward positive
- Acceleration components
  - $a_y = -g$ and $a_x = 0$
- Initial velocity components
  - $v_{xi} = v_i \cos\theta$ and $v_{yi} = v_i \sin\theta$

Verifying the Parabolic Trajectory, cont

- Displacements
  - $x_f = v_{xi}\, t = (v_i \cos\theta)\, t$
  - $y_f = v_{yi}\, t + ½ a_y t^2 = (v_i \sin\theta) t - ½ g t^2$
- Combining the equations gives:

$$y = (\tan\theta_i)\, x - \left(\frac{g}{2v_i^2 \cos^2\theta_i}\right) x^2$$

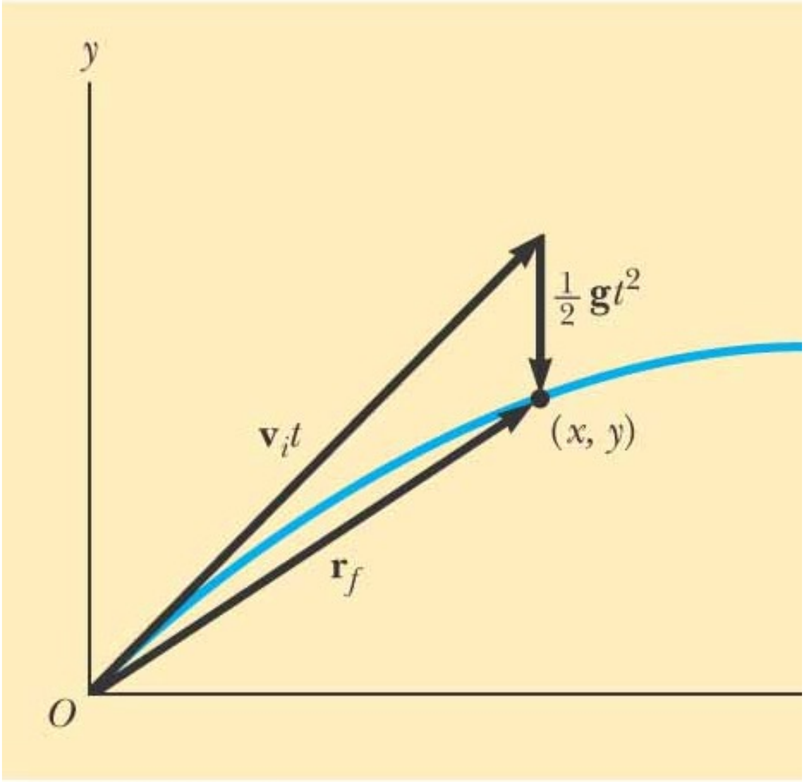
  - This is in the form of $y = ax - bx^2$ which is the standard form of a parabola

Analyzing Projectile Motion

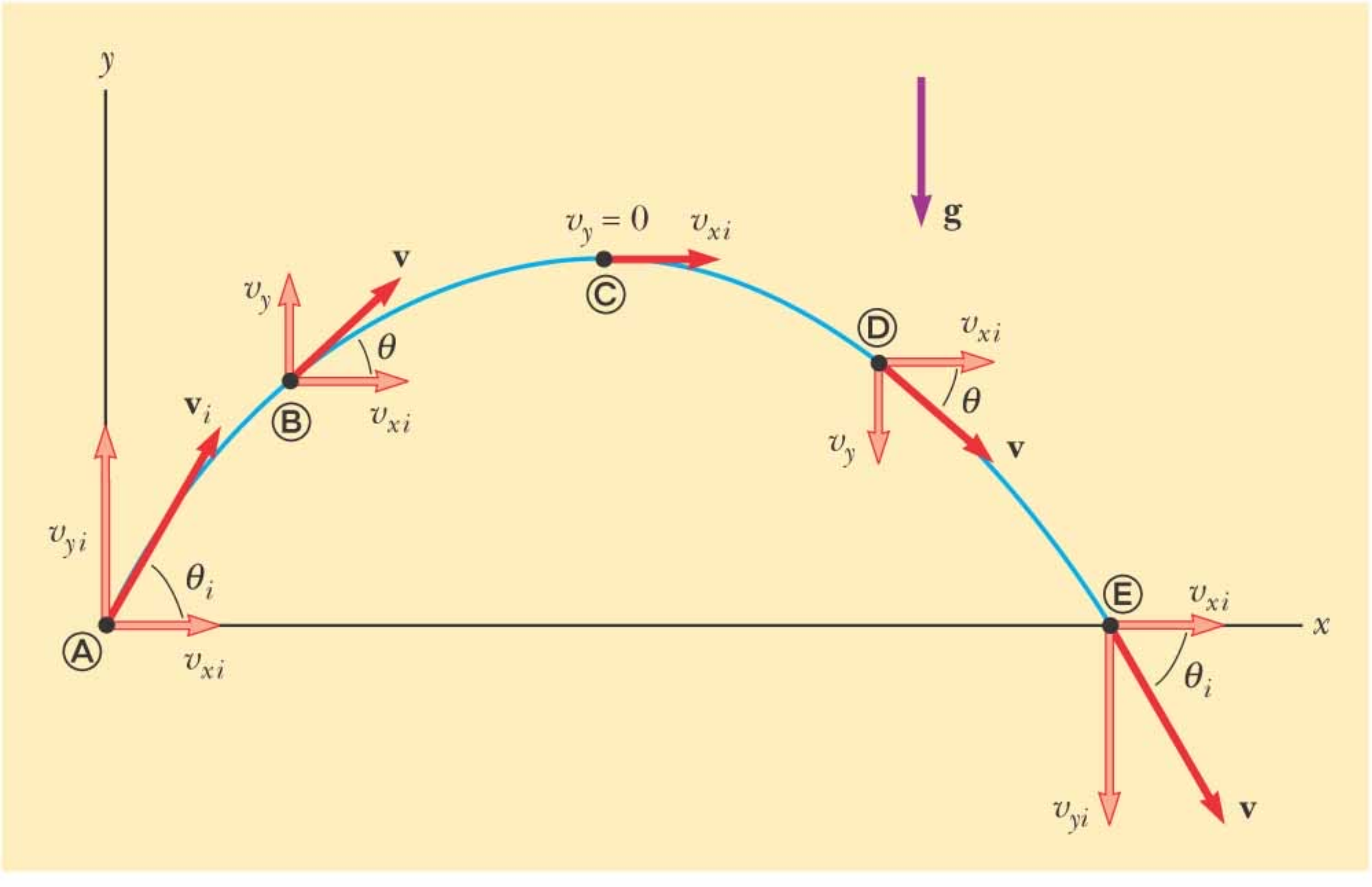
- Consider the motion as the superposition of the motions in the *x*- and *y*-directions
- The *x*-direction has constant velocity
  - $a_x = 0$
- The y-direction is free fall
  - $a_y = -g$
- The actual position at any time is given by: $\mathbf{r}_f = \mathbf{r}_i + \mathbf{v}_i t + \frac{1}{2}\mathbf{g}t^2$

Projectile Motion Vectors

- $\mathbf{r}_f = \mathbf{r}_i + \mathbf{v}_i\, t + ½\, \mathbf{g}\, t^2$
- The final position is the vector sum of the initial position, the position resulting from the initial velocity and the position resulting from the acceleration



Projectile Motion Diagram

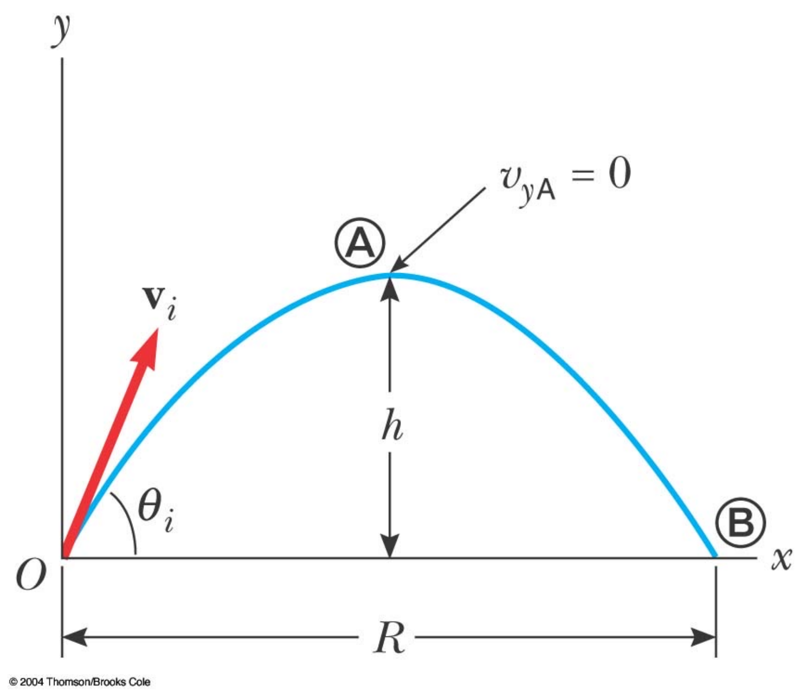


Projectile Motion – Implications

- The *y*-component of the velocity is zero at the maximum height of the trajectory
- The accleration stays the same throughout the trajectory

Range and Maximum Height of a Projectile

- When analyzing projectile motion, two characteristics are of special interest
- The range, *R*, is the horizontal distance of the projectile
- The maximum height the projectile reaches is *h*



Height of a Projectile, equation

- The maximum height of the projectile can be found in terms of the initial velocity vector:

$$h = \frac{v_i^2 \sin^2 \theta_i}{2g}$$

- This equation is valid only for symmetric motion

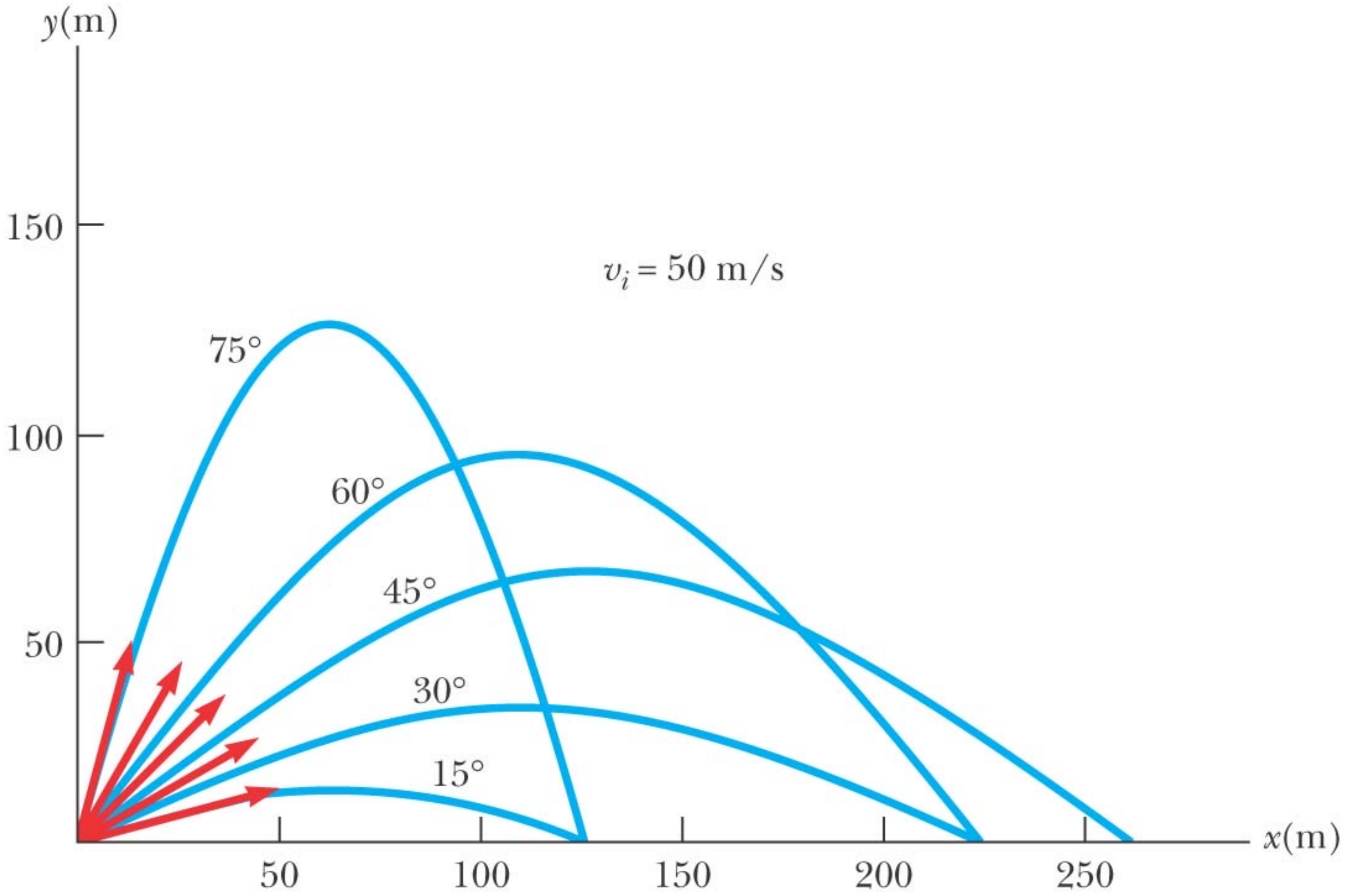
Range of a Projectile, equation

- The range of a projectile can be expressed in terms of the initial velocity vector:

$$R = \frac{v_i^2 \sin 2\theta_i}{g}$$

- This is valid only for symmetric trajectory

More About the Range of a Projectile



Range of a Projectile, final

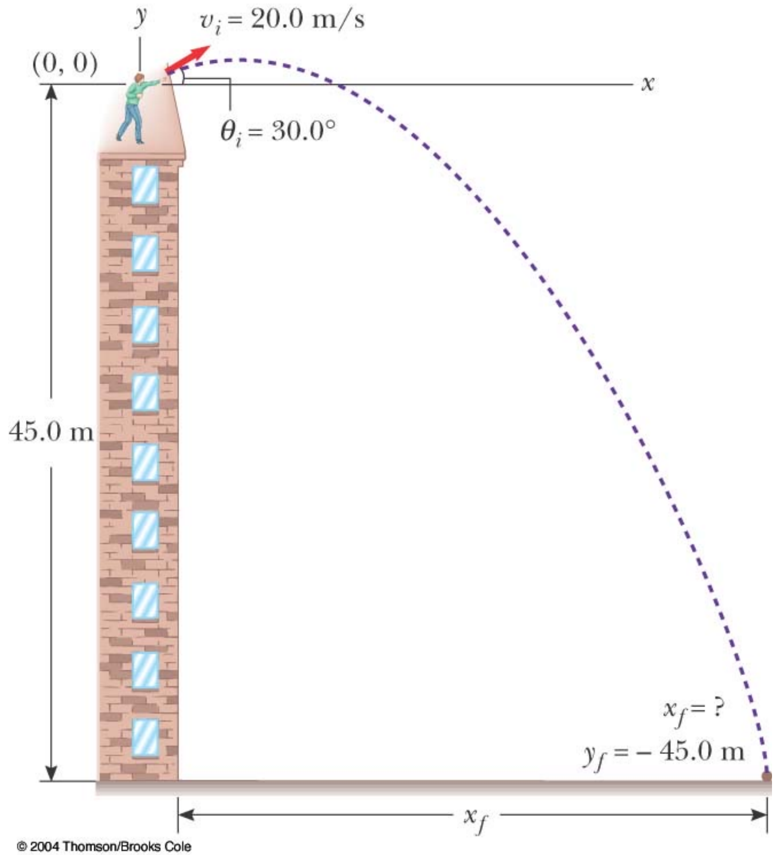
- The maximum range occurs at $\theta_i = 45^o$
- Complementary angles will produce the same range
  - The maximum height will be different for the two angles
  - The times of the flight will be different for the two angles

Projectile Motion – Problem Solving Hints

- Select a coordinate system
- Resolve the initial velocity into *x* and *y* components
- Analyze the horizontal motion using constant velocity techniques
- Analyze the vertical motion using constant acceleration techniques
- Remember that both directions share the same time

Non-Symmetric Projectile Motion

- Follow the general rules for projectile motion
- Break the *y*-direction into parts
  - up and down *or*
  - symmetrical back to initial height and then the rest of the height
- May be non-symmetric in other ways

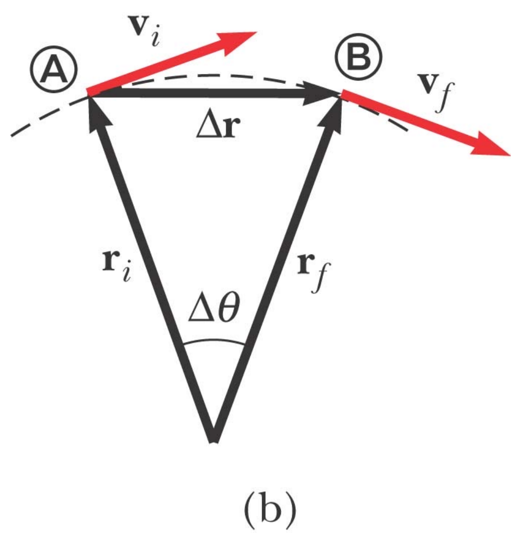


Uniform Circular Motion

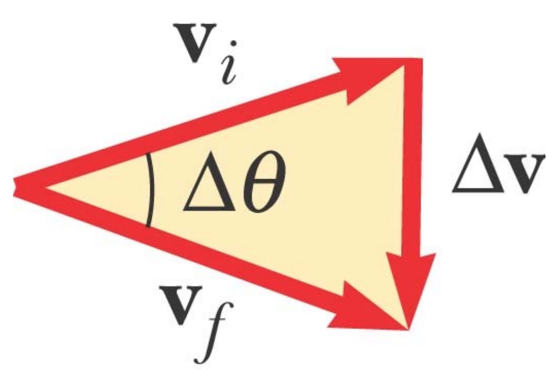
- ***Uniform circular motion*** occurs when an object moves in a circular path with a constant speed
- An acceleration exists since the *direction* of the motion is changing
  - This change in velocity is related to an acceleration
- The velocity vector is always tangent to the path of the object

Changing Velocity in Uniform Circular Motion

- The change in the velocity vector is due to the change in direction



- The vector diagram shows $\Delta\mathbf{v} = \mathbf{v}_f - \mathbf{v}_i$



Centripetal Acceleration

- The acceleration is always perpendicular to the path of the motion
- The acceleration always points toward the center of the circle of motion
- This acceleration is called the ***centripetal acceleration***

Centripetal Acceleration, cont

- The magnitude of the centripetal acceleration vector is given by

$$a_C = \frac{v^2}{r}$$

- The direction of the centripetal acceleration vector is always changing, to stay directed toward the center of the circle of motion

Period

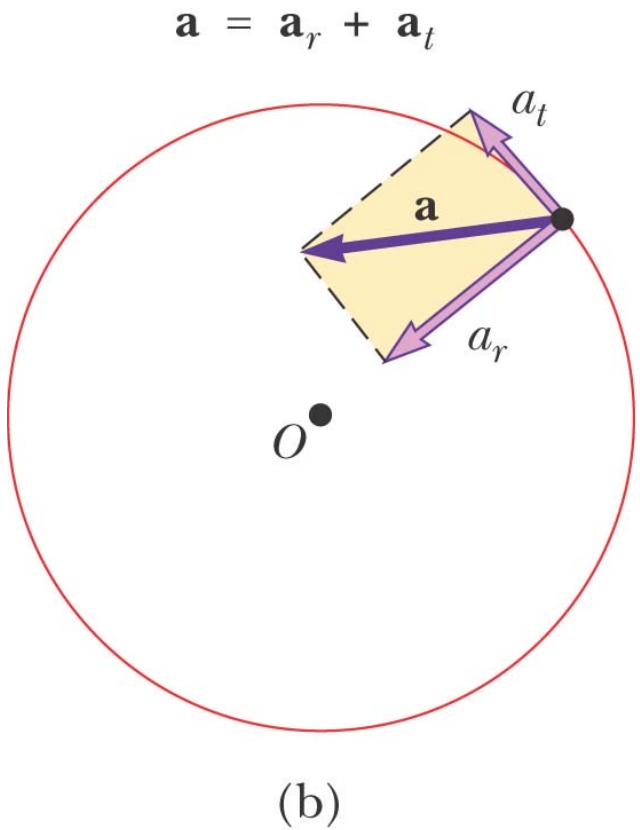
- The ***period***, *T*, is the time required for one complete revolution
- The speed of the particle would be the circumference of the circle of motion divided by the period
- Therefore, the period is $T \equiv \frac{2\pi r}{v}$

Tangential Acceleration

- The magnitude of the velocity could also be changing
- In this case, there would be a ***tangential acceleration***

Total Acceleration

- The tangential acceleration causes the change in the speed of the particle
- The radial acceleration comes from a change in the direction of the velocity vector



Total Acceleration, equations

- The tangential acceleration: $a_t = \frac{d\left|\mathbf{v}\right|}{dt}$

- The radial acceleration: $a_r = -a_C = -\frac{v^2}{r}$

- The total acceleration:
  - Magnitude $a = \sqrt{a_r^2 + a_t^2}$

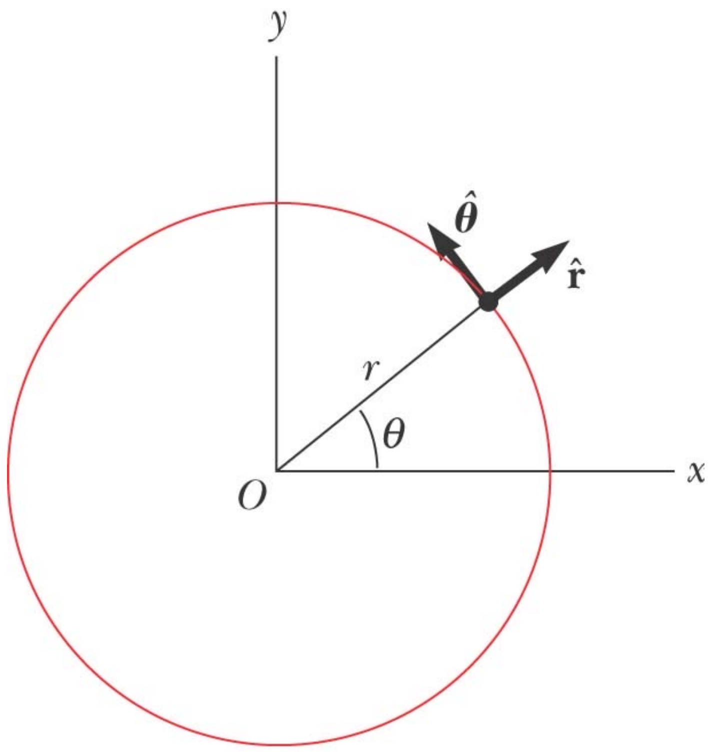
Total Acceleration, In Terms of Unit Vectors

- Define the following unit vectors

$$\hat{\mathbf{r}} \text{ and } \hat{\theta}$$

  - r lies along the radius vector
  - θ is tangent to the circle
- The total acceleration is

$$\mathbf{a} = \mathbf{a}_t + \mathbf{a}_r = \frac{d|\mathbf{v}|}{dt}\hat{\theta} - \frac{v^2}{r}\hat{\mathbf{r}}$$



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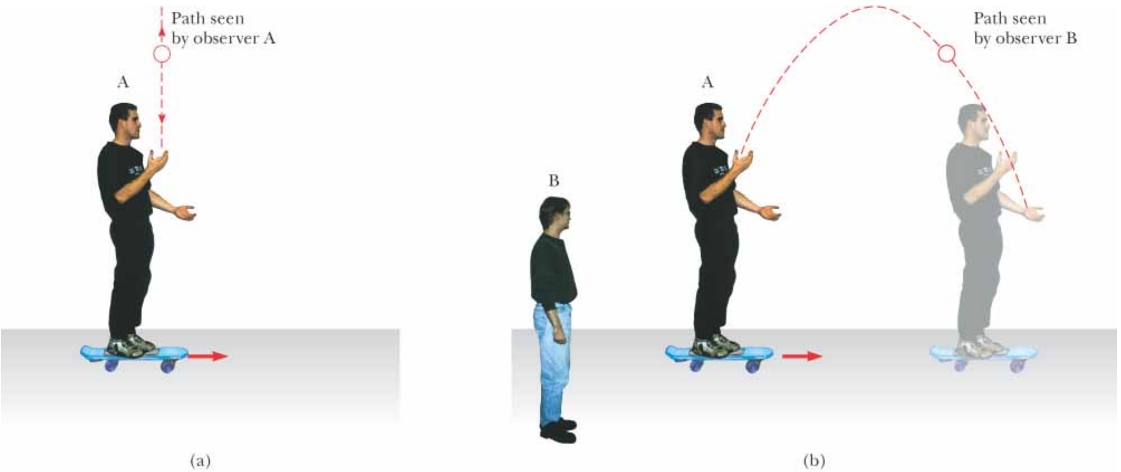
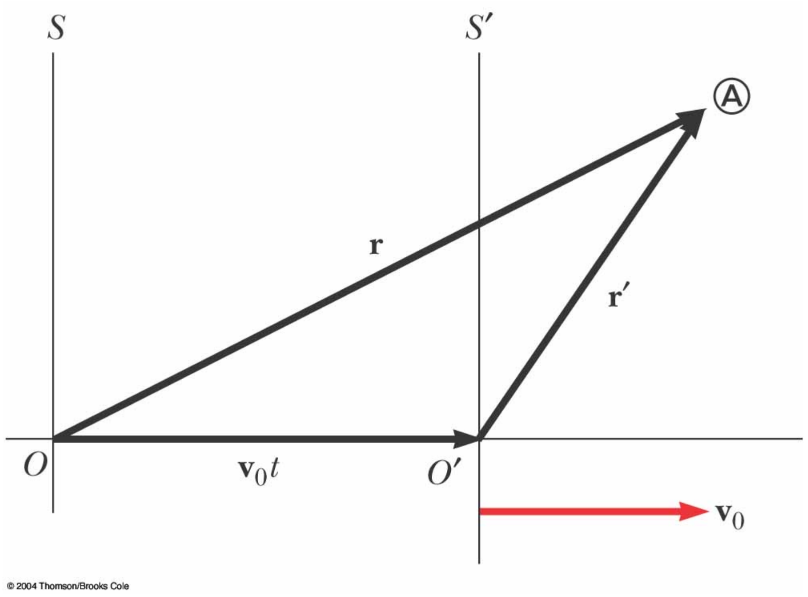
Relative Velocity

- Two observers moving relative to each other generally do not agree on the outcome of an experiment
- For example, observers A and B below see different paths for the ball



Relative Velocity, generalized

- Reference frame *S* is stationary
- Reference frame *S'* is moving at $\mathbf{v}_o$
  - This also means that *S* moves at $-\mathbf{v}_o$ relative to *S'*
- Define time $t = 0$ as that time when the origins coincide



Relative Velocity, equations

- The positions as seen from the two reference frames are related through the velocity
  - $\mathbf{r}' = \mathbf{r} - \mathbf{v}_o\, t$
- The derivative of the position equation will give the velocity equation
  - $\mathbf{v}' = \mathbf{v} - \mathbf{v}_o$
- These are called the **Galilean transformation equations**

Acceleration in Different Frames of Reference

- The derivative of the velocity equation will give the acceleration equation
- The acceleration of the particle measured by an observer in one frame of reference is the same as that measured by any other observer moving at a *constant velocity* relative to the first frame.