[Solution Key] MATH-244 (Linear Algebra); Mid-term Exam; Semester 432

Question 1:

a) **Find** the *values* of λ for which the *matrix* $\begin{bmatrix} 1 & 0 & \lambda \\ 2 & 1 & 2+\lambda \\ 2 & 3 & \lambda^2 \end{bmatrix}$ is *invertible*.

Solution: $\begin{bmatrix} 1 & 0 & \lambda \\ 2 & 1 & 2+\lambda \\ 2 & 3 & \lambda^2 \end{bmatrix}^{-1}$ exists $\Leftrightarrow \begin{vmatrix} 1 & 0 & \lambda \\ 2 & 1 & 2+\lambda \\ 2 & 3 & \lambda^2 \end{vmatrix} = \lambda^2 + \lambda - 6 \neq 0 \Leftrightarrow \lambda \in \mathbb{R} - \{-3, 2\}$.

b) **By using properties of the determinants, show** that:

$$\begin{vmatrix} a+b+c & b & a \\ d+e+f & e & d \\ g+h+i & h & g \end{vmatrix} = \begin{vmatrix} c & b & a \\ f & e & d \\ i & h & g \end{vmatrix}.$$

Solution: $\begin{vmatrix} a+b+c & b & a \\ d+e+f & e & d \\ g+h+i & h & g \end{vmatrix} = \begin{vmatrix} a+b+c & d+e+f & g+h+i \\ b & e & h \\ a & d & g \end{vmatrix}$ (*by taking transpose*)

$= \begin{vmatrix} c & b & a \\ f & e & d \\ i & h & g \end{vmatrix}$ (*by the row operations* $R_1 + (-1)R_2,\ R_1 + (-1)R_3$).

c) Let $A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & -2 & 1 \\ 1 & -1 & 0 \end{bmatrix}$. **Find** $adj(A)$ and A^{-1}.

Solution: $adj(A) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -2 \\ 1 & 1 & -3 \end{bmatrix}$.

Since $|A| = 1$, we get $A^{-1} = |A|^{-1}adj(A) = adj(A) = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & -2 \\ 1 & 1 & -3 \end{bmatrix}$.

Question 2:

a) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 2 & 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$. **Show** that the *linear system* **AX = B** has a *unique solution* for any fixed $\alpha, \beta, \gamma \in \mathbb{R}$.

Solution: Since $|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 2 & 1 & 3 \end{vmatrix} = 5 \neq 0$, A^{-1} exists. So, the linear system has the unique solution X$= A^{-1}B = A^{-1}\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix}$

for any fixed $\alpha, \beta, \gamma \in \mathbb{R}$.

b) **Solve** the following *system of linear equations* **by using the Cramer's rule**:

$$\begin{aligned} x - y + z &= 0 \\ x + y + z &= 2 \\ x + 2y + 4z &= 3. \end{aligned}$$

Solution: $|A| = 6$, $|A_x| = 6$, $|A_y| = 6$ and $|A_z| = 0$. Hence, $x = \frac{|A_x|}{|A|} = \frac{6}{6} = 1$, $y = \frac{|A_y|}{|A|} = \frac{6}{6} = 1$ and $z = \frac{|A_z|}{|A|} = \frac{0}{6} = 0$.

c) **Use any of the elimination methods to show** that the following *system of linear equations* is *inconsistent*:

$$-x + 2y - 5z = 3$$

$$x - 3y + z = 4$$

$$5x - 13y + 13z = 8.$$

Solution: Since $\left[\begin{array}{ccc|c} -1 & 2 & -5 & 3 \\ 1 & -3 & 1 & 4 \\ 5 & -13 & 13 & 8 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -3 & 1 & 4 \\ 0 & 1 & 4 & -7 \\ 0 & 0 & 0 & 2 \end{array}\right]$, the given linear system is inconsistent.

Question 3:

a) **Let** $\{\boldsymbol{v_1}, \boldsymbol{v_2}, \boldsymbol{v_3}\}$ be a *linearly independent subset* of *vector space* V. **Show** that the *subset* $\{\boldsymbol{w_1}, \boldsymbol{w_2}, \boldsymbol{w_3}\}$ is *linearly independent in* V, where $\boldsymbol{w_1} = \boldsymbol{v_1} + 2\boldsymbol{v_3}$, $\boldsymbol{w_2} = \boldsymbol{v_1} + \boldsymbol{v_2} + \boldsymbol{v_3}$ and $\boldsymbol{w_3} = \boldsymbol{v_2} + \boldsymbol{v_3}$.

Solution: If $0 = \alpha_1 w_1 + \alpha_2 w_2 + \alpha_3 w_3 = \alpha_1(v_1 + 2v_3) + \alpha_2(v_1 + v_2 + v_3) + \alpha_3(v_2 + v_3)$

$$= (\alpha_1 + \alpha_2)v_1 + (\alpha_2 + \alpha_3)v_2 + (2\alpha_1 + \alpha_2 + \alpha_3)v_3\,,$$

then, by the linear independence of $\{v_1, v_2, v_3\}$**, we get** $\alpha_1 + \alpha_2 = 0$, $\alpha_2 + \alpha_3 = 0$ **and** $2\alpha_1 + \alpha_2 + \alpha_3 = 0$;

which gives $\alpha_1 = \alpha_2 = \alpha_3 = 0$. **Hence,** $\{\boldsymbol{w_1}, \boldsymbol{w_2}, \boldsymbol{w_3}\}$ is *linearly independent in* ***V***.

b) **Show** that $\boldsymbol{F} \coloneqq \{(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \in \mathbb{R}^3 \,|\, \boldsymbol{y} - \boldsymbol{z} = \boldsymbol{0}, \boldsymbol{y} + \boldsymbol{z} = \boldsymbol{0}\}$ is a *vector subspace* of *Euclidean space* $\mathbb{R}^3$. Then **find** a *basis* and *dimension* of $\boldsymbol{F}$.

Solution: $(x, y, z) \in F \Leftrightarrow (x, y, z) = (0,0,z) = z(0,0,1)$. So, $F = \boldsymbol{span}(\{(0,0,1)\})$; which is a vector subspace of $\mathbb{R}^3$. [:

Hence, $\{(0,0,1)\}$ is a basis of ***F* and so** $\dim(F) = 1$.

c) **Show** that $\boldsymbol{B} \coloneqq \{\boldsymbol{t^2} + \boldsymbol{2}, -\boldsymbol{t} + \boldsymbol{1}, \boldsymbol{2t} - \boldsymbol{1}\}$ is a *basis* of the *real vector space* $\boldsymbol{P_2(t)}$ of all *polynomials* in *real variable* $\boldsymbol{t}$ having *degree* $\leq \boldsymbol{2}$. Then **find** the *coordinate vector* of the *polynomial* $\boldsymbol{t^2} + \boldsymbol{3t} + \boldsymbol{3}$ *with respect to* the *basis* $\boldsymbol{B}$.

Solution: **If** $\boldsymbol{0} = \alpha(t^2 + 2) + \beta(-t + 1) + \gamma(2t - 1) = \alpha t^2 + (2\gamma - \beta)t + 2\alpha + \beta - \gamma$ **then** $\alpha = \beta = \gamma = 0$**. So, the set** B **is linearly independent in the vector space** $P_2(t)$**. However,** $\dim(P_2(t).) = 3$**. So,** B **is a basis of** $P_2(t)$**.**

Now, if $t^2 + 3t + 3 = \alpha(t^2 + 2) + \beta(-t + 1) + \gamma(2t - 1) = \alpha t^2 + (2\gamma - \beta)t + 2\alpha + \beta - \gamma$, **then**

$\alpha = 1$, $\beta = 5$ **and** $\gamma = 4$**. Hence,** $[t^2 + 3t + 3\,]_B = \begin{bmatrix} 1 \\ 5 \\ 4 \end{bmatrix}$.

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