KING FAHD UNIVERSITY OF PETROLEUM AND MINERALS

ELECTRICAL ENGINEERING DEPARTMENT

Probabilistic Methods in Electrical Engineering
EE 315
Semester 071
FIRST MAJOR

DATE : November 6, 2007

TIME:6:30-8:00 pm

Name: Solution

ID :

Section # : 04

QUESTION	MARK
1	/20
2	/20
3	/10
4	/10
TOTAL	/60

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Problem 1:

The lifetime (in years) of a device behaves as a random variable with exponential density

$$f_X(x) = e^{-x} u(x).$$

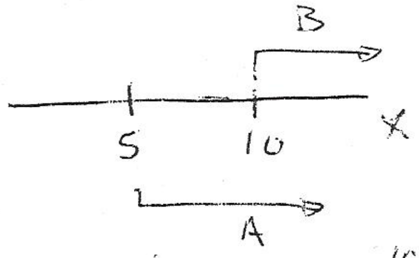
Let A be the event "device lifetime greater than 5 years", and B be the event "device lifetime greater than 10 years". Find:

(5) a) $P(A \cap B)$.

(5) b) $P(A \cap \bar{B})$.

(5) c) $P(A \cup B)$.

(5) d) $P(B \mid A)$.



a) $P(A \cap B) = P(B) = P_X(X > 10)$

$= 1 - P(X \leq 10) = 1 - F_X(10) = 1 - \int_0^{10} e^{-x} = 1 + [e^{-x}]_0^{10} = e^{-10}$

b) $P(A \cap \bar{B}) = P(5 < X \leq 10) = F_X(10) - F_X(5)$

$= \int_0^{10} e^{-x} dx - \int_0^{5} e^{-x} dx = -e^{-x}\Big|_0^{10} + e^{-x}\Big|_0^{5} = -e^{-10} + 1 + e^{-5} - 1$

$= e^{-5} - e^{-10}$

c) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= P(X > 5) + P(X > 10) - P(A \cap B)$

$= [1 - F_X(5)] + [1 - F_X(10)] - P(A \cap B)$

$= 1 - \int_0^{5} e^{-x} dx + 1 - \int_0^{10} e^{-x} dx - e^{-10}$

$= 1 + e^{-5} - 1 + e^{-10} - 1 - e^{-10} = e^{-5}$

d)

$P(B \mid A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{e^{-10}}{P(X > 5)} = \dfrac{e^{-10}}{1 - F_X(5)} = \dfrac{e^{-10}}{1 - \int_0^{5} e^{-x} dx}$

$= \dfrac{e^{-10}}{e^{-5}} = e^{-5}$

Pg. 2

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Problem 2:
A random variable X has a propbabilty density function (pdf) defined by:

$$f_X(x) = \begin{cases} cx(1-x), & 0 \le x \le 1, \\ 0, & \text{elsewhere} \end{cases}$$

1. Find c such that $f_X(x)$ is a valid pdf. (4)
2. Find $F_X(x)$ and sketch it. (8) → 8
3. Find b such that $P[|X|<b] = \frac{5b}{8}$. (4)
4. Find $P[X>0.5 \mid 0<X<1]$. (4) → 4

1 - c such that $f_X(x)$ is a valid pdf.

$$\int_{-\infty}^{+\infty} f_X(x)\,dx = 1$$

$$\int_0^1 c[x - x^2]\,dx = 1 \quad (2)$$

$$c\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 1 \quad (2)$$

$$c\left[\frac{1}{2} - \frac{1}{3}\right] = 1 \Rightarrow \boxed{c = 6}$$

2 - $F_X(x)$ and its graph.

* for $x < 0$:

$f_X(x) = 0$

$$\therefore F_X(x) = \int_{-\infty}^{x} f_X(\xi)\,d\xi = 0$$

* for $0 \le x \le 1$:

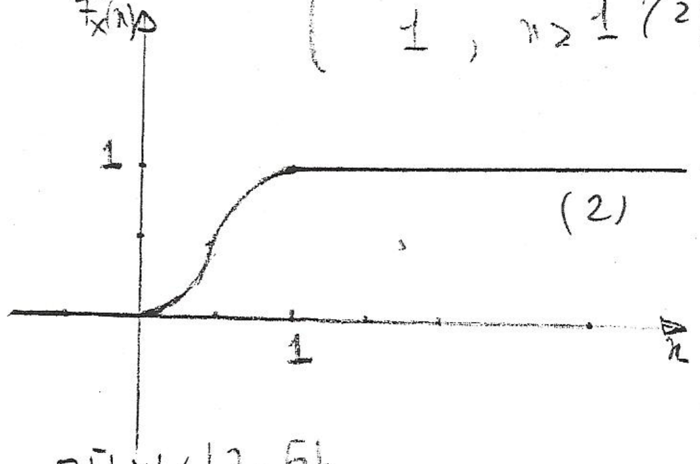
$$\therefore F_X(x) = \int_{-\infty}^{x} f_X(\xi)\,d\xi = \int_{-\infty}^{0} f_X(\xi)\,d\xi + \int_0^x f_X(\xi)\,d\xi = 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]$$

* for $x > 1$:

$$\therefore F_X(x) = \int_{-\infty}^{x} f_X(\xi)\,d\xi = \int_{-\infty}^{0} f_X(\xi)\,d\xi + \int_0^1 f_X(\xi)\,d\xi + \int_1^x f_X(\xi)\,d\xi = 1$$

Therefore:

$$F_X(x) = \begin{cases} 0, & x < 0 \quad (2) \\ 6\left(\frac{x^2}{2} - \frac{x^3}{3}\right), & 0 \le x \le 1 \quad (2) \\ 1, & x > 1 \quad (2) \end{cases}$$



(2)

3 - $P[|X|<b] = \frac{5b}{8}$

$P[-b<X<b] = \frac{5b}{8}$, since $x \in [0,1]$

then $P[X<b] = \frac{5b}{8}$

$$\int_0^b 6(x - x^2)\,dx = \frac{5b}{8}$$

$$6\left(\frac{b^2}{2} - \frac{b^3}{3}\right) = \frac{5b}{8}, \quad \text{since } b \ne 0$$

then $16b^2 - 24b + 5 = 0$ ⟨ $b = 1/4$, $b = 1.25$ rej.

$b = \frac{1}{4}$

4. $P[X > 0.5 \mid 0 < X < 1]$

$$= \frac{P[X > 0.5 \cap 0 < X < 1]}{P[0 < X < 1]}$$

$$= \frac{P[0.5 < X < 1]}{P[0 < X < 1]}$$

$$= \frac{\int_{0.5}^{1} f_X(x)\,dx}{\int_{0}^{1} f_X(x)\,dx}$$

$$= \frac{6\left(x^2/2 - x^3/3\right)\Big|_{0.5}^{1}}{4}$$

$$= 6\left(\frac{1}{2} - \frac{1}{3}\right) - 6\left(\frac{0.5^2}{2} - \frac{0.5^3}{3}\right)$$

$$= 1 - 0.5^2\left[3 - \frac{2 \times 0.5}{1}\right]$$

$$= 1 - (0.25 * 2)$$

~~= 0.75~~

$$= 0.5$$

Pg. 4

Problem 3:

A random voltage V has the density function $f_V(v) = \frac{1}{4} u(v) e^{-v/4}$

a) Calculate the mean value of V.

b) If the voltage is passed through a device that generates the voltage $Y = V^3$, then calculate the expected value of Y.

(5) $\bar{V} = E[V] = \int_0^\infty \frac{v}{4} e^{-v/4} dv = -4 e^{-v/4} \Big|_0^\infty = 4$

(5) b) $E[Y] = E[v^3] = \int_0^\infty \frac{1}{4} v^3 e^{-v/4} dv$

Integrating by parts:

$E[Y] = -v^3 e^{-v/4} \Big|_0^\infty - 12 v^2 e^{-v/4} \Big|_0^\infty - 96 v e^{-v/4} \Big|_0^\infty - 384 e^{-v/4} \Big|_0^\infty$

$= 0 + 0 + 0 + 384 = 384$

1 Problem 4:

An audio amplifier contains six transistors. A technician has determined that two transistors are defective, but he does not know which two. The technician removes three transistors at random and inspects them. Let X be the number of defective transistors that the technician finds, where X may be 0, 1, or 2.

1. Find the probability density function for X.
2. Now if the technician decides to inspect the six transistors by picking one transistor at a time and inspecting it, what is the probability that he will be successful in finding a defective transistor from the second inspection?

2 Solution Part 1:

1 / 0.5

The probability of defectiveness is equal to $\frac{2}{6} = \frac{1}{3}$. The sample space has $\binom{6}{3} = 20$ elements.

- The probability of picking three transistors and none of them are defected is:

$$P[x_0 = 0] = \frac{\binom{2}{0} \times \binom{4}{3}}{\binom{6}{3}} = \frac{1 \times 4}{20} = \frac{4}{20}.$$

1.5 2

- The probability of picking three transistors and one is defected is:

$$P[x_1 = 1] = \frac{\binom{2}{1} \times \binom{4}{2}}{\binom{6}{3}} = \frac{2 \times 6}{20} = \frac{12}{20}.$$

1.5 2 4.5

- The probability of picking three transistors and two are defected is: 2

$$P[x_2 = 2] = \frac{\binom{2}{2} \times \binom{4}{1}}{\binom{6}{3}} = \frac{1 \times 4}{20} = \frac{4}{20}.$$

1.5

- The probability density function for X is:

$$f_X(x) = \frac{4}{20}\delta(x) + \frac{12}{20}\delta(x-1) + \frac{4}{20}\delta(x-2).$$

2

3 Solution Part 2:

The probability of being successful in finding a defective transistor from the second inspection, meaning that the first inspected transistor is not found defected, is:

2

$$P[\text{successful in second inspection}] = \frac{\binom{4}{1} \times \binom{2}{1}}{\binom{6}{2}} = \frac{8}{15}.$$

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