PHYS 502
HANDOUT 4 - *Laplace Transform*

1. Prove the following elementary Laplace transformation.

$F(t)$	1	$\exp(kt)$	$\cosh(kt)$	$\sinh(kt)$	$\cos(kt)$	$\sin(kt)$
$f(s)$	$1/s$	$1/(s-k)$	$s/\left(s^2-k^2\right)$	$k/\left(s^2-k^2\right)$	$s/\left(s^2+k^2\right)$	$k/\left(s^2+k^2\right)$

(Arf. p. 825)

2. Apply the partial fractions expansion method to find the inverse Laplace transform of the function:

$$f(s)=\frac{k^2}{s\left(s^2+k^2\right)}.$$

(Arf. p. 827)

3. Evaluate the function

$$F(t)=\int_0^\infty \frac{\sin tx}{x}dx$$

(Arf. p. 828)

4. Apply the method of Laplace transform to solve the problem of the classical simple harmonic oscillator.

(Arf. p. 832)

5. Find the Laplace transform of the delta function. Apply this to study the effect of an impulsive force on a particle.

(Arf. p. 835)

6. Use Laplace transforms to study the motion of a damped harmonic oscillator. Assume the following initial conditions: $x(0)=x_0,\ x'(0)=0$.

(Arf. p. 836)

7. Use Laplace transforms to study the current in an *RLC* circuit.

(Arf. p. 839)

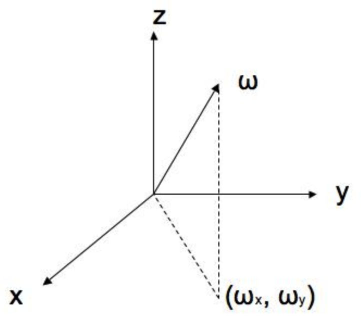
8. Use Laplace transforms to study the driven oscillator with damping. Assume that there is a damping force $F_d=-bv$ (where v is the velocity of the body) and that the driving force is of impulsive character: $F(t)=P\delta(t)$. Assume also that $x(0)=x'(0)=0$

(Arf. p. 851)

9. *The Earth's Nutation:* If we treat the earth as a rigid (oblate) spheroid the Euler's equation of motion reduce to

$$\frac{dX}{dt}=-aY,\qquad \frac{dY}{dt}=+aX,$$

where $a \equiv \left[\left(I_z - I_x\right)/I_z\right]\omega_z$,
$X = \omega_x$, $Y = \omega_y$ with angular velocity vector $\boldsymbol{\omega} = \left(\omega_x, \omega_y, \omega_z\right)$, I_x, I_y, I_z are the moment of inertia about the *x, y* and *z* axis respectively. Solve the equations of motion.



(Arf. p. 833)

10. A mass *m* is attached to one end of an unstreched spring, spring constant *k*. At time *t*=0 the free end of the spring experiences a constant acceleration *a*, away from the mass. Using Laplace transforms,

(a) Find the position *x* of *m* as a function of time.
(b) Determine the limiting form of *x*(*t*) for small *t*.

(Arf. p. 836)

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11. Show the following transforms:

a) $L\left\{\cosh at \cos at\right\} = \dfrac{s^3}{s^4 + 4a^4}$,

b) $L\left\{\cosh at \sin at\right\} = \dfrac{as^2 + 2a^3}{s^4 + 4a^4}$,

c) $L\left\{\sinh at \cos at\right\} = \dfrac{as^2 - 2a^3}{s^4 + 4a^4}$,

d) $L\left\{\sinh at \sin at\right\} = \dfrac{2a^2 s}{s^4 + 4a^4}$

(Arf. p. 848)

12. Show the following transforms:

a) $L^{-1}\left\{\left(s^2 + a^2\right)^{-2}\right\} = \dfrac{1}{2a^3}\sin at - \dfrac{1}{2a^2} t \cos at$,

b) $L^{-1}\left\{s\left(s^2 + a^2\right)^{-2}\right\} = \dfrac{1}{2a} t \sin at$,

c) $L^{-1}\left\{s^2\left(s^2 + a^2\right)^{-2}\right\} = \dfrac{1}{2a}\sin at + \dfrac{1}{2} t \cos at$,

d) $L^{-1}\left\{s^3\left(s^2+a^2\right)^{-2}\right\} = \cos at - at\sin at$.

$$g(t) = \begin{cases} \sin\omega_0 t, & |t| < N\pi/\omega_0 \\ 0, & |t| > N\pi/\omega_0 \end{cases}$$

(Arf. p. 801)

13. Use the calculus of residues to calculate the inverse Laplace transform of $f(s) = a/\left(s^2 - a^2\right)$.

(Arf. p. 855)

14. Use the calculus of residues to calculate the inverse Laplace transform of $f(s) = \left(1 - e^{-as}\right)/s$.

(Arf. p. 855)

15. In a resonant cavity an electromagnetic oscillation of frequency ω_0 dies out as:

$$A(t) = A_0 e^{-\omega_0 t/2Q} e^{-i\omega_0 t}, \quad t > 0$$

(Take $A(t) = 0$ for $t < 0$.)
This parameter Q is a measure of the ratio of stored energy to energy loss per cycle. Calculate the frequency distribution of the oscillation, $a^*(\omega)a(\omega)$, where $a(\omega)$ is the Fourier transform of $A(t)$.

(Arf. p. 805)

16. Solve the differential equation, $dy/dt + 3y = e^{-2t}$, provided that $y(0)=4$.

(Adv. p. 117)

17. Solve the simultaneous differential equations:

$$\frac{dx}{dt} + \frac{dy}{dt} + x = 0$$
$$\frac{d^2x}{dt^2} + \frac{dy}{dt} + 3x + y = 0$$

provided the following initial conditions:

$$x(0) = 0,\ y(0) = 1,\ x'(0) = 3.$$

(Adv. p. 118)

Lecture in Laplace Transform
https://www.youtube.com/watch?v=an5E940fqZQ