1 • $\frac{d}{dx}\left(x^{n} J_n(x)\right) = x^{n} J_{n-1}(x) \Rightarrow n x^{n-1} J_n(x) + x^{n} J_n'(x) = x^{n} J_{n-1}(x)$

$\underset{/x^n}{\Rightarrow} \frac{n x^{n-1}}{x^n} J_n(x) + \frac{x^n}{x^n} J_n'(x) = \frac{x^n}{x^n} J_{n-1}(x) \Rightarrow$

$\Rightarrow \boxed{\frac{n J_n(x)}{x} + J_n'(x) = J_{n-1}(x)}$ (1)

• $\frac{d}{dx}\left(x^{-n} J_n(x)\right) = -x^{-n} J_{n+1}(x) \Rightarrow -n x^{-n-1} J_n(x) + x^{-n} J_n'(x) = -x^{-n} J_{n+1}(x)$

$\underset{/x^{-n}}{\Rightarrow} -\frac{n x^{-n-1}}{x^{-n}} J_n(x) + \frac{x^{-n}}{x^{-n}} J_n'(x) = -\frac{x^{-n}}{x^{-n}} J_{n+1}(x) \Rightarrow$

$\Rightarrow -n x^{-1} J_n(x) + J_n'(x) = -J_{n+1}(x) \Rightarrow \boxed{\frac{n J_n(x)}{x} - J_n'(x) = J_{n+1}(x)}$ (2)

By adding (1) + (2)

$\boxed{\frac{2n J_n(x)}{x} = J_{n-1}(x) + J_{n+1}(x)}$ (3)

By subtracting 1 − 2 we get:

$$J_n'(x) + J_n'(x) = J_{n-1}(x) - J_{n+1}(x) \Rightarrow 2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$$

$$\Rightarrow \boxed{J_n'(x) = \frac{1}{2}\left[J_{n-1}(x) - J_{n+1}(x)\right]} \quad (4)$$

(2) $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ (1) $\quad y(0,t) = y(L,t) = 0 \quad t>0$

$\dfrac{\partial y}{\partial t}(x,0) = 0$

Let $y(x,t) = X(x)T(t)$ (2) then by inserting (2) into (1) we get:

$X\ddot{T} = c^2 T X'' \Rightarrow \dfrac{X\ddot{T}}{XT} = \dfrac{c^2 T X''}{TX} \Rightarrow \dfrac{\ddot{T}}{T} = \dfrac{c^2 X''}{X} \Rightarrow \dfrac{\ddot{T}}{c\,T} = \dfrac{X''}{X} = -\lambda^2 \quad (\lambda>0)$

$\Rightarrow \left.\begin{array}{l} X'' + \lambda^2 X = 0 \\ \ddot{T} + \lambda^2 c^2 T = 0 \end{array}\right\} \quad \begin{array}{ll} X(x) = X_1 \cos(\lambda x) + X_2 \sin(\lambda x) & (3) \\ T(t) = T_1 \cos(\lambda c t) + T_2 \sin(\lambda c t) & (4) \end{array}$

but since $y(0,t) = 0 \Rightarrow X(0) = 0 \Rightarrow \boxed{X_1 = 0}$ (5a)

also since $y(L,t) = 0 \Rightarrow X(L) = 0 \Rightarrow \sin(\lambda L) = \sin(n\pi) \Rightarrow \boxed{\lambda_n = n\pi/L}$ (5b)

Thus $y(x,t) = \left(T_1 \cos(\lambda_n c t) + T_2 \sin(\lambda_n c t)\right) \sin(\lambda_n x)$

but $\dfrac{\partial y}{\partial t}(x,0) = 0 \Rightarrow \left[-T_1 \lambda_n c \sin(\lambda_n c t) + \lambda_n c T_2 \cos(\lambda_n c t)\right]_{t=0} \sin(\lambda_n x) = 0$

$\Rightarrow \boxed{T_2 = 0}$ (6)

So the partial solution is given by

$$y_n(x,t) = B_n \cos(\lambda_n c t)\sin(\lambda_n x) \Rightarrow \boxed{y_n(x,t) = B_n \cos(\omega_n t)\sin\left(\frac{n\pi x}{L}\right)} \quad (7)$$

$$\left(\text{with } \omega_n = \frac{n\pi c}{L}\right)$$

Thus the general solution is given by:

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} B_n \cos(\omega_n t)\sin\left(\frac{n\pi x}{L}\right)} \quad (8)$$

To find B_n we apply the initial condition

$$y(x,0) = f(x) \Rightarrow \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) = f(x)$$

Thus

$$B_n = \frac{\left(f(x), \sin\left(\frac{n\pi x}{L}\right)\right)}{\left(\sin\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)\right)} \Rightarrow B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \Rightarrow$$

$$\Rightarrow B_n = \frac{2}{L}\left\{\int_0^{L/2} x\sin\left(\frac{n\pi x}{L}\right)dx + \int_{L/2}^{L}(-x+L)\sin\left(\frac{n\pi x}{L}\right)dx\right\}$$

$$= \frac{2}{L}\left\{\frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) + \frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) + \frac{L^2}{n^2\pi^2}(n\pi)\cos(n\pi) - \frac{L^2}{n\pi}\cos(n\pi)\right\}$$

$$B_n = \frac{2}{L}\left\{ \frac{2L^2}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) \right\} \Rightarrow \boxed{B_n = \frac{4L}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right)} \quad (9)$$

Thus the solution gets the form

$$\boxed{y(x,t) = \frac{4L}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\left(\frac{n\pi x}{L}\right) \cos(\omega_n t)} \quad (10)$$

3 a) $\sigma \nabla^2 u = \frac{\partial u}{\partial t}$ $\quad u(0,t) = u(L,t) = 0 \quad u(x,0) = f(x)$

$\nabla^2 u = \frac{1}{\sigma}\frac{\partial u}{\partial t} \Longrightarrow$ (since the problem is one-dimensional) $\boxed{\frac{\partial^2 u}{\partial x^2} = \frac{1}{\sigma}\frac{\partial u}{\partial t}}$ (1)

Let $u(x,t) = X(x)T(t)$ (2). Putting (2) $\to$ (1) we get

$$T X'' = \frac{1}{\sigma} X \dot{T} \Longrightarrow \frac{T X''}{XT} = \frac{1}{\sigma}\frac{X\dot{T}}{XT} \Rightarrow \frac{X''}{X} = \frac{1}{\sigma}\frac{\dot{T}}{T} = -\lambda \quad (\lambda > 0)$$

$$\Rightarrow \begin{cases} X'' + \lambda X = 0 & (3) \\ \dot{T} + \sigma\lambda T = 0 & (4) \end{cases}$$

Solving (3) $\quad X(x) = X_1 \cos\sqrt{\lambda}x + X_2 \sin\sqrt{\lambda}x$ (5)

Solving (4) $\quad T(t) = T_1 e^{-\sigma\lambda t}$ (6)

Since $u(0,t) = 0 \Rightarrow X(0) = 0 \Rightarrow \boxed{X_1 = 0}$ (7a)

Since $u(L,t) = 0 \Rightarrow X(L) = 0 \Rightarrow \sqrt{\lambda} = \frac{n\pi}{L} \Rightarrow \boxed{\lambda = \frac{n^2\pi^2}{L^2}}$ (7b)

Thus the partial solution of (1) is : $u_n(x,t) = B_n e^{-\frac{n^2\pi^2}{L^2}\sigma t} \sin\left(\frac{n\pi x}{L}\right)$

and the general solution is given by :

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\frac{n^2\pi^2}{L^2}\sigma t} \sin\left(\frac{n\pi x}{L}\right)} \qquad (8)$$

(b) Applying the initial condition $u(x,0) = f(x)$ we get

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) \qquad (9)$$

$$B_n = \frac{\left(f(x), \sin\left(\frac{n\pi x}{L}\right)\right)}{\left(\sin\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)\right)} \Rightarrow B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

$$\Rightarrow B_n = \frac{2}{L}\int_0^L x \sin\left(\frac{n\pi x}{L}\right) dx \Rightarrow B_n = \frac{2}{L}\left\{ \frac{\sin\frac{n\pi x}{L}}{\left(\frac{n\pi}{L}\right)^2}\Bigg|_0^L - \frac{x\cos\left(\frac{n\pi x}{L}\right)}{(n\pi/L)}\Bigg|_0^L \right\}$$

$$\Rightarrow B_n = \frac{2}{L}\left\{(0-0) - \frac{L\cos(n\pi)}{(n\pi/L)} + 0\right\} \Rightarrow \boxed{B_n = -\frac{2L}{n\pi}\cos(n\pi)} \qquad (10)$$

Now since $L=2$ and $\sigma = 3$ we have:

$$B_n = -\frac{4}{n\pi}\cos(n\pi)$$ and thus

$$u(x,t) = -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(n\pi)}{n}\, e^{-\frac{3n^2\pi^2}{4}t}\sin\left(\frac{n\pi x}{2}\right)$$

(1)