# Method of Solving Linear System

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Last week we studied the Gaussian Elimination Method, which is to do some row operations on the augmented matrix in order to get the following form:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} A_{11} & A_{12} & A_{13} & B_1 \\ 0 & A_{22} & A_{23} & B_2 \\ 0 & 0 & A_{33} & B_3 \end{bmatrix}$$

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The Gauss-Jordan Elimination Method is by using row operations to find the following form:

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**Q:Solve the following linear system by Gauss-Jordan method:**

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 8 \\ -x_1 - 2x_2 + 3x_3 &= 1 \\ 3x_1 - 7x_2 + 4x_3 &= 10. \end{aligned}$$

**Solution:**

$$\begin{bmatrix} 1 & 1 & 2 & 8 \\ -1 & -2 & 3 & 1 \\ 3 & -7 & 4 & 10 \end{bmatrix}$$

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Equivalent system of equations is:

$$\begin{aligned} x_1 &= 3 \\ x_2 &= 1 \\ x_3 &= 2. \end{aligned}$$

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.

**Example:** Use Gauss-Jordan method to solve the system of linear equation:

$$\begin{aligned} x - y + 2z - w &= -1 \\ 2x + y - 2z - 2w &= -2 \\ -x + 2y - 4z + w &= 1 \\ 3x \qquad\qquad - 3w &= -3. \end{aligned}$$

**Solution:** Gauss-Jordan method is same as to reduced row echelon form:

$$\begin{bmatrix} 1 & -1 & 2 & -1 & -1 \\ 2 & 1 & -2 & -2 & -2 \\ -1 & 2 & -4 & 1 & 1 \\ 3 & 0 & 0 & -3 & -3 \end{bmatrix}$$

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As the number of equations are less than the number of the variables (unknowns), then there are infinite solutions. Choosing $z = t$, where $t \in \mathbb{R}$, then the solution is

$$\begin{aligned} z &= t \\ y &= \tfrac{10}{7} + 2t \\ x &= 4 + 3t - 4t - \tfrac{20}{7} = -t + \tfrac{8}{7}. \end{aligned}$$

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**Case II:** When $a = -4$, then $0z = -8$, therefore, there is no solution.

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**Case III:** When $a \neq \pm 4$, we have only one solution. For example when $a = 0$, the solution is:

$$\begin{aligned} z &= \frac{1}{4} \\ y &= \tfrac{10}{7} - 2(\frac{1}{4}) = \frac{13}{14} \\ x &= 4 - 2(\frac{13}{14}) + 3(\frac{1}{4}) = \frac{39}{28}. \quad \text{(check!!)} \end{aligned}$$

**Example:** What conditions must $a$, $b$ and $c$ satisfy in order for the linear system:

$$\begin{aligned} x + y + 2z &= a \\ x \quad\;\; + z &= b \\ 2x + y + 3z &= c \end{aligned}$$

to be consistent?.

**Q:** For what value of $\lambda$ does the system of equations have

(i) have infinitely many solutions?

(ii) have no solution?

(iii) have just one solution?.

$$\begin{aligned} 3x + \lambda z &= 2 \\ 3x + 3y + 4z &= 4 \\ y + 2z &= 3. \end{aligned}$$