Chapter 10: One- and Two-Sample Tests of Hypotheses:

Consider a population with some unknown parameter θ. We are interested in testing (confirming or denying) some conjectures about θ. For example, we might be interested in testing the conjecture that $\theta = \theta_o$, where θ_o is a given value.

10.1-10.3: Introduction +:
- A statistical hypothesis is a conjecture concerning (or a statement about) the population.
- For example, if θ is an unknown parameter of the population, we may be interested in testing the conjecture that $\theta > \theta_o$ for some specific value θ_o.
- We usually test the null hypothesis:
  $$H_o: \theta = \theta_o \qquad \text{(Null Hypothesis)}$$
  Against one of the following alternative hypotheses:

$$H_1: \begin{cases} \theta \neq \theta_o \\ \theta > \theta_o \\ \theta < \theta_o \end{cases} \quad \begin{array}{l}\text{(Alternative Hypothesis} \\ \text{or Research Hypothesis)}\end{array}$$

- Possible situations in testing a statistical hypothesis:

	H_o is true	H_o is false
Accepting H_o	Correct Decision	Type II error (β)
Rejecting H_o	Type I error (α)	Correct Decision

Type I error = Rejecting H_o when H_o is true
Type II error = Accepting H_o when H_o is false
P(Type I error) = P(Rejecting H_o | H_o is true) = α
P(Type II error) = P(Accepting H_o | H_o is false) = β

- The level of significance of the test:
  α = P(Type I error) = P(Rejecting H_o | H_o is true)

- One-sided alternative hypothesis:

$H_o: \theta = \theta_o$ or $\quad H_o: \theta = \theta_o$

$H_1: \theta > \theta_o$ $\qquad H_1: \theta < \theta_o$

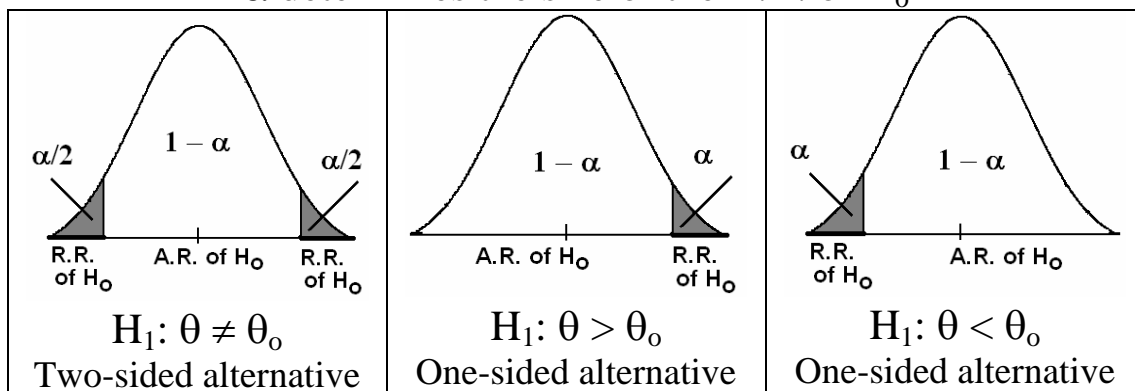- Two-sided alternative hypothesis:

$H_o: \theta = \theta_o$

$H_1: \theta \neq \theta_o$

- The test procedure for rejecting H_o (accepting H_1) or accepting H_o (rejecting H_1) involves the following steps:

1. Determining a test statistic (T.S.)
2. Determining the significance level α

$\alpha = 0.01, 0.025, 0.05$, or 0.10

3. Determining the rejection region (R.R.) and the acceptance region (A.R.) of H_o .

R.R. of H_o depends on H_1 and α

- H_1 determines the direction of the R.R. of H_o
- α determines the size of the R.R. of H_o

$\alpha/2$ $\quad 1-\alpha$ $\quad \alpha/2$	$1-\alpha$ $\quad \alpha$	α $\quad 1-\alpha$
R.R. A.R. of H_o R.R. of H_o of H_o	A.R. of H_o R.R. of H_o	R.R. A.R. of H_o of H_o
$H_1: \theta \neq \theta_o$	$H_1: \theta > \theta_o$	$H_1: \theta < \theta_o$
Two-sided alternative	One-sided alternative	One-sided alternative

4. Decision:

We reject H_o (accept H_1) if the value of the T.S. falls in the R.R. of H_o , and vice versa.

10.5: Single Sample: Tests Concerning a Single Mean (Variance Known):

Suppose that $X_1, X_2, \ldots, X_n$ is a random sample of size n from distribution with mean μ and (known) variance σ^2.

Recall:

- $E(\overline{X}) = \mu_{\overline{X}} = \mu$

- $Var(\overline{X}) = \sigma_{\overline{X}}^2 = \dfrac{\sigma^2}{n}$

- $\overline{X} \sim N\left(\mu, \dfrac{\sigma}{\sqrt{n}}\right) \iff Z = \dfrac{\overline{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$

- Let μ_o be a given known value.
- Test Procedure:

Hypotheses	$H_o: \mu = \mu_o$ $H_1: \mu \neq \mu_o$	$H_o: \mu = \mu_o$ $H_1: \mu > \mu_o$	$H_o: \mu = \mu_o$ $H_1: \mu < \mu_o$
Test Statistic (T.S.)	$Z = \dfrac{\overline{X} - \mu_o}{\sigma/\sqrt{n}} \sim N(0,1)$	
R.R. and A.R. of H_o			
Decision:	Reject H_o (and accept H_1) at the significance level α if:	
	$Z > Z_{\alpha/2}$ or $Z < -Z_{\alpha/2}$ Two-Sided Test	$Z > Z_{\alpha}$ One-Sided Test	$Z < -Z_{\alpha}$ One-Sided Test

Example 10.3:

A random sample of 100 recorded deaths in the United States during the past year showed an average of 71.8 years. Assuming a population standard deviation of 8.9 year, does this seem to indicate that the mean life span today is greater than 70 years? Use a 0.05 level of significance.

Solution:

$.n=100, \quad \overline{X}=71.8, \quad \sigma=8.9$

μ=average (mean) life span

$\mu_o=70$

Hypotheses:

H_o: $\mu = 70$

H_1: $\mu > 70$

T.S. :

$$Z = \frac{\overline{X} - \mu_o}{\sigma / \sqrt{n}} = \frac{71.8 - 70}{8.9 / \sqrt{100}} = 2.02$$

Level of significance:

$\alpha = 0.05$
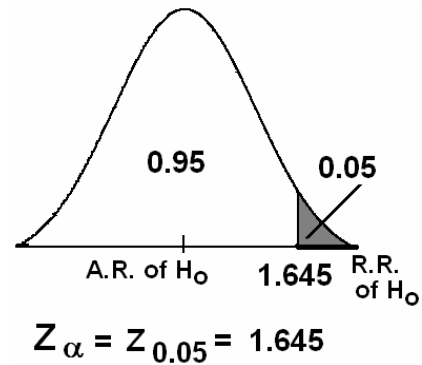
R.R.:

$Z_\alpha = Z_{0.05} = 1.645$

$Z > Z_\alpha = Z_{0.05} = 1.645$

Decision:

Since $Z = 2.02 \in$ R.R., i.e., $Z = 2.02 > Z_{0.05}$, we reject H_o at $\alpha = 0.05$ and accept H_1: $\mu > 70$. Therefore, we conclude that the mean life span today is greater than 70 years.

Example 10.4:

A manufacturer of sports equipment has developed a new synthetic fishing line that he claims has a mean breaking strength of 8 kilograms with a standard deviation of 0.5 kilograms. Test the hypothesis that $\mu = 8$ kg against the alternative that $\mu \neq 8$ kg if a random sample of 50 lines is tested and found to have a mean breaking strength of 7.8 kg. Use a 0.01 level of significance.

Solution:

.$n = 50$, $\overline{X} = 7.8$, $\sigma = 0.5$,

$\alpha = 0.01$, $\alpha/2 = 0.005$

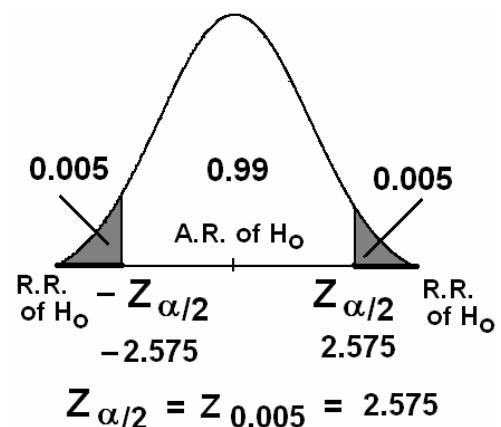$\mu =$ mean breaking strength

$\mu_o = 8$

Hypotheses:

H_o: $\mu = 8$

H_1: $\mu \neq 8$

T.S. :

$$Z = \frac{\overline{X} - \mu_o}{\sigma / \sqrt{n}} = \frac{7.8 - 8}{0.5 / \sqrt{50}} = -2.83$$

$Z_{\alpha/2} = Z_{0.005} = 2.575$ and $-Z_{\alpha/2} = -Z_{0.005} = -2.575$
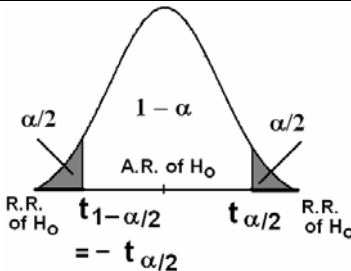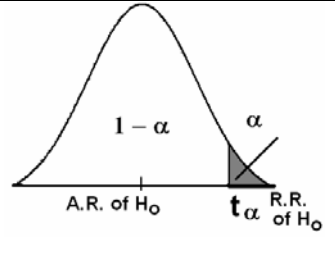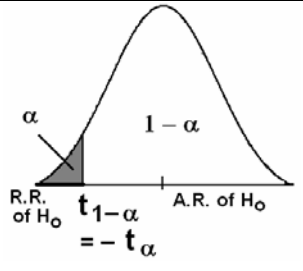
Decision:

Since $Z = -2.83 \in$ R.R., i.e., $Z = -2.83 < -Z_{0.005}$, we reject H_o at $\alpha = 0.01$ and accept H_1: $\mu \neq 8$. Therefore, we conclude that the claim is not correct.

10.7: Single Sample: Tests on a Single Mean (Variance Unknown):

Suppose that $X_1, X_2, \ldots, X_n$ is a random sample of size n from normal distribution with mean μ and unknown variance σ^2.
Recall:

- $T = \dfrac{\overline{X} - \mu}{S/\sqrt{n}} \sim t(\nu)$; $\nu = n-1$

- $S = \sqrt{\sum\limits_{i=1}^{n}(X_i - \overline{X})^2 /(n-1)}$

- Test Procedure:

Hypotheses	H_o: $\mu = \mu_o$ H_1: $\mu \neq \mu_o$	H_o: $\mu = \mu_o$ H_1: $\mu > \mu_o$	H_o: $\mu = \mu_o$ H_1: $\mu < \mu_o$
Test Statistic (T.S.)	$T = \dfrac{\overline{X} - \mu_o}{S/\sqrt{n}} \sim t(n-1)$		
R.R. and A.R. of H_o			
Decision:	Reject H_o (and accept H_1) at the significance level α if:		
	$T > t_{\alpha/2}$ or $T < -t_{\alpha/2}$ Two-Sided Test	$T > t_{\alpha}$ One-Sided Test	$T < -t_{\alpha}$ One-Sided Test

Example 10.5:

… If a random sample of 12 homes included in a planned study indicates that vacuum cleaners expend an average of 42 kilowatt-hours per year with a standard deviation of 11.9 kilowatt-hours, does this suggest at the 0.05 level of significance that the vacuum cleaners expend, on the average, less than 46

kilowatt-hours annually? Assume the population of kilowatt-hours to be normal.

Solution:

.$n=12$, $\overline{X}=42$, S=11.9, $\alpha=0.05$

μ=average (mean) kilowatt-hours annual expense of a vacuum cleaner
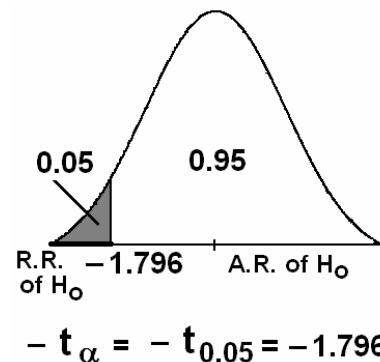
$\mu_o=46$

Hypotheses:

H_o: $\mu = 46$

H_1: $\mu < 46$

T.S. :

$$T = \frac{\overline{X} - \mu_o}{S/\sqrt{n}} = \frac{42 - 46}{11.9/\sqrt{12}} = -1.16 \; ;$$

ν=df= $n-1$=11

$-t_\alpha = -t_{0.05} = -1.796$

0.05   0.95

R.R. −1.796   A.R. of H_o
of H_o

$-t_\alpha = -t_{0.05} = -1.796$

Decision:

Since T=$-1.16 \notin$R.R. (T=$-1.16 \in$A.R.), we do not reject H_o at α=0.05 (i.e, accept H_o: $\mu = 46$) and reject H_1: $\mu < 46$. Therefore, we conclude that μ is not less than 46 kilowatt-hours.

10.8: Two Samples: Tests on Two Means:

Recall: For two independent samples:

- If σ_1^2 and σ_2^2 are known, then we have:

$$Z = \frac{(\overline{X}_1 - \overline{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}} \sim N(0,1)$$

- If σ_1^2 and σ_2^2 are unknown but $\sigma_1^2=\sigma_2^2=\sigma^2$, then we have:

$$T = \frac{(\overline{X}_1 - \overline{X}_2) - (\mu_1 - \mu_2)}{S_p\sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}} \sim t(n_1+n_2-2)$$

Where the pooled estimate of σ^2 is

$$S_p^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2}$$

The degrees of freedom of S_p^2 is $\nu=n_1+n_2-2$.

Now, suppose we need to test the null hypothesis

$$H_o: \mu_1 = \mu_2 \qquad \Leftrightarrow \qquad H_o: \mu_1 - \mu_2 = 0$$

Generally, suppose we need to test

$$H_o: \mu_1 - \mu_2 = d \qquad \text{(for some specific value } d)$$

Against one of the following alternative hypothesis

$$H_1: \begin{cases} \mu_1 - \mu_2 \neq d \\ \mu_1 - \mu_2 > d \\ \mu_1 - \mu_2 < d \end{cases}$$

Hypotheses	$H_o: \mu_1 - \mu_2 = d$ $H_1: \mu_1 - \mu_2 \neq d$	$H_o: \mu_1 - \mu_2 = d$ $H_1: \mu_1 - \mu_2 > d$	$H_o: \mu_1 - \mu_2 = d$ $H_1: \mu_1 - \mu_2 < d$
Test Statistic (T.S.)	$Z = \dfrac{(\bar{X}_1 - \bar{X}_2) - d}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}} \sim N(0,1)$ {if σ_1^2 and σ_2^2 are known} or $T = \dfrac{(\bar{X}_1 - \bar{X}_2) - d}{S_p \sqrt{\dfrac{1}{n_1} + \dfrac{1}{n_2}}} \sim t(n_1+n_2-2)$ {if $\sigma_1^2 = \sigma_2^2 = \sigma^2$ is unknown}		
R.R. and A.R. of H_o	Or	Or	Or
Decision:	Reject H_o (and accept H_1) at the significance level α if:		
	T.S. $\in$ R.R. Two-Sided Test	T.S. $\in$ R.R. One-Sided Test	T.S. $\in$ R.R. One-Sided Test

Example10.6:

An experiment was performed to compare the abrasive wear of

two different laminated materials. Twelve pieces of material 1 were tested by exposing each piece to a machine measuring wear. Ten pieces of material 2 were similarly tested. In each case, the depth of wear was observed. The samples of material 1 gave an average wear of 85 units with a sample standard deviation of 4, while the samples of materials 2 gave an average wear of 81 and a sample standard deviation of 5. Can we conclude at the 0.05 level of significance that the mean abrasive wear of material 1 exceeds that of material 2 by more than 2 units? Assume populations to be approximately normal with equal variances.

Solution:

Material 1	material 2
$n_1=12$	$n_2=10$
$\bar{X}_1=85$	$\bar{X}_2=81$
$S_1=4$	$S_2=5$

Hypotheses:

$H_o: \mu_1 = \mu_2 + 2$ ($d=2$)

$H_1: \mu_1 > \mu_2 + 2$

Or equivalently,

$H_o: \mu_1 - \mu_2 = 2$ ($d=2$)

$H_1: \mu_1 - \mu_2 > 2$

Calculation:

$\alpha=0.05$

$$S_p^2 = \frac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1+n_2-2} = \frac{(12-1)(4)^2 + (10-1)(5)^2}{12+10-2} = 20.05$$
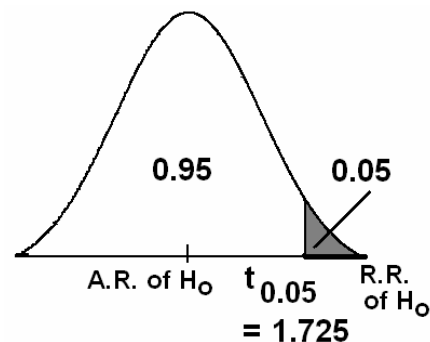
$S_p=4.478$
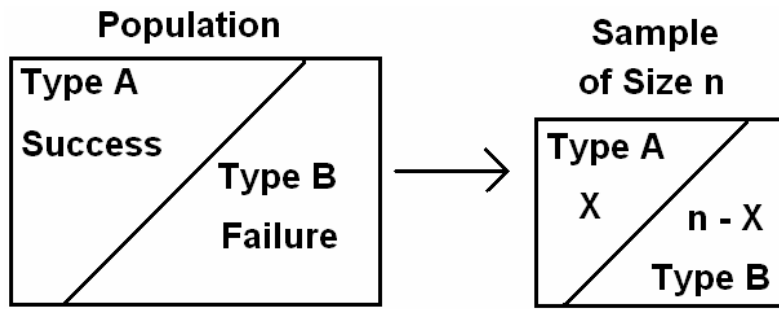
$v = n_1+n_2-2 = 12+10-2 = 20$

$t_{0.05} = 1.725$

T.S.:

$$T = \frac{(\bar{X}_1 - \bar{X}_2) - d}{S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = \frac{(85-81)-2}{(4.478)\sqrt{\frac{1}{12} + \frac{1}{10}}} = 1.04$$

Decision:

Since T=1.04 $\in$ A.R. (T=1.04< $t_{0.05}$ = 1.725), we accept (do not reject) H_o and reject $H_1: \mu_1 - \mu_2 > 2$ at $\alpha=0.05$.

10.11 One Sample: Tests on a Single Proportion:

Recall:

- $.p$ = Population proportion of elements of Type A in the population
- $= \dfrac{A}{A+B} = \dfrac{no.\ of\ elements\ of\ type\ A\ in\ the\ population}{Total\ no.\ of\ elements}$
- $.n$ = sample size
- X = no. of elements of type A in the sample of size n.
- $\hat{p}$ = Sample proportion elements of Type A in the sample
  $= \dfrac{X}{n}$
- For large n, we have
  $Z = \dfrac{\hat{p} - p}{\sqrt{\dfrac{pq}{n}}} = \dfrac{X - np}{\sqrt{npq}} \sim N(0,1)$　　(Approximately, $q=1-p$)

Hypotheses	H_o: $p = p_o$ H_1: $p \neq p_o$	H_o: $p = p_o$ H_1: $p > p_o$	H_o: $p = p_o$ H_1: $p < p_o$
Test Statistic (T.S.)	$Z = \dfrac{\hat{p} - p_o}{\sqrt{\dfrac{p_o q_o}{n}}} = \dfrac{X - np_o}{\sqrt{np_o q_o}} \sim N(0,1)$　　$(q_o = 1 - p_o)$		
R.R. and A.R. of H_o			
Decision:	Reject H_o (and accept H_1) at the significance level α if:		
	$Z > Z_{\alpha/2}$ or $Z < -Z_{\alpha/2}$ Two-Sided Test	$Z > Z_\alpha$ One-Sided Test	$Z < -Z_\alpha$ One-Sided Test

Example 10.10:

A builder claims that heat pumps are installed in 70% of all homes being constructed today in the city of Richmond. Would you agree with this claim if a random survey of new homes in the city shows that 8 out of 15 homes had heat pumps installed? Use a 0.10 level of significance.

Solution:

.p = Proportion of homes with heat pumps installed in the city.

.$n=15$

X= no. of homes with heat pumps installed in the sample = 8

$\hat{p}$ = proportion of homes with heat pumps installed in the

$$\text{sample} = \frac{X}{n} = \frac{8}{15} = 0.5333$$

Hypotheses:

$\quad$ H_o: $p = 0.7$ $\quad$ ($p_o=0.7$)

$\quad$ H_1: $p \neq 0.7$

Level of significance:

$\quad$ $\alpha=0.10$

T.S.:

$$Z = \frac{\hat{p} - p_o}{\sqrt{\dfrac{p_o q_o}{n}}} = \frac{0.5333 - 0.70}{\sqrt{\dfrac{(0.7)(0.3)}{15}}} = -1.41$$
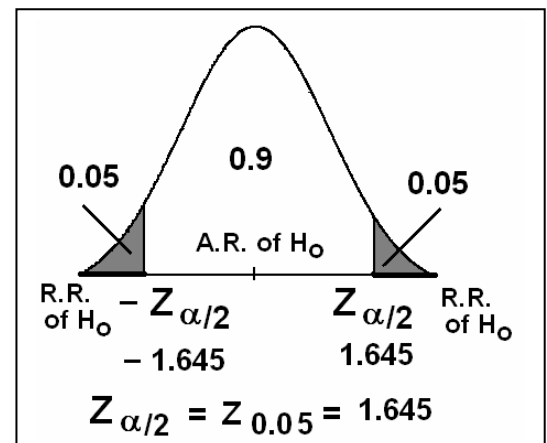
or

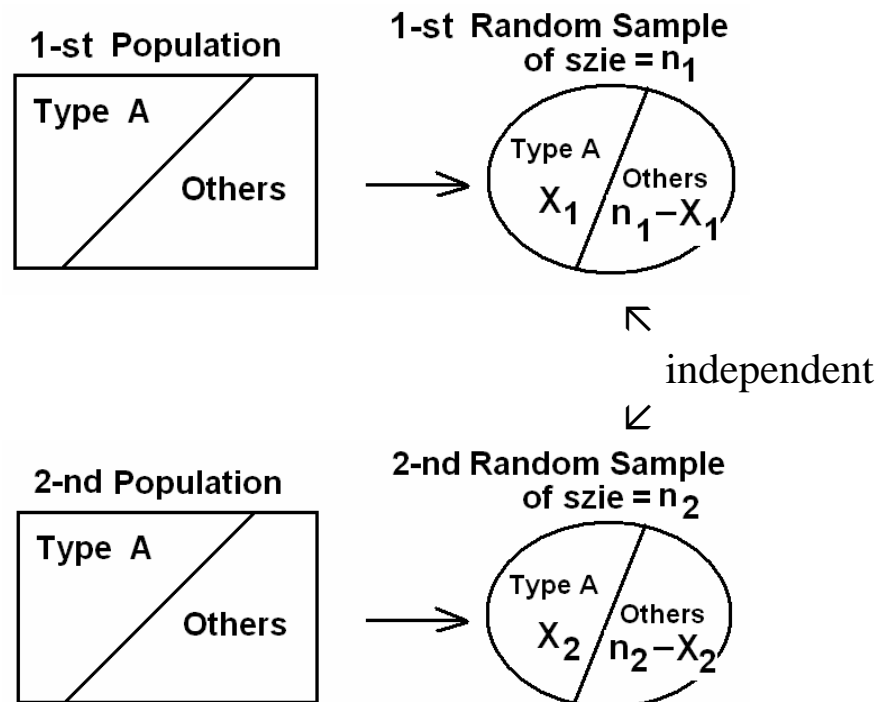$$Z = \frac{X - np_o}{\sqrt{np_o q_o}} = \frac{8 - (15)(0.7)}{\sqrt{(15)(0.7)(0.3)}} = -1.41$$

$Z_{\alpha/2} = Z_{0.05} = 1.645$

Decision:

$\quad$ Since Z= -1.41 $\in$A.R., we accept (do not reject) H_o: $p=0.7$ and reject H_1: $p \neq 0.7$ at $\alpha=0.1$. Therefore, we agree with the claim.

Example 10.11: Reading Assignment

10.12 Two Samples: Tests on Two Proportions:

Suppose that we have two populations:

- $.p_1$ = proportion of the 1-st population.
- $.p_2$ = proportion of the 2-nd population.
- We are interested in comparing p_1 and p_2, or equivalently, making inferences about $p_1 - p_2$.
- We independently select a random sample of size n_1 from the 1-st population and another random sample of size n_2 from the 2-nd population:
- Let X_1 = no. of elements of type A in the 1-st sample.
- Let X_2 = no. of elements of type A in the 2-nd sample.
- $\hat{p}_1 = \dfrac{X_1}{n_1}$ = proportion of the 1-st sample
- $\hat{p}_2 = \dfrac{X_2}{n_2}$ = proportion of the 2-nd sample
- The sampling distribution of $\hat{p}_1 - \hat{p}_2$ is used to make inferences about $p_1 - p_2$.
- For large n_1 and n_2, we have

$$Z = \frac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\dfrac{p_1\,q_1}{n_1} + \dfrac{p_2\,q_2}{n_2}}} \sim N(0,1) \quad \text{(Approximately)}$$

Suppose we need to test:

H_o: $p_1 = p_2$

H_1: $\begin{cases} p_1 \neq p_2 \\ p_1 > p_2 \\ p_1 < p_2 \end{cases}$

Or, equivalently,

H_o: $p_1 - p_2 = 0$

H_1: $\begin{cases} p_1 - p_2 \neq 0 \\ p_1 - p_2 > 0 \\ p_1 - p_2 < 0 \end{cases}$

Note, under H_o: $p_1 = p_2 = p$, the pooled estimate of the proportion p is:

$$\hat{p} = \frac{X_1 + X_2}{n_1 + n_2} \qquad (\hat{q} = 1 - \hat{p})$$

The test statistic (T.S.) is

$$Z = \frac{(\hat{p}_1 - \hat{p}_2)}{\sqrt{\hat{p}\hat{q}\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}} \sim N(0,1)$$

Hypotheses	H_o: $p_1 - p_2 = 0$ H_1: $p_1 - p_2 \neq 0$	H_o: $p_1 - p_2 = 0$ H_1: $p_1 - p_2 > 0$	H_o: $p_1 - p_2 = 0$ H_1: $p_1 - p_2 < 0$
Test Statistic (T.S.)	$Z = \dfrac{(\hat{p}_1 - \hat{p}_2)}{\sqrt{\hat{p}\hat{q}\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}} \sim N(0,1)$		
R.R. and A.R. of H_o			
Decision:	Reject H_o (and accept H_1) at the significance level α if:		

	$Z > Z_{\alpha/2}$ or $Z < -Z_{\alpha/2}$ Two-Sided Test	$Z > Z_{\alpha}$ One-Sided Test	$Z < -Z_{\alpha}$ One-Sided Test

Example 10.12:

A vote is to be taken among the residents of a town and the surrounding county to determine whether a proposed chemical plant should be constructed. The construction site is within the town limits and for this reason many voters in the county feel that the proposal will pass because of the large proportion of town voters who favor the construction. To determine if there is a significant difference in the proportion of town voters and county voters favoring the proposal, a poll is taken. If 120 of 200 town voters favor the proposal and 240 of 500 county voters favor it, would you agree that the proportion of town voters favoring the proposal is higher than the proportion of county voters? Use a 0.025 level of significance.

Solution:

$.p_1$ = proportion of town voters favoring the proposal

$.p_2$ = proportion of county voters favoring the proposal

$\hat{p}_1$ = sample proportion of town voters favoring the proposal

$\hat{p}_2$ = sample proportion of county voters favoring the proposal

Town

$n_1 = 200$

$X_1 = 120$

$\hat{p}_1 = \dfrac{X_1}{n_1} = \dfrac{120}{200} = 0.60$

$\hat{q}_1 = 1 - 0.60 = 0.40$

County

$n_2 = 500$

$X_2 = 240$

$\hat{p}_2 = \dfrac{X_2}{n_2} = \dfrac{240}{500} = 0.48$

$\hat{q}_2 = 1 - 0.48 = 0.52$

The pooled estimate of the proportion p is:

$\hat{p} = \dfrac{X_1 + X_2}{n_1 + n_2} = \dfrac{120 + 240}{200 + 500} = 0.51$

$\hat{q} = 1 - 0.51 = 0.49$

Hypotheses:

   H_o: $p_1 = p_2$

H_1: $p_1 > p_2$
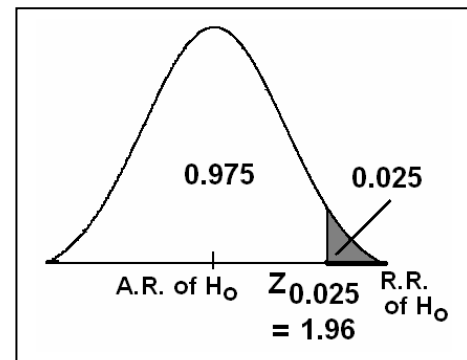
or

H_o: $p_1 - p_2 = 0$
H_1: $p_1 - p_2 > 0$

Level of significance:
$\alpha = 0.025$
$Z_\alpha = Z_{0.025} = 1.96$
T.S.:

0.975   0.025

A.R. of H_o $Z_{0.025}$ R.R. of H_o
= 1.96

$$Z = \frac{(\hat{p}_1 - \hat{p}_2)}{\sqrt{\hat{p}\hat{q}\left(\dfrac{1}{n_1} + \dfrac{1}{n_2}\right)}} = \frac{(0.60 - 0.48)}{\sqrt{(0.51)(0.49)\left(\dfrac{1}{200} + \dfrac{1}{500}\right)}} = 2.869$$

Decision:
        Since Z= 2.869 $\in$R.R. (Z= 2.869> Z_α= $Z_{0.025}$= 1.96), we reject H_o: $p_1 = p_2$ and accept H_1: $p_1 > p_2$ at α=0.025. Therefore, we agree that the proportion of town voters favoring the proposal is higher than the proportion of county voters.