PHYS 502 – 2^{nd} Homework Solutions

③ $$g(t) = \begin{cases} 1, & -d/2 < t < d/2 \\ 0 & \text{otherwise} \end{cases}$$

$$G(\omega) = F(g(t)) = \int_{-d/2}^{d/2} e^{-i\omega t}\, dt = -\frac{1}{i\omega} e^{-i\omega t}\Big|_{-d/2}^{d/2} =$$

$$= -\frac{1}{i\omega}\left(e^{-i\omega d/2} - e^{i\omega d/2}\right)$$

$$= -\frac{i}{\omega}\left(e^{i\omega d/2} - e^{-i\omega d/2}\right) = -\frac{i}{\omega} 2i \sin(\omega d/2) =$$

$$= \frac{d \sin(\omega d/2)}{(\omega d/2)}$$

② $$g(t) = e^{-\alpha|t|} \quad (\alpha > 0) \Rightarrow g(t) = \begin{cases} e^{-\alpha t} & t \geq 0 \\ e^{\alpha t} & t < 0 \end{cases}$$

$$G(\omega) = F(g(t)) = \int_{-\infty}^{0} e^{\alpha t} e^{-i\omega t}\, dt + \int_{0}^{\infty} e^{-\alpha t} e^{-i\omega t}\, dt$$

$$= \int_{-\infty}^{0} e^{t(\alpha - i\omega)}\, dt + \int_{0}^{\infty} e^{t(-\alpha - i\omega)}\, dt$$

$$= \frac{1}{(\alpha - i\omega)}\left[e^{t(\alpha - i\omega)}\Big|_{-\infty}^{0}\right] + \left[e^{t(-\alpha - i\omega)}\Big|_{0}^{\infty}\right]\frac{1}{(-\alpha - i\omega)}$$

$$= \frac{1}{\alpha - i\omega} + \frac{1}{\alpha + i\omega} = \frac{2\alpha}{\alpha^2 + \omega^2}$$

① $g(x) = \begin{cases} h(1-a|x|), & |x| < 1/a \\ 0, & |x| > 1/a \end{cases}$

The function gets the analytic form

$$g(x) = \begin{cases} h(1-ax), & 0 < x < 1/a \\ h(1+ax), & -1/a < x < 0 \\ 0, & x < -1/a \text{ or } x > 1/a \end{cases}$$

$$G(\omega) = \mathcal{F}\{g(x)\} = \int_{-\infty}^{\infty} g(x) e^{-i\omega x} dx =$$

$$= \int_{-1/a}^{0} h(1+ax) e^{-i\omega x} dx + \int_{0}^{1/a} h(1-ax) e^{-i\omega x} dx$$

$$= \int_{-1/a}^{0} h e^{-i\omega x} dx + \int_{-1/a}^{0} hax e^{-i\omega x} dx + \int_{0}^{1/a} h e^{-i\omega x} dx$$

$$- \int_{0}^{1/a} ax e^{-i\omega x} dx =$$

$$= h \Big\{ \int_{-1/a}^{0} e^{-i\omega x} dx + a \int_{-1/a}^{0} x e^{-i\omega x} dx + \int_{0}^{1/a} e^{-i\omega x} dx$$

$$- a \int_{0}^{1/a} x e^{-i\omega x} dx \Big\}$$

$$= h \Big\{ \int_{-1/a}^{1/a} e^{-i\omega x} + a \int_{-1/a}^{0} x e^{-i\omega x} dx - a \int_{0}^{1/a} x e^{-i\omega x} dx \Big\}$$

$$= h\left\{ -\frac{1}{i\omega}\left[e^{-i\omega/a} - e^{i\omega/a} \right] + \frac{\alpha(1+i\omega x)}{\omega^2} e^{-i\omega x} \Big|_{-1/a}^{0} \right.$$

$$\left. - \frac{\alpha(1+i\omega x) e^{-i\omega x}}{\omega^2} \Big|_{0}^{1/a} \right\}$$

$$= h\left\{ \frac{2}{\omega}\sin(\omega/a) + \frac{\alpha}{\omega^2}\left[1 - e^{i\omega/a}\right] + \frac{i}{\omega} e^{i\omega/a} \right.$$

$$\left. - \frac{\alpha}{\omega^2} e^{-i\omega/a} + \frac{\alpha}{\omega^2} - \frac{i}{\omega} e^{-i\omega/a} \right\}$$

$$= h\left\{ \frac{2}{\omega^2} - \frac{2}{\omega^2}\cos(\omega/a) \right\} = \frac{2h}{\omega^2}\left\{ 1 - \cos(\omega/a) \right\}$$

$$= \frac{4h}{\omega^2}\sin^2(\omega/2a)$$