Chapter 9: One- and Two-Sample Estimation Problems:

9.1 Introduction:

- Suppose we have a population with some unknown parameter(s).
  Example: Normal(μ,σ)
  μ and σ are parameters.
- We need to draw conclusions (make inferences) about the unknown parameters.
- We select samples, compute some statistics, and make inferences about the unknown parameters based on the sampling distributions of the statistics.

* Statistical Inference
(1) Estimation of the parameters (Chapter 9)
$\rightarrow$ Point Estimation
$\rightarrow$ Interval Estimation (Confidence Interval)
(2) Tests of hypotheses about the parameters (Chapter 10)

9.3 Classical Methods of Estimation:

Point Estimation:

A point estimate of some population parameter θ is a single value $\hat{\theta}$ of a statistic $\hat{\Theta}$. For example, the value $\bar{x}$ of the statistic $\bar{X}$ computed from a sample of size n is a point estimate of the population mean μ.

Interval Estimation (Confidence Interval = C.I.):

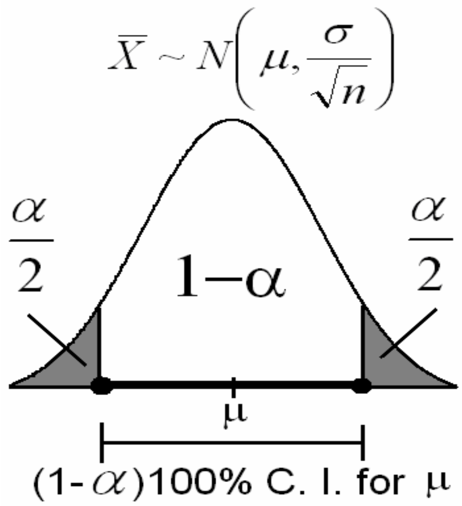
An interval estimate of some population parameter θ is an interval of the form $(\hat{\theta}_L, \hat{\theta}_U)$, i.e, $\hat{\theta}_L < \theta < \hat{\theta}_U$. This interval contains the true value of θ "with probability $1-\alpha$", that is $P(\hat{\theta}_L < \theta < \hat{\theta}_U) = 1-\alpha$.

- $(\hat{\theta}_L, \hat{\theta}_U) = \hat{\theta}_L < \theta < \hat{\theta}_U$ is called a $(1-\alpha)100\%$ confidence interval (C.I.) for θ.
- $1-\alpha$ is called the confidence coefficient
- $\hat{\theta}_L$ = lower confidence limit

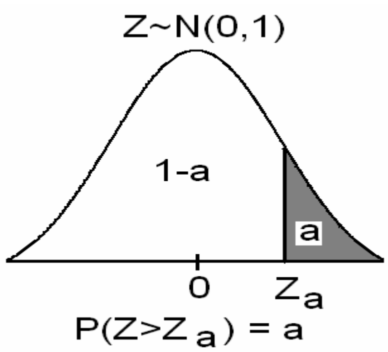
- $\hat{\theta}_U =$ upper confidence limit
- α=0.1, 0.05, 0.025, 0.01 ($0<\alpha<1$)

9.4 Single Sample: Estimation of the Mean (μ):

Recall:



- $E(\bar{X}) = \mu_{\bar{X}} = \mu$
- $Var(\bar{X}) = \sigma^2_{\bar{X}} = \dfrac{\sigma^2}{n}$
- $\bar{X} \sim N\left(\mu, \dfrac{\sigma}{\sqrt{n}}\right)$
- $Z = \dfrac{\bar{X} - \mu}{\sigma / \sqrt{n}}$ ~N(0,1)
  (σ^2 is known)
- $T = \dfrac{\bar{X} - \mu}{S / \sqrt{n}}$ ~ t(n–1)
  (σ^2 is unknown)
- We use the sampling distribution of $\bar{X}$ to make inferences about μ.



Notation:
Z_a is the Z-value leaving an area of a to the right; i.e., $P(Z>Z_a)=a$ or equivalently, $P(Z<Z_a)=1-a$

Point Estimation of the Mean (μ):

- The sample mean $\bar{X} = \sum_{i=1}^{n} X_i / n$ is a "good" point estimate for μ.

Interval Estimation (Confidence Interval) of the Mean (μ):

(i) First Case: σ^2 is known:

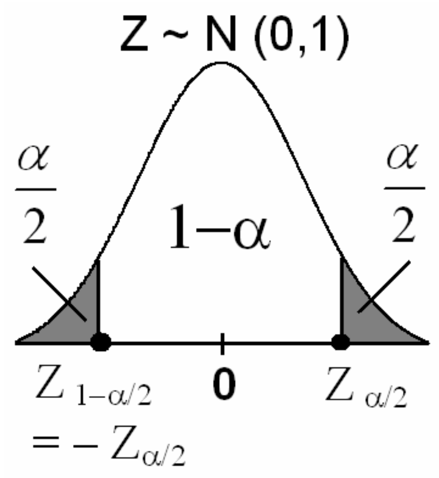
Result:

If $\bar{X} = \sum_{i=1}^{n} X_i / n$ is the sample mean of a random sample of size n from a population (distribution) with mean μ and known variance σ^2, then a $(1-\alpha)100\%$ confidence interval for μ is :

$$(\overline{X} - Z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\ ,\ \overline{X} + Z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}})$$

$$\Leftrightarrow\ \overline{X} \pm Z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}$$

$$\Leftrightarrow\ \overline{X} - Z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} < \mu < \overline{X} + Z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}$$



where $Z_{\frac{\alpha}{2}}$ is the Z-value leaving an area of α/2 to the right; i.e., P(Z>$Z_{\frac{\alpha}{2}}$)=α/2, or equivalently, P(Z<$Z_{\frac{\alpha}{2}}$)=1−α/2.

Note:

We are (1−α)100% confident that μ $\in (\overline{X} - Z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\ ,\ \overline{X} + Z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}})$

Example 9.2:

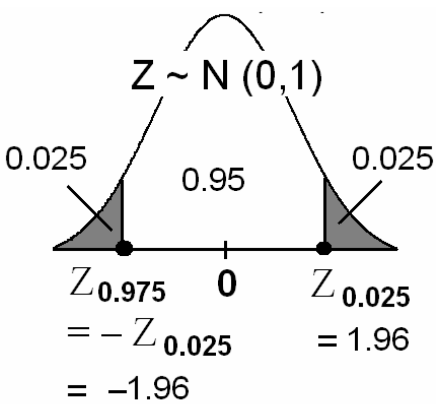
The average zinc concentration recorded from a sample of zinc measurements in 36 different locations is found to be 2.6 gram/milliliter. Find a 95% and 99% confidence interval (C.I.) for the mean zinc concentration in the river. Assume that the population standard deviation is 0.3.

Solution:

μ=the mean zinc concentration in the river.
(unknown parameter)

Population	Sample
μ=??	n=36
σ=0.3	$\overline{X}$ =2.6

First, a point estimate for μ is $\overline{X}$ =2.6.



(a) We want to find 95% C.I. for μ.
α = ??
95% = (1−α)100%
⇔ 0. 95 = (1−α)
⇔ α=0.05 ⇔ α/2 = 0.025

$Z_{\frac{\alpha}{2}} = Z_{0.025}$

$= 1.96$

A 95% C.I. for μ is

$$\overline{X} \pm Z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$$

$$\Leftrightarrow \overline{X} - Z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} < \mu < \overline{X} + Z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$$

$$\Leftrightarrow 2.6 - (1.96)\left(\frac{0.3}{\sqrt{36}}\right) < \mu < 2.6 + (1.96)\left(\frac{0.3}{\sqrt{36}}\right)$$

$\Leftrightarrow 2.6 - 0.098 < \mu < 2.6 + 0.098$

$\Leftrightarrow 2.502 < \mu < 2.698$

$\Leftrightarrow \mu \in (2.502, 2.698)$

We are 95% confident that $\mu \in (2.502, 2.698)$.

(b) Similarly, we can find that (Homework) A 99% C.I. for μ is

$2.471 < \mu < 2.729$

$\Leftrightarrow \mu \in (2.471, 2.729)$

We are 99% confident that $\mu \in (2.471, 2.729)$

Notice that a 99% C.I. is wider that a 95% C.I..

Note:

Error

|-----------|

______|________________ $\overline{X}$ ______ μ ___________|_____

$\overline{X} - Z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$ $\overline{X} + Z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$

|---------------------|-------------------------|

$Z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$ $Z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$

Theorem 9.1:

If $\overline{X}$ is used as an estimate of μ, we can then be (1–α)100% confident that the error (in estimation) will <u>not exceed</u> $Z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$.

Note:

max error of estimation = $Z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}$ with (1−α)100% confidence.

Example:

In Example 9.2, we are 95% confident that the sample mean $\overline{X}=2.6$ differs from the true mean μ by an amount less than $Z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}=(1.96)\left(\frac{0.3}{\sqrt{36}}\right)=0.098$.

Note:

Let e be the maximum amount of the error, that is $e=Z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}$,

then:

$$e=Z_{\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}} \Leftrightarrow \sqrt{n}=Z_{\frac{\alpha}{2}}\frac{\sigma}{e} \Leftrightarrow n=\left(Z_{\frac{\alpha}{2}}\frac{\sigma}{e}\right)^2$$

Theorem 9.2:

If $\overline{X}$ is used as an estimate of μ, we can then be (1−α)100% confident that the error (in estimation) will not exceed a specified amount e when the sample size is $n=\left(Z_{\frac{\alpha}{2}}\frac{\sigma}{e}\right)^2$.

Note:

1. All fractional values of $n=(Z_{\frac{\alpha}{2}}\sigma/e)^2$ are rounded up to the next whole number.
2. If σ is unknown, we could take a preliminary sample of size $n \geq 30$ to provide an estimate of σ. Then using $S=\sqrt{\sum_{i=1}^{n}(X_i-\overline{X})^2/(n-1)}$ as an approximation for σ in Theorem 9.2 we could determine approximately how many observations are needed to provide the desired degree of accuracy.

Example 9.3:

How large a sample is required in Example 9.2 if we want to be 95% confident that our estimate of μ is off by less than 0.05?

Solution:

We have σ= 0.3 , $Z_{\frac{\alpha}{2}}=1.96$, e=0.05. Then by Theorem 9.2,

$$n=\left(Z_{\frac{\alpha}{2}}\frac{\sigma}{e}\right)^2=\left(1.96\times\frac{0.3}{0.05}\right)^2=138.3\approx 139$$

Therefore, we can be 95% confident that a random sample of size n=139 will provide an estimate $\overline{X}$ differing from μ by an amount less than e=0.05.

<u>Interval Estimation (Confidence Interval) of the Mean (μ):</u>

<u>(ii) Second Case: σ^2 is unknown:</u>

Recall:

- $T=\frac{\overline{X}-\mu}{S/\sqrt{n}}$ ~ t(n–1)

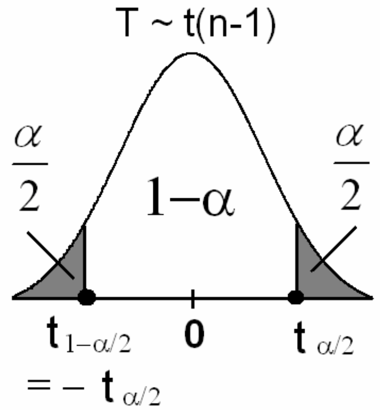
<u>Result:</u>

If $\overline{X}=\sum_{i=1}^{n}X_i/n$ and $S=\sqrt{\sum_{i=1}^{n}(X_i-\overline{X})^2/(n-1)}$ are the sample mean and the sample standard deviation of a random sample of size n from <u>a normal</u> population (distribution) with unknown variance σ^2, then a (1–α)100% confidence interval for μ is :

$$(\overline{X}-t_{\frac{\alpha}{2}}\frac{S}{\sqrt{n}}\ ,\ \overline{X}+t_{\frac{\alpha}{2}}\frac{S}{\sqrt{n}})$$

$$\Leftrightarrow\ \overline{X}\pm t_{\frac{\alpha}{2}}\frac{S}{\sqrt{n}}$$

$$\Leftrightarrow\ \overline{X}-t_{\frac{\alpha}{2}}\frac{S}{\sqrt{n}}<\mu<\overline{X}+t_{\frac{\alpha}{2}}\frac{S}{\sqrt{n}}$$



where $t_{\frac{\alpha}{2}}$ is the t-value with ν=n–1 degrees of freedom leaving an area of α/2 to the right; i.e., P(T>$t_{\frac{\alpha}{2}}$)=α/2, or equivalently, P(T<$t_{\frac{\alpha}{2}}$)=1–α/2.

Example 9.4:

The contents of 7 similar containers of sulfuric acid are 9.8, 10.2, 10.4, 9.8, 10.0, 10.2, and 9.6 liters. Find a 95% C.I. for the mean of all such containers, assuming an approximate normal distribution.

Solution:

.n=7 $\overline{X}=\sum_{i=1}^{n}X_i/n=10.0$ $S=\sqrt{\sum_{i=1}^{n}(X_i-\overline{X})^2/(n-1)}=0.283$

First, a point estimate for μ is $\overline{X}=\sum_{i=1}^{n}X_i/n=10.0$

Now, we need to find a confidence interval for μ.

α = ??

95%=(1−α)100% ⇔ 0. 95=(1−α) ⇔ α=0.05 ⇔ α/2=0.025

$t_{\frac{\alpha}{2}}$ = $t_{0.025}$ =2.447 (with ν=n−1=6 degrees of freedom)

	0.025
6	↓ → $t_{0.025}$=2.447

A 95% C.I. for μ is

$$\overline{X}\pm t_{\frac{\alpha}{2}}\frac{S}{\sqrt{n}}$$

$$\Leftrightarrow \overline{X}-t_{\frac{\alpha}{2}}\frac{S}{\sqrt{n}}<\mu<\overline{X}+t_{\frac{\alpha}{2}}\frac{S}{\sqrt{n}}$$

$$\Leftrightarrow 10.0-(2.447)\left(\frac{0.283}{\sqrt{7}}\right)<\mu<10.0+(2.447)\left(\frac{0.283}{\sqrt{7}}\right)$$

⇔ 10.0 – 0.262< μ < 10.0 + 0.262

⇔ 9.74 < μ < 10.26

⇔ μ ∈(9.74 , 10.26)

We are 95% confident that μ ∈(9.74 , 10.26).

9.5 Standard Error of a Point Estimate:

- The standard error of an estimator is its standard deviation.
- We use $\overline{X}=\sum_{i=1}^{n}X_i/n$ as a point estimator of μ, and we used the sampling distribution of $\overline{X}$ to make a (1−α)100% C.I. for μ.
- The standard deviation of $\overline{X}$, which is $\sigma_{\overline{X}}=\sigma/\sqrt{n}$, is called the standard error of $\overline{X}$. We write *s.e.*($\overline{X}$)= $\sigma/\sqrt{n}$.

- Note: a (1−α)100% C.I. for μ, when σ^2 is known, is

$$\overline{X} \pm Z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}} = \overline{X} \pm Z_{\frac{\alpha}{2}} s.e(\overline{X}).$$

- Note: a (1−α)100% C.I. for μ, when σ^2 is unknown and the distribution is normal, is

$$\overline{X} \pm t_{\frac{\alpha}{2}} \frac{S}{\sqrt{n}} = \overline{X} \pm t_{\frac{\alpha}{2}} \hat{s}.e(\overline{X}). \quad (\nu = n-1 \text{ df})$$

9.7 Two Samples: Estimating the Difference between Two Means ($\mu_1 - \mu_2$):

Recall: For two independent samples:

- $\mu_{\overline{X}_1-\overline{X}_2} = \mu_1 - \mu_2 = E(\overline{X}_1 - \overline{X}_2)$
- $\sigma^2_{\overline{X}_1-\overline{X}_2} = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2} = Var(\overline{X}_1 - \overline{X}_2)$
- $\sigma_{\overline{X}_1-\overline{X}_2} = \sqrt{\sigma^2_{\overline{X}_1-\overline{X}_2}} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$
- $Z = \frac{(\overline{X}_1 - \overline{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0,1)$

Point Estimation of $\mu_1 - \mu_2$:

- $\overline{X}_1 - \overline{X}_2$ is a "good" point estimate for $\mu_1 - \mu_2$.

Confidence Interval of $\mu_1 - \mu_2$:

(i) First Case: σ_1^2 and σ_2^2 are known:

- $Z = \frac{(\overline{X}_1 - \overline{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}} \sim N(0,1)$
- Result:

a (1−α)100% confidence interval for $\mu_1 - \mu_2$ is :

$$(\overline{X}_1 - \overline{X}_2) - Z_{\frac{\alpha}{2}} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} < \mu_1 - \mu_2 < (\overline{X}_1 - \overline{X}_2) + Z_{\frac{\alpha}{2}} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

or $(\bar{X}_1 - \bar{X}_2) \pm Z_{\frac{\alpha}{2}} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$

or $\left((\bar{X}_1 - \bar{X}_2) - Z_{\frac{\alpha}{2}} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}, (\bar{X}_1 - \bar{X}_2) + Z_{\frac{\alpha}{2}} \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}} \right)$

(ii) Second Case: $\sigma_1^2 = \sigma_2^2 = \sigma^2$ is unknown:

- If σ_1^2 and σ_2^2 are unknown but $\sigma_1^2 = \sigma_2^2 = \sigma^2$, then the pooled estimate of σ^2 is

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$$

where S_1^2 is the variance of the 1-st sample and S_2^2 is the variance of the 2-nd sample. The degrees of freedom of S_p^2 is $\nu = n_1 + n_2 - 2$.

- Result:

a $(1-\alpha)100\%$ confidence interval for $\mu_1 - \mu_2$ is :

$$(\bar{X}_1 - \bar{X}_2) - t_{\frac{\alpha}{2}} \sqrt{\frac{S_p^2}{n_1} + \frac{S_p^2}{n_2}} < \mu_1 - \mu_2 < (\bar{X}_1 - \bar{X}_2) + t_{\frac{\alpha}{2}} \sqrt{\frac{S_p^2}{n_1} + \frac{S_p^2}{n_2}}$$

or

$$(\bar{X}_1 - \bar{X}_2) - t_{\frac{\alpha}{2}} S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} < \mu_1 - \mu_2 < (\bar{X}_1 - \bar{X}_2) + t_{\frac{\alpha}{2}} S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

or $(\bar{X}_1 - \bar{X}_2) \pm t_{\frac{\alpha}{2}} \; S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$

or $\left((\bar{X}_1 - \bar{X}_2) - t_{\frac{\alpha}{2}} \; S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}, (\bar{X}_1 - \bar{X}_2) + t_{\frac{\alpha}{2}} \; S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} \right)$

where $t_{\frac{\alpha}{2}}$ is the t-value with $\nu = n_1 + n_2 - 2$ degrees of freedom.

Example 9.6: (1st Case: σ_1^2 and σ_2^2 are known)

An experiment was conducted in which two types of engines, *A* and *B*, were compared. Gas mileage in miles per gallon was measured. 50 experiments were conducted using engine type *A* and 75 experiments were done for engine type *B*. The gasoline used and other conditions were held constant. The average gas mileage for engine *A* was 36 miles per gallon and the average for engine *B* was 42 miles per gallon. Find 96% confidence interval for $\mu_B - \mu_A$, where μ_A and μ_B are population mean gas mileage for engines *A* and *B*, respectively. Assume that the population standard deviations are 6 and 8 for engines *A* and *B*, respectively.

Solution:

Engine *A*	Engine *B*
n_A=50	n_B=75
$\bar{X}_A$=36	$\bar{X}_B$=42
σ_A=6	σ_B=8

A point estimate for $\mu_B - \mu_A$ is $\bar{X}_B - \bar{X}_A$=42–36=6.

Confidence interval:

α = ??

96% = (1–α)100% $\Leftrightarrow$ 0. 96 = (1–α) $\Leftrightarrow$ α=0.04 $\Leftrightarrow$ α/2 = 0.02

$Z_{\frac{\alpha}{2}} = Z_{0.02} = 2.05$

A 96% C.I. for $\mu_B - \mu_A$ is

$$(\bar{X}_B - \bar{X}_A) - Z_{\frac{\alpha}{2}}\sqrt{\frac{\sigma_B^2}{n_B}+\frac{\sigma_A^2}{n_A}} < \mu_B - \mu_A < (\bar{X}_B - \bar{X}_A) + Z_{\frac{\alpha}{2}}\sqrt{\frac{\sigma_B^2}{n_B}+\frac{\sigma_A^2}{n_A}}$$

$$(\bar{X}_B - \bar{X}_A) \pm Z_{\frac{\alpha}{2}}\sqrt{\frac{\sigma_B^2}{n_B}+\frac{\sigma_A^2}{n_A}}$$

$$(42-36) \pm Z_{0.02}\sqrt{\frac{8^2}{75}+\frac{6^2}{50}}$$

$$6 \pm (2.05)\sqrt{\frac{64}{75}+\frac{36}{50}}$$

$$6 \pm 2.571$$

3.43 < $\mu_B - \mu_A$ < 8.57

We are 96% confident that $\mu_B - \mu_A \in$(3.43, 8.57).

Example 9.7: (2nd Case: $\sigma_1^2 = \sigma_2^2$ unknown) Reading Assignment

Example: (2nd Case: $\sigma_1^2 = \sigma_2^2$ unknown)

To compare the resistance of wire *A* with that of wire *B*, an experiment shows the following results based on two independent samples (original data multiplied by 1000):

Wire *A*: 140, 138, 143, 142, 144, 137

Wire *B*: 135, 140, 136, 142, 138, 140

Assuming equal variances, find 95% confidence interval for $\mu_A - \mu_B$, where μ_A (μ_B) is the mean resistance of wire *A* (*B*).

Solution:

Wire *A*	Wire *B*
$n_A=6$	$n_B=6$
$\overline{X}_A=140.67$	$\overline{X}_B=138.50$
$S^2_A=7.86690$	$S^2_B=7.10009$

A point estimate for $\mu_A - \mu_B$ is $\overline{X}_A - \overline{X}_B = 140.67 - 138.50 = 2.17$.

Confidence interval:

$95\% = (1-\alpha)100\% \Leftrightarrow 0.95 = (1-\alpha) \Leftrightarrow \alpha = 0.05 \Leftrightarrow \alpha/2 = 0.025$

$\nu = \text{df} = n_A + n_B - 2 = 10$

$t_{\frac{\alpha}{2}} = t_{0.025} = 2.228$

$$S_p^2 = \frac{(n_A - 1)S_A^2 + (n_B - 1)S_B^2}{n_A + n_B - 2}$$

$$= \frac{(6-1)(7.86690) + (6-1)(7.10009)}{6+6-2} = 7.4835$$

$$S_p = \sqrt{S_p^2} = \sqrt{7.4835} = 2.7356$$

A 95% C.I. for $\mu_A - \mu_B$ is

$$(\overline{X}_A - \overline{X}_B) - t_{\frac{\alpha}{2}} S_p \sqrt{\frac{1}{n_A} + \frac{1}{n_B}} < \mu_A - \mu_B < (\overline{X}_A - \overline{X}_B) + t_{\frac{\alpha}{2}} S_p \sqrt{\frac{1}{n_A} + \frac{1}{n_B}}$$

or $(\overline{X}_A - \overline{X}_B) \pm t_{\frac{\alpha}{2}} S_p \sqrt{\frac{1}{n_A} + \frac{1}{n_B}}$

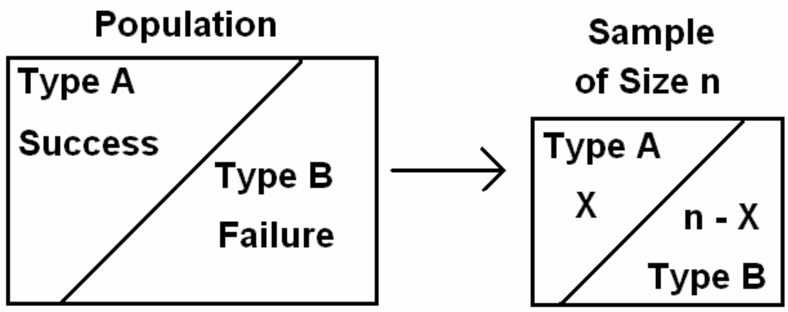
$$(140.67 - 138.50) \pm (2.228)(2.7356)\sqrt{\frac{1}{6} + \frac{1}{6}}$$

$$2.17 \pm 3.51890$$

$-1.35 < \mu_A - \mu_B < 5.69$

We are 95% confident that $\mu_A - \mu_B \in (-1.35, 5.69)$

9.9 Single Sample: Estimating of a Proportion:



.p = Population proportion of successes (elements of Type A) in the population

$$= \frac{A}{A+B} = \frac{\textit{no. of elements of type A in the population}}{\textit{Total no. of elements}}$$

.n = sample size

X = no. of elements of type A in the sample of size n.

$\hat{p}$ = Sample proportion of successes (elements of Type A) in the sample

$$= \frac{X}{n}$$

Recall that:

(1) X ~ Binomial (n, p)

(2) $E(\hat{p}) = E(\frac{X}{n}) = p$

(3) $Var(\hat{p}) = Var(\frac{X}{n}) = \frac{pq}{n}$; $q = 1 - p$

(4) For large n, we have

$$\hat{p} \sim N(p, \sqrt{\frac{pq}{n}}) \quad \text{(Approximately)}$$

$$Z = \frac{\hat{p} - p}{\sqrt{\frac{pq}{n}}} \sim N(0,1) \quad \text{(Approximately)}$$

Point Estimation for *p*:

A good point estimator for the population proportion p is given by the statistic (sample proportion):

$$\hat{p} = \frac{X}{n}$$

Confidence Interval for *p*:

Result:

For large *n*, an approximate $(1-\alpha)100\%$ confidence interval for *p* is :

$$\hat{p} \pm Z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}\,\hat{q}}{n}} \quad ; \; \hat{q} = 1 - \hat{p}$$

or

$$\left(\hat{p} - Z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}\,\hat{q}}{n}} \;,\; \hat{p} + Z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}\,\hat{q}}{n}} \right)$$

or

$$\hat{p} - Z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}\,\hat{q}}{n}} < p < \hat{p} + Z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}\,\hat{q}}{n}}$$

Example 9.10:

In a random sample of n=500 families owing television sets in the city of Hamilton, Canada, it was found that x=340 subscribed to HBO. Find 95% confidence interval for the actual proportion of families in this city who subscribe to HBO.

Solution:

.p = proportion of families in this city who subscribe to HBO.

.n = sample size
= 500

X = no. of families in the sample who subscribe to HBO.
= 340

$\hat{p}$ = proportion of families in the sample who subscribe to HBO.

$= \frac{X}{n} = \frac{340}{500} = 0.68$

$\hat{q} = 1 - \hat{p} = 1 - 0.68 = 0.32$

A point estimator for *p* is

$$\hat{p} = \frac{X}{n} = \frac{340}{500} = 0.68$$

Now,

$95\% = (1-\alpha)100\% \Leftrightarrow 0.95 = (1-\alpha) \Leftrightarrow \alpha = 0.05 \Leftrightarrow \alpha/2 = 0.025$

$Z_{\frac{\alpha}{2}} = Z_{0.025} = 1.96$

A 95% confidence interval for p is:

$$\hat{p} \pm Z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}\,\hat{q}}{n}} \quad ; \ \hat{q} = 1 - \hat{p}$$

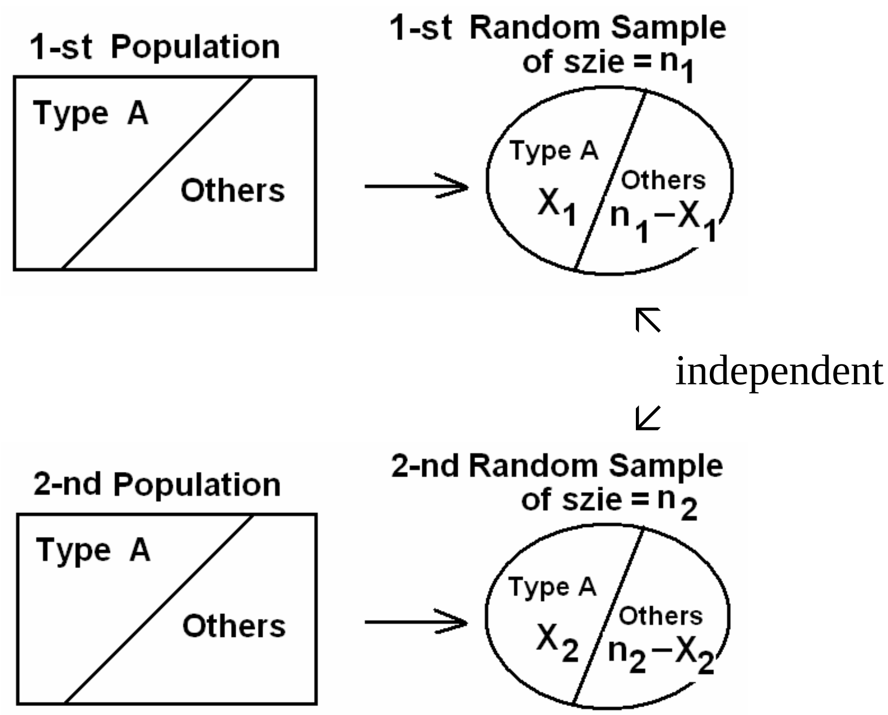
$$0.68 \pm 1.96\sqrt{\frac{(0.68)(0.32)}{500}}$$

$$0.68 \pm 0.04$$

$$0.64 < p < 0.72$$

We are 95% confident that $p \in (0.64, 0.72)$.

9.10 Two Samples: Estimating the Difference between Two Proportions:



Suppose that we have two populations:

- .p_1 = proportion of the 1-st population.
- .p_2 = proportion of the 2-nd population.
- We are interested in comparing p_1 and p_2, or equivalently, making inferences about $p_1 - p_2$.
- We independently select a random sample of size n_1 from the 1-st population and another random sample of size n_2 from the 2-nd population:
- Let X_1 = no. of elements of type A in the 1-st sample.

  $X_1 \sim \text{Binomial}(n_1, p_1)$

$E(X_1) = n_1\, p_1$

$Var(X_1) = n_1\, p_1\, q_1 \quad (q_1 = 1 - p_1)$

- Let X_2 = no. of elements of type A in the 2-nd sample.

  $X_2 \sim Binomial(n_2, p_2)$

  $E(X_2) = n_2\, p_2$

  $Var(X_2) = n_2\, p_2\, q_2 \quad (q_2 = 1 - p_2)$
- $\hat{p}_1 = \frac{X_1}{n_1}$ = proportion of the 1-st sample
- $\hat{p}_2 = \frac{X_2}{n_2}$ = proportion of the 2-nd sample
- The sampling distribution of $\hat{p}_1 - \hat{p}_2$ is used to make inferences about $p_1 - p_2$.

Result:

(1) $E(\hat{p}_1 - \hat{p}_2) = p_1 - p_2$

(2) $Var(\hat{p}_1 - \hat{p}_2) = \frac{p_1\, q_1}{n_1} + \frac{p_2\, q_2}{n_2} \quad ; q_1 = 1 - p_1,\ q_2 = 1 - p_2$

(3) For large n_1 and n_2, we have

$$\hat{p}_1 - \hat{p}_2 \sim N(p_1 - p_2, \sqrt{\frac{p_1\, q_1}{n_1} + \frac{p_2\, q_2}{n_2}}) \quad \text{(Approximately)}$$

$$Z = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\frac{p_1\, q_1}{n_1} + \frac{p_2\, q_2}{n_2}}} \sim N(0,1) \quad \text{(Approximately)}$$

Point Estimation for $p_1 - p_2$:

A good point estimator for the difference between the two proportions, $p_1 - p_2$, is given by the statistic:

$$\hat{p}_1 - \hat{p}_2 = \frac{X_1}{n_1} - \frac{X_2}{n_2}$$

Confidence Interval for $p_1 - p_2$:

Result:

For large n_1 and n_2, an approximate $(1-\alpha)100\%$ confidence interval for $p_1 - p_2$ is :

$$(\hat{p}_1 - \hat{p}_2) \pm Z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}_1\, \hat{q}_1}{n_1} + \frac{\hat{p}_2\, \hat{q}_2}{n_2}}$$

or

$$\left((\hat{p}_1 - \hat{p}_2) - Z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}_1 \hat{q}_1}{n_1} + \frac{\hat{p}_2 \hat{q}_2}{n_2}},\ (\hat{p}_1 - \hat{p}_2) + Z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}_1 \hat{q}_1}{n_1} + \frac{\hat{p}_2 \hat{q}_2}{n_2}} \right)$$

or

$$(\hat{p}_1 - \hat{p}_2) - Z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}_1 \hat{q}_1}{n_1} + \frac{\hat{p}_2 \hat{q}_2}{n_2}} < p_1 - p_2 < (\hat{p}_1 - \hat{p}_2) + Z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}_1 \hat{q}_1}{n_1} + \frac{\hat{p}_2 \hat{q}_2}{n_2}}$$

Example 9.13:

A certain change in a process for manufacture of component parts is being considered. Samples are taken using both existing and the new procedure to determine if the new process results in an improvement. If 75 of 1500 items from the existing procedure were found to be defective and 80 of 2000 items from the new procedure were found to be defective, find 90% confidence interval for the true difference in the fraction of defectives between the existing and the new process.

Solution:

.p_1 = fraction (proportion) of defectives of the existing process

.p_2 = fraction (proportion) of defectives of the new process

$\hat{p}_1$ = sample fraction of defectives of the existing process

$\hat{p}_2$ = sample fraction of defectives of the new process

Existing Process	New Process
$n_1 = 1500$	$n_2 = 2000$
$X_1=75$	$X_2=80$
$\hat{p}_1 = \frac{X_1}{n_1} = \frac{75}{1500} = 0.05$	$\hat{p}_2 = \frac{X_2}{n_2} = \frac{80}{2000} = 0.04$
$\hat{q}_1 = 1 - 0.05 = 0.95$	$\hat{q}_2 = 1 - 0.04 = 0.96$

Point Estimation for $p_1 - p_2$:

A point estimator for the difference between the two proportions, $p_1 - p_2$, is:

$$\hat{p}_1 - \hat{p}_2 = 0.05 - 0.04 = 0.01$$

Confidence Interval for $p_1 - p_2$:

A 90% confidence interval for $p_1 - p_2$ is :

$$(\hat{p}_1 - \hat{p}_2) \pm Z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}_1\,\hat{q}_1}{n_1} + \frac{\hat{p}_2\,\hat{q}_2}{n_2}}$$

$$(\hat{p}_1 - \hat{p}_2) \pm Z_{0.05} \sqrt{\frac{\hat{p}_1\,\hat{q}_1}{n_1} + \frac{\hat{p}_2\,\hat{q}_2}{n_2}}$$

$$0.01 \pm 1.645 \sqrt{\frac{(0.05)(0.95)}{1500} + \frac{(0.04)(0.96)}{2000}}$$

$$0.01 \pm 0.01173$$

$$-0.0017 < p_1 - p_2 < 0.0217$$

We are 90% confident that $p_1 - p_2 \in (-0.0017, 0.0217)$.

Note:

Since $0 \in$ 90% confidence interval=(–0.0017, 0.0217), there is no reason to believe that the new procedure produced a significant decrease in the proportion of defectives over the existing method ($p_1 - p_2 \approx 0 \Leftrightarrow p_1 \approx p_2$).