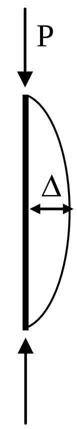
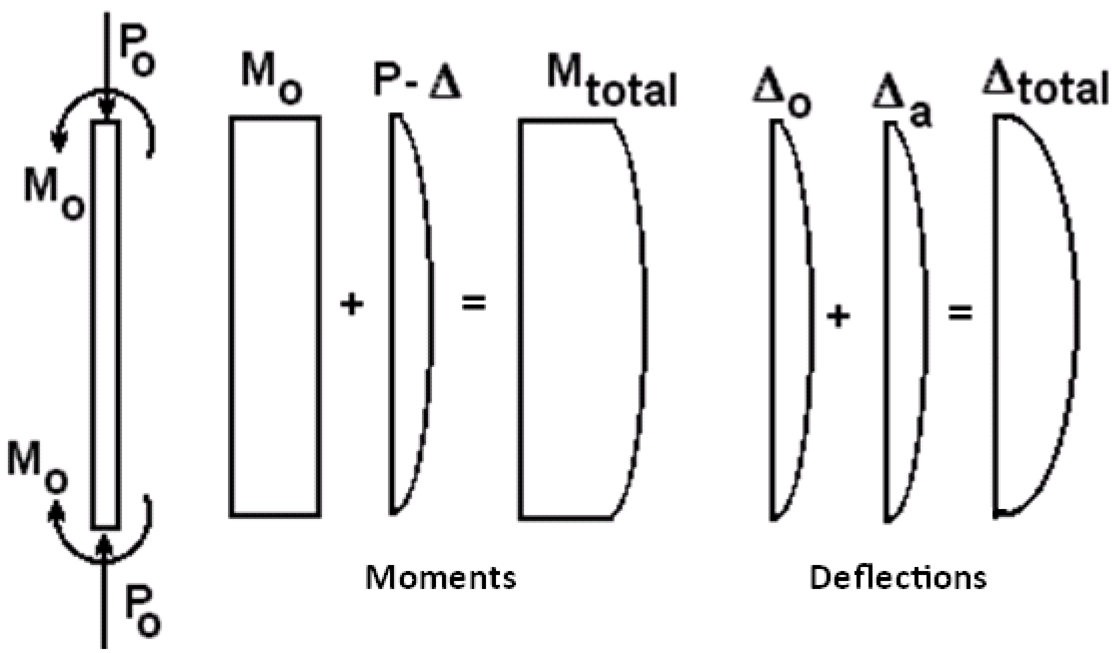
# Slender columns

Slender columns are subjected to buckling risk and their stability must be checked.
Instability is caused by the interaction between bending and the axial force.
If the deflection $\Delta$ caused by the bending moment is important it will interact with the axial force P and cause an extra moment equal to $P.\Delta$



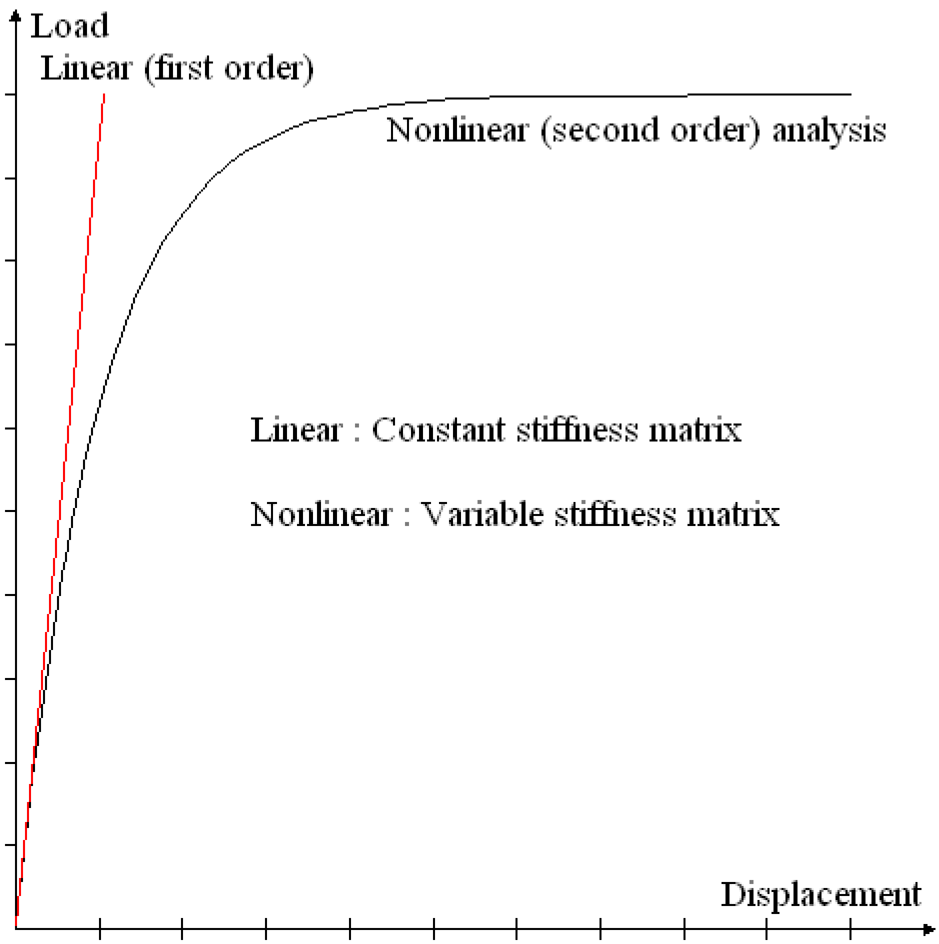

The slenderness problem is also called P-delta effect.
In linear elastic (first order) analysis, the deflections are considered so small that this P-delta effect is neglected but for slender members it is no longer true and second order structural analysis with stability check must be performed.
Because of the coupling between the deflection, axial force and bending moment, the second order structural analysis is nonlinear and complex.
ACI and SBC codes allow using simple methods to account for P-delta effects and check column slenderness at design stage (moment magnification method).
The slender column is designed using a magnified moment.
Moment magnification method can only be used if the slenderness ratio is less than 100.0




Typical slenderness effects

$\Delta_0$ : First order deflection caused by moment $M_0$

$\Delta_a$ : Second order (additional) deflection caused by axial compression force $P$



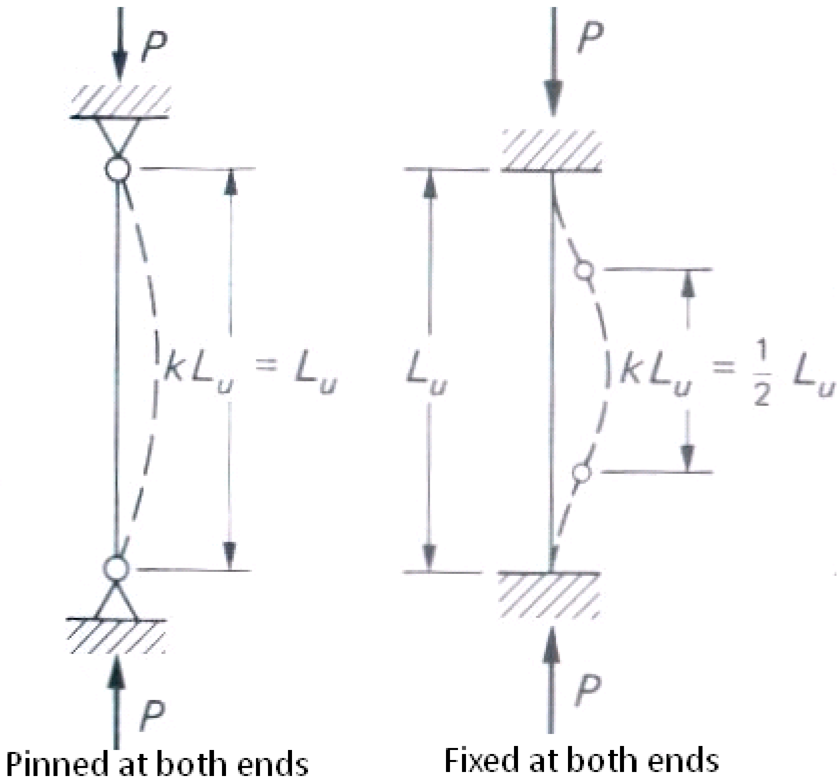

Linear and nonlinear structural analysis

Recall Euler critical load for a pinned column: $P_c = \dfrac{EI\pi^2}{L_u^2}$

More generally the critical load depends on the end conditions: $P_c = \dfrac{EI\pi^2}{(kL_u)^2}$

$L_u$ is the unsupported (clear) length of the column.

$kL_u$ is the effective length and $k$ is the effective length factor depending on the fixity conditions of the column ends. $k$ represents the ratio of the distance between points of inflexion (zero moment). $k$ is therefore equal to unity for a doubly pinned column (original Euler problem) and 0.5 for a doubly fixed column.




Effective length in columns

In real structures, fixity depends on ratio of beam to column stiffness and sway or non-sway conditions as well as the possible bracing. Braced or un-braced columns depend on the stability index defined as:

$$Q = \frac{\sum P_u \Delta_0}{V_u l_c}$$

$Q \leq 0.05$ : Story is braced (non-sway)

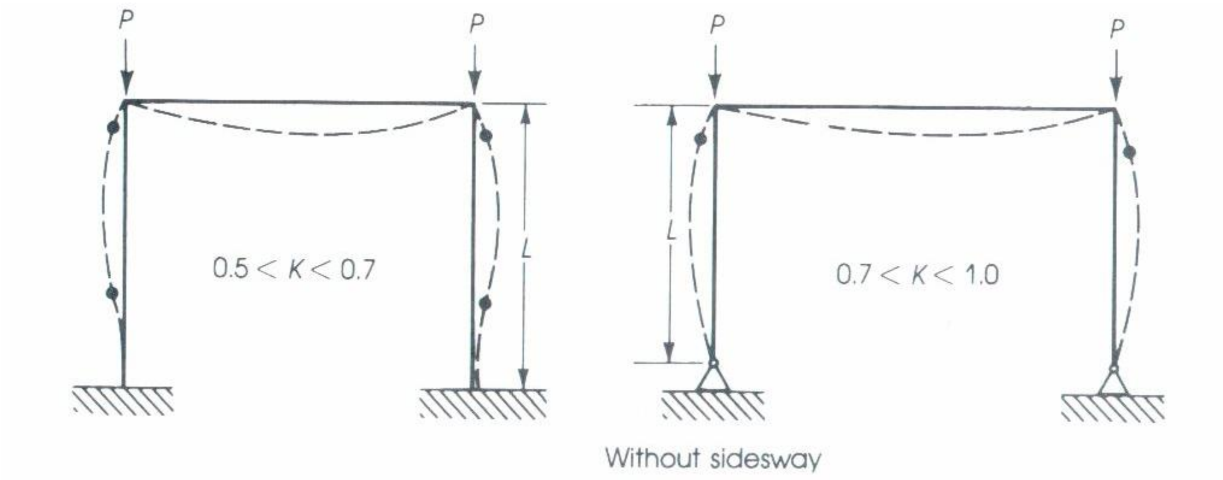
$Q > 0.05$ : Story is un-braced (sway)

$\sum P_u$ : Total vertical load in all columns and walls of story

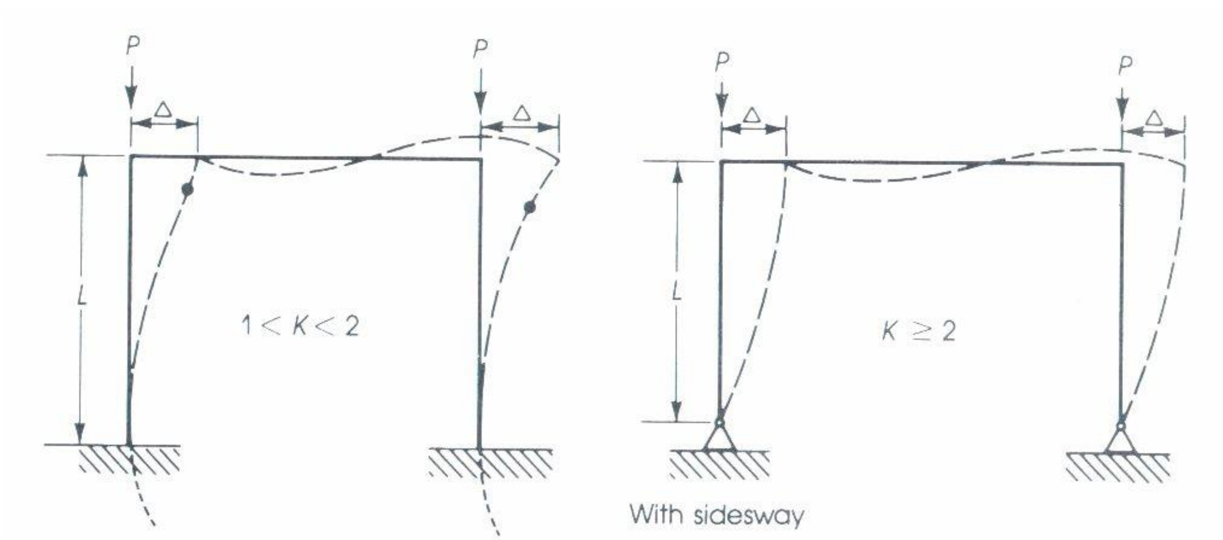
$V_u$ : Shear in story due to lateral loads

$\Delta_0$ : First order relative deflection between top and bottom of story due to $V_u$

$l_c$ : Height of story (center to center)




Braced (non-sway) frames $0.5 \leq k \leq 1.0$




Unbraced (sway) frames $k > 1.0$

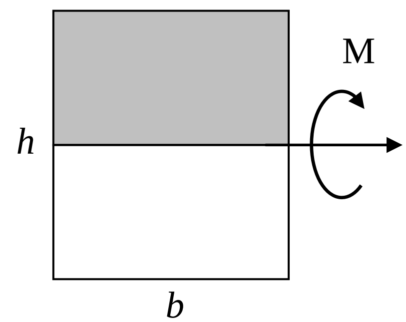
# ACI / SBC method of moment magnification for slender columns

## 1. Basis of the method

The moment magnification method is used at the design stage for slender columns for which the slenderness ratio is greater than a certain minimum limit and less or equal to a maximum limit equal to 100.0. Above this value, the method cannot be used and second order effects should be included in the structural analysis step (second order structural analysis).

Slenderness must be checked in each of the two bending directions *X* and *Y*.

The magnification method is presented for a bending moment as shown in the figure. The second direction is treated in a similar fashion.




The slenderness ratio is defined as the ratio of the effective length to the radius of gyration.

$$\text{Slenderness ratio} = \frac{kL_u}{r} \quad \textbf{(1)}$$

$L_u$ is the unsupported height of the column while $r$ is its radius of gyration defined as:

$$r = \sqrt{\frac{I_g}{A_g}} \quad \textbf{(2a)}$$

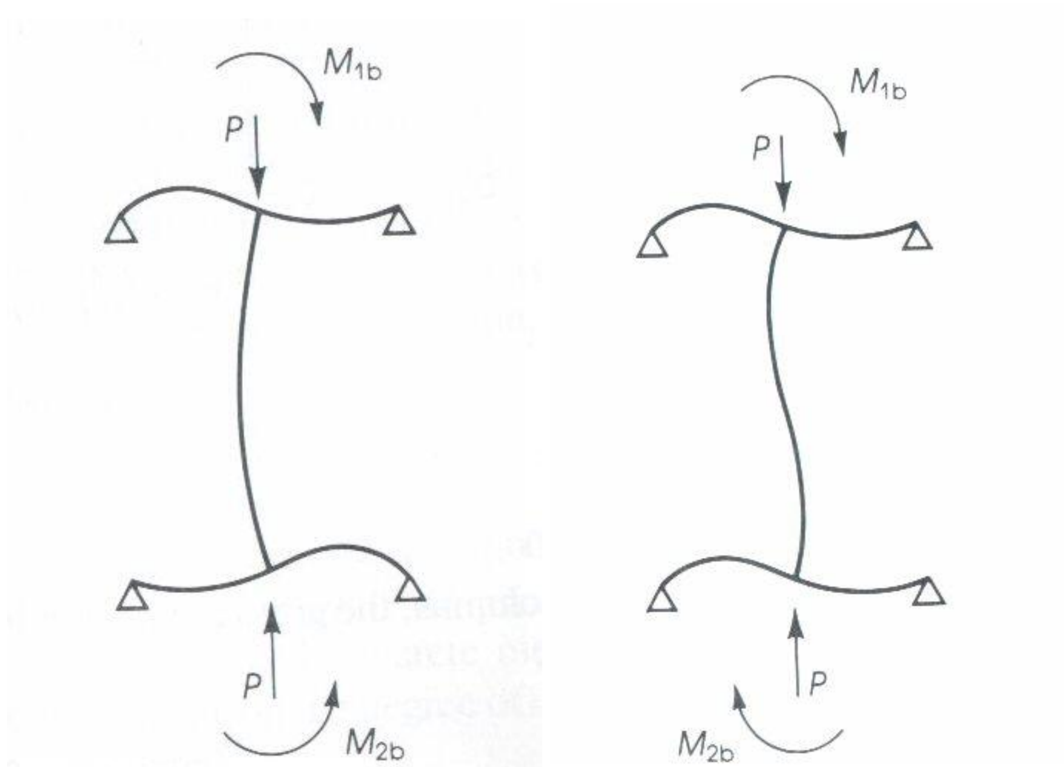
For a rectangular section $b \times h$, the radius of gyration becomes: $r \approx 0.3h$ **(2b)**

$k$ is the effective length factor, to be defined later, which depends on the type of column (braced or un-braced) and its end conditions.

ACI and SBC codes specify that column moments are magnified by a factor to account for slenderness effects. The magnification factor depends on the ultimate axial force as well as the critical buckling load and is computed differently for braced and sway columns.

**Sign convention for column moments**

The top and bottom moments $M_1$ and $M_2$ may cause either single curvature or double curvature as shown in the figure.




**Single curvature** **Double curvature**

The ratio $\frac{M_1}{M_2}$ is considered positive in single curvature and negative in double curvature.

**2. Moment magnification for slender braced columns**

For braced columns the effective length factor *k* is less or equal to unity: $0.5 \le k \le 1.0$

It is therefore conservative to use a value of unity.

The slenderness effects will be considered if the slenderness ratio is greater than or equal to the limit defined as follows:

If $$\frac{kL_u}{r} \ge 34.0 - 12\frac{M_1}{M_2} : \quad \text{The column is slender} \qquad \textbf{(3)}$$

$M_1$ and $M_2$ are the smaller and larger factored moments at the column ends. They have the same sign if the column is bent in single curvature.

The ratio $\frac{M_1}{M_2}$ must always be greater or equal to (-0.5), otherwise if the slenderness ratio is greater or equal to 40, the column is considered as slender whatever the value of ($34.0 - 12\frac{M_1}{M_2}$)

The moment magnification method cannot however be used if the slenderness ratio exceeds the upper value of 100. In this case a second order structural analysis is required.

$$\text{If } \frac{kL_u}{r} > 100: \qquad \text{Use second order analysis}$$

$L_u$ is the unsupported height of the column while $r$ is its radius of gyration given by equations (2a) and (2b).

The minimum moment to be considered is:

$$M_{\min} = P_u(15 + 0.03h) \quad \text{with } h \text{ in } mm. \tag{4}$$

The critical (magnified) moment is then given by:

$$M_c = \delta \, x \, Max(M_2, M_{\min}) \tag{5}$$

The moment magnification factor is given by:

$$\delta = Max\left(\frac{c_m}{1.0 - \dfrac{P_u}{0.75P_c}}, 1.0\right) \tag{6}$$

where the moment coefficient $c_m$ is given by:

$$c_m = Max\left(0.6 + 0.4\frac{M_1}{M_2}, 0.4\right) \tag{7}$$

The critical buckling load is defined as: $$P_c = \frac{EI\pi^2}{(kL_u)^2} \tag{8}$$

The long term column stiffness $EI$ is computed as: $$EI = \frac{0.40E_cI_g}{1+\beta_d} \tag{9}$$

$E_c$ is the concrete modulus of elasticity given by: $$E_c = 4700\sqrt{f_c^{'}} \tag{10}$$

$\beta_d$ is the ratio of sustained axial force to ultimate axial force. $$\beta_d = \frac{1.4P_D}{P_u} \tag{11}$$

If data is unavailable, a default value of 0.6 for $\beta_d$ may be used.

$I_g$ is the moment of inertia of the column gross section

### 3. Computation of the effective length factor *k* for slender braced columns:

The effective length factor *k* depends on the fixity conditions of the column. At each end, the fixity ratio is related to the ratio $\psi$ of the sum of stiffnesses of columns and beams connected to it as:

$$\psi = \frac{\left(\sum \frac{EI}{L}\right)_{columns}}{\left(\sum \frac{EI}{L}\right)_{beams}} \tag{12}$$

*I* is the moment of inertia of the cracked section, defined with respect to the gross moment of inertia $I_g$:

$$\text{Beams: } I_b = 0.35\, I_{gb} \qquad \text{Columns: } I_c = 0.70\, I_{gc} \tag{13}$$

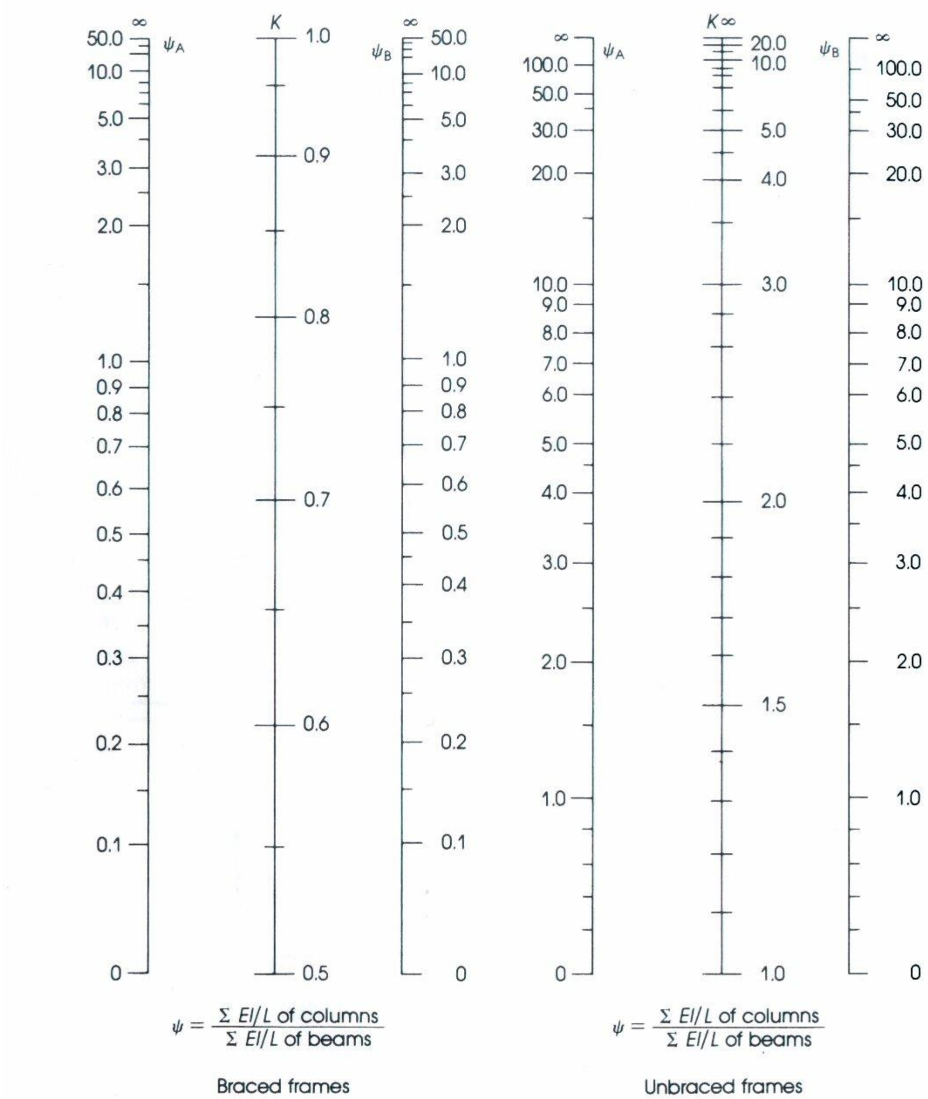
If the same material is used, then *E* is constant and equation (12) becomes:

$$\psi = 2\frac{\left(\sum \frac{I_g}{L}\right)_{columns}}{\left(\sum \frac{I_g}{L}\right)_{beams}} \tag{14}$$

For braced columns, the effective length factor *k* is computed as:

$$k = Min \begin{cases} 0.7 + 0.05(\psi_A + \psi_B) \\ 0.85 + 0.05\psi_{\min} \\ 1.0 \end{cases} \qquad \text{with } k \geq 0.5 \tag{15}$$

$\psi_A$ and $\psi_B$ are values of $\psi$ at top and bottom ends of the column while $\psi_{\min}$ is the smaller of the two. At each end, $\psi$ is computed by relations (12) or (14). The effective length factor *k* may also be read from Tables or nomographs (in ACI / SBC codes) when values $\psi_A$ and $\psi_B$ are determined, instead of using equation (15). Considering a unit value for *k* is always conservative for braced columns.




Nomographs for reading values of *k*

**4. Moment magnification for slender un-braced (sway) columns**

For un-braced columns the effective length factor $k$ is greater than or equal to unity: $k \geq 1.0$

The slenderness effects must be considered if the slenderness ratio is greater or equal to 22.0:

$$\text{If} \quad \frac{kL_u}{r} \geq 22.0 : \quad \text{The column is slender} \tag{16}$$

The moment magnification factor cannot however be used if the slenderness ratio exceeds the upper value of 100.0.

The sway moment magnification factor is computed as:

$$\delta_s = Max\left( \frac{1}{1.0 - \mu \dfrac{P_u}{0.75P_c}}, 1.0 \right) \tag{17}$$

where $\mu$ is the following floor to column ratio given by:

$$\mu = \frac{\sum P_u \Big/ \sum P_c}{P_u / P_c} \tag{18}$$

$M_{1b}$ and $M_{2b}$ are the smaller and larger factored braced moments at the column ends. They have the same sign if the column is bent in single curvature.

$M_{1s}$ and $M_{2s}$ are the smaller and larger factored sway moments at the column ends. They have the same sign if the column is bent in single curvature.

If $\dfrac{l_u}{r} \leq \dfrac{35.0}{\sqrt{\dfrac{P_u}{f_c^{'} A_g}}}$ then the magnified moments are:

$$M_{1c} = M_{1b} + \delta_s M_{1s} \qquad M_{2c} = M_{2b} + \delta_s M_{2s} \tag{19}$$

If $\dfrac{l_u}{r} > \dfrac{35.0}{\sqrt{\dfrac{P_u}{f_c^{'} A_g}}}$ : then the braced moment magnification factor $\delta$ is computed as for braced columns but with $\beta_d = 0$ and the magnified moments are:

$$M_{1c} = \delta(M_{1b} + \delta_s M_{1s}) \qquad M_{2c} = \delta(M_{2b} + \delta_s M_{2s}) \tag{20}$$

**5. Computation of the effective length factor *k* for sway columns:**

For un-braced columns, the effective length factor *k* is computed as follows:

If both ends are restrained and $\psi_m < 2$ : $$k = Max\left(\frac{0.05(20-\psi_m)}{\sqrt{1+\psi_m}}, 1.0\right) \qquad \textbf{(21)}$$

If both ends are restrained and $\psi_m \geq 2$ : $$k = Max\left(0.9\sqrt{1+\psi_m}, 1.0\right) \qquad \textbf{(22)}$$

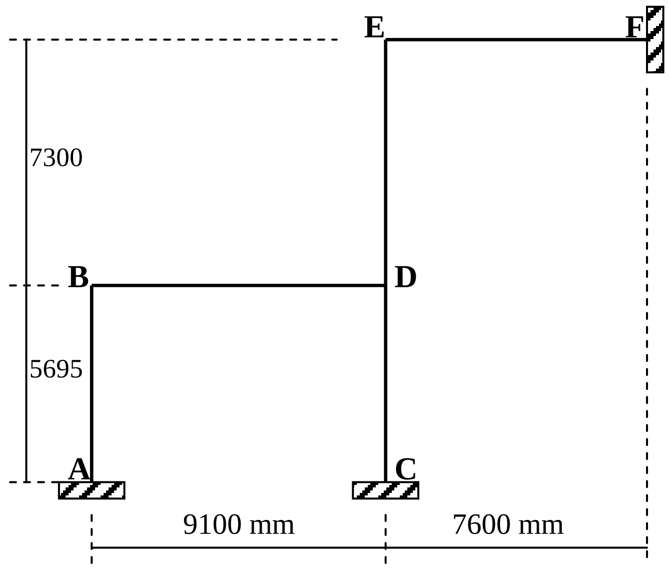
If the upper end is hinged: $$k = 2 + 0.3\psi_B \qquad \textbf{(23)}$$

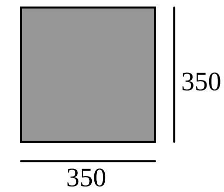
If the lower end is hinged: $$k = 2 + 0.3\psi_A \qquad \textbf{(24)}$$

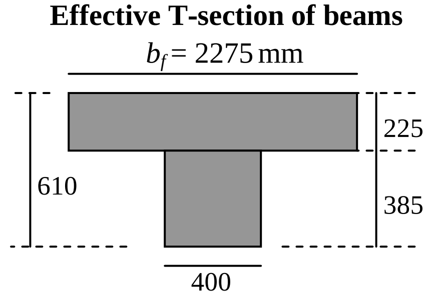
If both ends are hinged: $$k = 30 \qquad \textbf{(25)}$$

$\psi_A$ and $\psi_B$ are values of $\psi$ at top and bottom ends of the column while $\psi_m$ is the average of the two. At each end, $\psi$ is computed using equations (12) or (14) and *k* may also be read in Tables or nomographs.

**Slender braced column example:** Material data: $f_c^{'} = 20\,MPa$ $f_y = 420\,MPa$



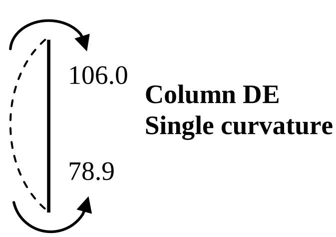



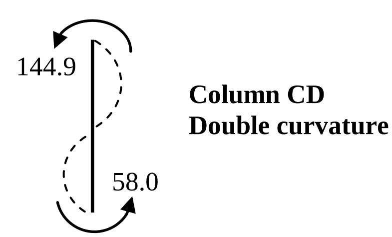




The structural analysis results for columns CD and DE are:

| | Column CD | | | Column DE | | |
|---|---|---|---|---|---|---|
| | Axial force (kN) | Top moment (kN.m) | Bottom Moment (kN.m) | Axial force (kN) | Top moment (kN.m) | Bottom Moment (kN.m) |
| **Dead** | 310.0 | -78.6 | -31.6 | 220.0 | 57.5 | -43.0 |
| **Live** | 85.5 | -20.5 | -8.1 | 60.0 | 15.0 | -11.0 |
| **Ultimate** | 579.4 | -144.9 | -58.0 | 410.0 | 106.0 | -78.9 |

The moment sign convention used in structural analysis is for a positive moment in clockwise direction. Columns CD and DE would therefore be subjected to moments as shown:



**Analysis of column DE:**

Column DE being deflected in single curvature, the ratio ($M_1 / M_2$) is therefore positive.

The values must be: $M_1$ = 78.9 kN.m and $M_2$ = 106.0 kN.m

The column unsupported length is: $L_u = 7300 - \frac{610}{2} - \frac{610}{2} = 6690\,mm$

The radius of gyration given by equation (2b) is: $r = 0.3\,x\,350 = 105\,mm$

The right hand term of (3) is: $34.0 - 12\frac{M_1}{M_2} = 34 - 12\frac{78.9}{106} = 25.07$

The slenderness ratio $\frac{kL_u}{r}$ is equal to $\frac{6690}{105} =$ 63.7 if $k$ =1.0, and equal to 31.85 if $k$ = 0.5

So $\frac{kL_u}{r} \geq 34.0 - 12\frac{M_1}{M_2}$ whether $k$ =1.0 or $k$ =0.5

This means that column CD is slender (since $k$ must be between 0.5 and 1.0)

On the other hand $\frac{kL_u}{r}$ is less than 100. This means that the moment magnification method can be used.

**Determination of *k*:** (this part is skipped if $k$ is known)

At top end E of the column, there is one beam (EF) and one column (DE). At bottom end D, there is one beam (BD) and two columns (DE and CD). The fixity ratios given by (14) at both ends are:

$$\psi_{Top} = 2\frac{\left(\frac{I_g}{L}\right)_{DE}}{\left(\frac{I_g}{L}\right)_{EF}} \qquad \psi_{Bot} = 2\frac{\left(\frac{I_g}{L}\right)_{DE} + \left(\frac{I_g}{L}\right)_{CD}}{\left(\frac{I_g}{L}\right)_{BD}}$$

The moment of inertia (about centroid axis) of the column gross section is $I_g = \frac{350^4}{12} = 1.25\,x10^9\,mm^4$

The moment of inertia of the shown effective T-section of the beam is $I_g = 15.07\,x10^9\,mm^4$

The fixity ratios are:

$$\psi_{Top} = 2\frac{\frac{1.25\,x10^9}{7300}}{\frac{15.07\,x10^9}{7600}} = 0.1727 \qquad \psi_{Bot} = 2\frac{\frac{1.25\,x10^9}{7300} + \frac{1.25\,x10^9}{5695}}{\frac{15.07\,x10^9}{9100}} = 0.4719$$

The effective length factor given by equation (15) is then: ***k* = 0.732**

**Sustained axial force ratio:** Given by (11): $\beta_d = \frac{1.4 P_D}{P_u} = \frac{1.4\,x\,220}{410} = 0.751$

**Concrete Young's modulus:** Given by (10): $E_c = 4700\sqrt{f_c^{'}} = 4700\sqrt{20} = 21019.0\,MPa$

**Long term flexural rigidity:**

Given by (9): $EI = \frac{0.40 E_c I_g}{1+\beta_d} = \frac{0.40\,x\,21019\;\;x 1.25\,x 10^9}{1+0.751} = 6.0\,x 10^{12}\;N.mm^2$

**Critical buckling load:**

Given by (8): $P_c = \frac{EI\pi^2}{(kL_u)^2} = \frac{6\,x 10^{12}\;x\,\pi^2}{(0.732\,x\,6690)^2} = 2469315.6\,N = 2469.3\,kN$

**Moment coefficient:**

Given by (7): $c_m = Max\left(0.6 + 0.4\frac{M_1}{M_2},\;0.4\right) = Max\left(0.6+0.4\frac{78.9}{106},\;0.4\right) = 0.8977$

**Moment magnification factor (6):**

$$\delta = Max\left(\frac{c_m}{1.0 - \frac{P_u}{0.75 P_c}},\;1.0\right) = Max\left(\frac{0.8977}{1.0-\frac{410}{0.75\,x\,2469.3}},\;1.0\right) = 1.153$$

**Minimum moment (4):**

$M_{\min} = P_u(15+0.03h) = 410(15+0.03\,x\,350) = 10455\,kN.mm = 10.455\,kN.m$

**Critical moment:** Given by (5): $M_c = \delta\,x\,Max(M_2, M_{\min}) = 1.153\,x\,106 = 122.2\,kN.m$

Column DE must therefore be designed for this critical moment (instead of $M_2$) and the same axial force.

**Analysis of column CD:** In double curvature the ratio ($M_1 / M_2$) is negative.

The right hand term of (3) is: $34.0 - 12\frac{M_1}{M_2} = 34 + 12\frac{58}{144.9} = 38.8$

The column unsupported length is: $L_u = 5695 - \frac{610}{2} = 5390\,mm$

It is less than that of column DE.

The fixity ratio at the top end is the same ratio at bottom end for column DE: $\psi_{Top} = 0.4719$

The bottom end is attached to the foundation which is much stiffer than beams. The bottom ratio would be smaller than other values obtained. $k$ for column CD would then be less than for column DE.

Assuming a value for $k$ of 0.732 (same as for column DE), slenderness ratio $kL_u/r$ will be equal to 37.6

Column CD is therefore short. No moment magnification is required.

In 3D analysis, with columns subjected to biaxial bending, slenderness effects are checked in two planes X-Z and Y-Z.

RC-TOOL and RC-BIAX software include moment magnification method for slender columns (braced and sway).