PHYS 502
1st Midterm Exam
Wednesday 10th April 2013

Instructor: Dr. V. Lempesis

Student Name: ………………………………………………..

Student ID Number…………………………..

Student Grade: ……./20

Please answer all questions

1. You are given the following function: $f(x)=x^2,\quad 0<x<2\pi$ and with period equal to 2π.

a) Sketch a rough plot of the function in the region $-4\pi<x<4\pi$.

(2 marks)

b) Expand $f(x)$ in a Fourier series.

(4 marks)

Solution

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(n\frac{2\pi x}{l}\right)+\sum_{n=1}^{\infty}b_n\sin\left(n\frac{2\pi x}{l}\right)$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}x^2\cos(nx)\,dx=\frac{1}{\pi}\left\{\left(x^2\right)\left(\frac{\sin nx}{n}\right)-(2x)\left(\frac{-\cos nx}{n^2}\right)+2\left(-\frac{\sin nx}{n^3}\right)\right\}\Bigg|_0^{2\pi}=\frac{4}{n^2}$$

$$a_0=\frac{1}{\pi}\int_0^{2\pi}x^2dx=\frac{8\pi^3}{3}$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}x^2\sin(nx)\,dx=\frac{1}{\pi}\left\{\left(x^2\right)\left(-\frac{\cos nx}{n}\right)-(2x)\left(\frac{-\sin nx}{n^2}\right)+2\left(\frac{\cos nx}{n^3}\right)\right\}\Bigg|_0^{2\pi}=\frac{-4\pi}{n}$$

Thus the function is given by

$$f(x)=x^2=\frac{4\pi^2}{3}+\sum_{n=1}^{\infty}\left(\frac{4}{n^2}\cos nx-\frac{4\pi}{n}\sin nx\right)$$

c) Use your result to prove that

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots=\frac{\pi^2}{6}$$

(2 marks)

Solution

At x=0 the Fourier series is $\frac{4\pi^2}{3}+\sum_{n=1}^{\infty}\left(\frac{4}{n^2}\right)$. Also at this point the series converges to $\left[f(0+)+f(0-)\right]/2=\left[(2\pi)^2-0^2\right]/2=2\pi^2$. So

$$2\pi^2=\frac{4\pi^2}{3}+\sum_{n=1}^{\infty}\left(\frac{4}{n^2}\right)\Rightarrow\frac{2\pi^2}{3}=\sum_{n=1}^{\infty}\left(\frac{4}{n^2}\right)\Rightarrow\frac{\pi^2}{6}=\sum_{n=1}^{\infty}\left(\frac{1}{n^2}\right)$$

Hint:

$$\int x^2\cos(nx)dx=\left\{\left(x^2\right)\left(\frac{\sin nx}{n}\right)-(2x)\left(\frac{-\cos nx}{n^2}\right)+2\left(-\frac{\sin nx}{n^3}\right)\right\}$$

$$\int x^2\sin(nx)dx=\left\{\left(x^2\right)\left(-\frac{\cos nx}{n}\right)-(2x)\left(\frac{-\sin nx}{n^2}\right)+2\left(\frac{\cos nx}{n^3}\right)\right\}$$

2. The equation of motion for a damped harmonic oscillator is given by:

$$m\frac{d^2x(t)}{dt^2}=-kx-b\frac{dx(t)}{dt}$$

Use Laplace transforms to find $x(t)$ for the following initial conditions: $x(0)=x_0$, $x^{'}(0)=0$ and for $m=1\,kg$, $k=2\,N/m$, $b=4\,kg/s$, $x_0=5\times10^{-3}m$.

(6 marks)

Solution

$$m\frac{d^2x(t)}{dt^2}=-kx-b\frac{dx(t)}{dt}\Rightarrow\frac{d^2x(t)}{dt^2}=-2x-4\frac{dx(t)}{dt}\Rightarrow$$
$$s^2x(s)-sx(0)-x^{'}(0)=-2x(s)-4sx(s)+4x(0)\Rightarrow$$
$$s^2x(s)-sx_0=-2x(s)-4sx(s)+4x_0\Rightarrow$$
$$x(s)=x_0\frac{(s+4)}{\left(s^2+4s+2\right)}=x_0\frac{(s+4)}{(s+2)^2-2}=x_0\frac{(s+4)}{(s+2)^2-\left(\sqrt{2}\right)^2}=$$
$$x_0\frac{(s+2)}{(s+2)^2-\left(\sqrt{2}\right)^2}+x_0\frac{2}{(s+2)^2-\left(\sqrt{2}\right)^2}$$

Thus

$$x(t)=L^{-1}\left\{x_0\frac{(s+2)}{(s+2)^2-\left(\sqrt{2}\right)^2}+x_0\frac{2}{(s+2)^2-\left(\sqrt{2}\right)^2}\right\}=$$

$$x_0L^{-1}\left\{\frac{(s+2)}{(s+2)^2-\left(\sqrt{2}\right)^2}\right\}+x_0L^{-1}\left\{\frac{2}{(s+2)^2-\left(\sqrt{2}\right)^2}\right\}=$$

$$x_0e^{-2t}\cosh\left(t\sqrt{2}\right)+x_0\sqrt{2}e^{-2t}\sinh\left(t\sqrt{2}\right)=$$

$$5\times10^{-3}\left[e^{-2t}\cosh\left(t\sqrt{2}\right)+\sqrt{2}e^{-2t}\sinh\left(t\sqrt{2}\right)\right]$$

3. Let a function $g(t)$ which has a Fourier transformed function $G(\omega)$.

(a) Prove the frequency shifting property:

$$F\left\{g(t)e^{-i\omega_0 t}\right\}=G(\omega+\omega_0)$$

(2 marks)

Solution:

$$F\left\{g(t)e^{-i\omega_0 t}\right\}=\int_{-\infty}^{\infty}g(t)e^{-i\omega_0 t}e^{-i\omega t}dt=\int_{-\infty}^{\infty}g(t)e^{-i(\omega_0+\omega t}dt=G(\omega+\omega_0)$$

(b) In a resonant cavity an electromagnetic oscillation of frequency ω_0 dies out as:

$$A(t)=A_0e^{-\omega_0 t/2Q}e^{-i\omega_0 t},\quad t>0$$

(Take $A(t)=0$ for $t<0$.)
This parameter Q is a measure of the ratio of stored energy to energy loss per cycle. Use the answer in question (a) to calculate the frequency distribution of the oscillation, $a^*(\omega)a(\omega)$, where $a(\omega)$ is the Fourier transform of $A(t)$.

(4 marks)

Solution:

$$A(t)=A_0e^{-\omega_0 t/2Q}e^{-i\omega_0 t}\Rightarrow A(t)=g(t)e^{-i\omega_0 t},\qquad \text{where } g(t)=A_0e^{-\omega_0 t/2Q}$$

The Fourier transform of $g(t)$ is

$$G(\omega)=\int_{-\infty}^{\infty}g(t)e^{-i\omega t}=\int_{0}^{\infty}A_0e^{-\omega_0 t/2Q}e^{-i\omega t}dt=A_0\int_{0}^{\infty}e^{t(-\omega_0/2Q-i\omega)}dt=\frac{A_0}{(\omega_0/2Q+i\omega)}$$

From question (a) we can write that

$$a(\omega) = G(\omega + \omega_0) = \frac{A_0}{\left[\omega_0 / 2Q + i(\omega + \omega_0)\right]}.$$

Thus

$$a^*(\omega) = \frac{A_0}{\left[\omega_0 / 2Q - i(\omega + \omega_0)\right]}$$

$$a(\omega)a^*(\omega) = \frac{A_0^2}{\left[(\omega_0 / 2Q)^2 + (\omega + \omega_0)^2\right]}$$