PHYS 502
Lecture 5: Laplace transforms

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Introduction-a

- The Laplace transform $f(s)$ of a function $F(t)$ is defined as

$$f(s) = L\{F(t)\} = \lim_{a \to \infty} \int_0^a e^{-st} F(t)\,dt$$

- Laplace transform may not exist if there are strong singularities in the function $F(t)$ as t goes to infinity. A necessary condition for the existence of this transform is

$$e^{-st} F(t)\Big|_{t \to \infty} = 0$$

Introduction-b

Linearity

Laplace transform is a *linear* transform.

$$L\left\{aF(t)+bG(t)\right\}=aL\left\{F(t)\right\}+bL\left\{G(t)\right\}$$

It is very easy to show the following Laplace transforms which are presented in the table below:

$F(t)$	$f(s)$
$1, \quad t > 0$	$1/s, \quad s > 0$
$e^{kt}, \; t > 0$	$1/(s-k), \; s > k$
$\cosh(kt)$	$s/\left(s^2 - k^2\right)$
$\sinh(kt)$	$k/\left(s^2 - k^2\right)$
t^n	$n!/s^{n+1}$

The inverse Laplace transform

To each Laplace transform there exist an inverse transform as follows:

$$L\left\{F\left(t\right)\right\} = f(s), \qquad L^{-1}\left\{f\left(s\right)\right\} = F(t)$$

Taken literally, this inverse transform is *not* unique. Two functions $F_1\left(t\right), F_2\left(t\right)$ may have the same transform, $f(s)$. However, in this case, for all positive t_0 :

$$\int_0^{t_0} N(t)dt = 0$$

This is known as Lerch's theorem. Therefore to the physicist and engineer $N(t)$ may almost always be taken as zero and the inverse operation becomes unique.

The partial fraction expansion

Frequently $f(s)$ occurs in the form $g(s)/h(s)$, where $g(s)$, $h(s)$ are polynomials with no common factors, $g(s)$ being of lower degree than $h(s)$. If the factors $h(s)$ are all linear and distinct, then by the theory of partial fractions we may write

$$f(s) = \frac{c_1}{s-a_1} + \frac{c_2}{s-a_2} + \ldots + \frac{c_n}{s-a_n}$$

Where c_i are independent of s. The a_i are the roots of $h(s)$. If one of them, say a_1 is multiple, (occuring m times) then the above expression takes the form

$$f(s) = \frac{c_{1,m}}{(s-a_1)^m} + \frac{c_{1,m-1}}{(s-a_1)^{m-1}} + \ldots + \frac{c_{1,1}}{s-a_1} + \sum_{i=2}^{n} \frac{c_i}{s-a_i}$$

The Laplace Transform

The transform of derivatives

Perhaps the main application of Laplace transforms is in converting differential equations into simpler forms that may be solved more easily. For example, coupled differential equations with constant coefficients transform to simultaneous linear algebraic equations. It is obvious that in such cases we need the transform of derivatives. It can be shown that:

$$L\left\{F^{'}(t)\right\} = sL\left\{F(t)\right\} - F(0+)$$

$$L\left\{F^{(n)}(t)\right\} = s^{n}L\left\{F(t)\right\} - s^{n-1}F(+0) - s^{n-2}F^{'}(+0) - \ldots F^{(n-1)}(+0)$$

The symbol (+0) means that zero is approached from the *positive* side.

The Laplace Transform
Other properties

- ***Substitution:*** $L\left\{e^{at}F(t)\right\}=f(s-a)$
- ***Translation:*** $L\left\{F(t-b)\right\}=e^{-bs}f(s)$
- ***Derivative of transform:*** $f^{(n)}(s)=L\left\{(-t)^{n}F(t)\right\}$
- ***Integral of transform:*** $\int_{s}^{\infty}f(x)dx=L\left\{F(t)/t\right\}$
- ***Convolution:*** $f_1(s)\cdot f_2(s)=L\left\{\int_{0}^{t}F_1(t-z)F_2(z)dz\right\}$

The Inverse Laplace Transform

The inverse Laplace transform is defined by the so called Bromwich integral (sometimes known as Fourier-Mellin theorem or integral).

$$F(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} f(s)\,ds$$

This integral may now be evaluated by the regular methods of contour integration.