Differential and Integral Calculus (MATH-205)

Final Exam/Sem III (2022-23) Time Allowed: 3 Hours

Date: Tuesday, June 13, 2023 **Maximum Marks:** 40

Note: Solve all **10** questions and give **DETAILED** solutions. Make sure your solutions are clearly written and contain all necessary details.

Question 1: **(4°)** Determine whether the following infinite series converges or diverges.

$$\sum_{n=1}^{\infty}\left(\frac{2n^2+3n}{\sqrt{n^5+5}}-\frac{5}{(2n+1)(n+3)}\right)$$

Question 2: **(4°)** Find the radius of convergence and the interval of convergence of the series

$$\sum_{n=0}^{\infty}\frac{n(x+2)^n}{3^{n+1}}$$

Question 3: **(3°)** Find the Maclaurin series representation of $f(x)=e^{-x^2}$. Using first 4 nonzero terms of the this series, find approximate value of the integral $\int_0^1 e^{-x^2}\,dx$ upto 4 decimal places.

Question 4: **(3°)** An object starts from rest at the point $(2,1,3)$ and moves with acceleration

$$\mathbf{a}(t)=\hat{\mathbf{i}}+2t\,\hat{\mathbf{j}}+3t^2\,\hat{\mathbf{k}},\quad t\geq 0.$$

Find the location of the object after 3 seconds.

Question 5: **(5°)** Show that the lines l_1 passing through $A(1,3,0)$ and $B(0,4,5)$ and l_2 passing through $C(-2,-1,2)$ and $D(5,1,0)$ are skew lines. Find the shortest distance between l_1 and l_2.

Question 6: **(4°)** Find the domain, extrema and saddle points of the function given by $f(x,y)=\frac{x^3}{3}+4xy-9x-y^2$.

--- PTO ---

Question 7: (3°) If $w = f(x, y)$, where $x = r\cos\theta$ and $y = r\sin\theta$, show that

$$w_x^2 + w_y^2 = w_r^2 + \frac{1}{r^2} w_\theta^2.$$

Question 8: (4°) The surface of a lake is represented by a region D in the xy−plane such that the depth (in feet) under the point $P(x, y)$ is $f(x, y) = 300 - 2x^2 - 3y^2$. In what direction should a boat at $P(4, 9)$ sail in order for the depth of the water to decrease most rapidly? Find a unit vector in the direction in which the depth remains constant?

Question 9: (5°) Evaluate the double integral $\iint\limits_R x^3 \cos xy \, dA$, where R is the region bounded by the graphs of $y = x^2$, $y = 0$, and $x = 2$. Sketch the region R.

Question 10: (5°) Find the volume V of the solid that lies under the graph of the equation $z = 3xy - x^2 - y^2 + 6$ and over the region R in the xy-plane bounded by the graphs of $x = 0$, $y = 0$, and $\frac{x}{2} + y = 1$. Sketch the region R.

—— Good Luck ——

P7

Q.1 4

Given infinite series $\sum_{n=1}^{\infty}\left(\frac{2n^2+3n}{\sqrt{n^5+5}} - \frac{5}{(2n+1)(n+3)}\right)$

Let $a_n = \frac{2n^2+3n}{\sqrt{n^5+5}}$, $b_n = \frac{5}{(2n+1)(n+3)}$, $n \geq 1$

For $\sum a_n$: let $c_n = \frac{n^2}{n^{5/2}} = \frac{1}{\sqrt{n}}$, $n \geq 1$, then $\sum c_n = \sum \frac{1}{\sqrt{n}}$ is a divergent p-series with $p = \frac{1}{2} < 1$.

Ratio: $\frac{a_n}{c_n} = \frac{2n^2+3n}{\sqrt{n^5+5}} \cdot \frac{n^{1/2}}{1} = \frac{\left(2+\frac{3}{n}\right)}{\sqrt{1+\frac{5}{n^5}}} \Rightarrow \lim_{n\to\infty} \frac{a_n}{c_n} = \lim_{n\to\infty} \frac{2+3/n}{\sqrt{1+\frac{5}{n^5}}} = 2 > 0$

$\because \sum c_n$ is a divg. series, $\therefore$ by LCT, $\sum a_n$ is also a divg series

1 1/2

For $\sum b_n$: let $c_n = \frac{1}{n^2}$, then $\sum c_n = \sum \frac{1}{n^2}$, $n \geq 1$, is a Convg. p-series with p-series with $p = 2 > 1$.

Ratio: $\frac{b_n}{c_n} = \frac{5}{2n^2+7n+3} \cdot \frac{n^2}{1} = \frac{5}{2+\frac{7}{n}+\frac{3}{n^2}}$, $\Rightarrow \lim_{n\to\infty} \frac{b_n}{c_n} = \lim_{n\to\infty} \frac{5}{2+\frac{7}{n}+\frac{3}{n^2}} = \frac{5}{2} > 0$

$\because \sum c_n$ is a Convg. series, $\therefore$ by LCT, $\sum b_n$ is also a convg. Series.

1 1/2

$\because \sum a_n$ is a divg. Series and $\sum b_n$ is a Convg. Series,

$\therefore \sum (a_n - b_n) = \sum_{n=1}^{\infty}\left[\frac{2n^2+3n}{\sqrt{n^5+5}} - \frac{5}{(2n+1)(n+3)}\right]$ is a divg. series.

1

Q.2 (4)

Given power series is $\sum_{n=0}^{\infty} \frac{n}{3^{n+1}} (x+2)^n$ —— (i)

let $u_n = \frac{n}{3^{n+1}}(x+2)^n, n \geq 0$, then $u_{n+1} = \frac{n+1}{3^{n+2}} \cdot (x+2)^{n+1}$

Ratio: $\frac{u_{n+1}}{u_n} = \frac{n+1}{3^{n+2}} \cdot \frac{3^{n+1}}{n} \cdot \frac{(x+2)^{n+1}}{(x+2)^n} = \frac{1}{3}\left(1+\frac{1}{n}\right).(x+2)$

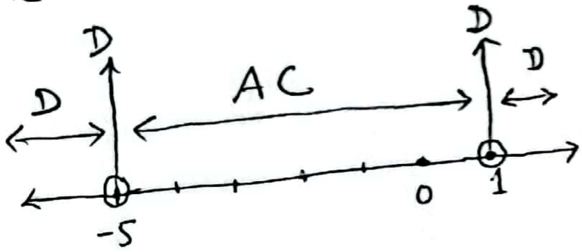
$\Rightarrow \lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \frac{1}{3}.|x+2|. \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = \frac{|x+2|}{3} = L$

By ratio test for AC, the given series converges absolutely if $\frac{|x+2|}{3} < 1$ i.e, $-3 < x+2 < 3 \Rightarrow -5 < x < 1$. (2)

Moreover the series will diverge if $\frac{|x+2|}{3} > 1$, i.e, $|x+2| > 3$

$\Rightarrow x+2 > 3$ or $x+2 < -3 \Rightarrow x > 1$, or $x < -5$.

For $\frac{|x+2|}{3} = 1$, i.e., $|x+2| = 3 \Rightarrow x = -2 \pm 3 = -5, 1$, the test fails. Therefore, we separately investigate the behavior at $x = -5, 1$.



For $x = 1$: (i) becomes

$\sum_{n=0}^{\infty} \frac{n}{3^{n+1}} \cdot 3^n = \sum_{n=0}^{\infty} \frac{n}{3}$, which diverges by n^{th} term test.

For $x = -5$: (i) becomes.

$\sum_{n=0}^{\infty} \frac{n}{3^{n+1}} \cdot (-1)^n \cdot 3^n = \sum_{n=0}^{\infty} \frac{n}{3} \cdot (-1)^n$, which is also divergent by AST.

$\therefore$ IC $= (-5, 1)$, RC $= 3$ (2)

Q.3 3 $R_{\neq}$

Here $f(x) = e^{-x^2}$. we know that the Maclaurin's series of e^x is given by

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\Rightarrow e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \frac{x^8}{4!} - \cdots = \sum_{n=}^{\infty} \frac{(-1)^n x^{2n}}{n!} \quad 1$$

Taking the first four non-zero terms of $f(x) = e^{-x^2}$, we get

$$e^{-x^2} \approx 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!}$$

$$\Rightarrow \int e^{-x^2} \cdot dx \approx x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} \quad 1$$

$$\Rightarrow \int_0^1 e^{-x^2} dx \approx \left| x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} \right|_0^1 = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} = 0.7429 \quad 1$$

Q.4 3

Given, acc. of the object, $\vec{a}(t) = \hat{i} + 2t\,\hat{j} + 3t^2 \cdot \hat{k}$, $t \geq 0$ —(i)

Initial velocity, $\vec{v}(0) = 0.\hat{i} + 0.\hat{j} + 0.\hat{k}$ —(ii)

Initial position, $\vec{r}(0) = 2\hat{i} + \hat{j} + 3\hat{k}$ —(iii)

Integrating (i), w.r.t. 't', on both sides, we get

$$\int \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) . dt = \dot{\vec{r}}(t) = t\hat{i} + t^2\hat{j} + t^3\hat{k} + \vec{b} \quad \text{—(iv)}$$

'$\vec{b}$' is constt of integration

At t=0 $\quad \dot{\vec{r}}(0) = \vec{v}(0) = \vec{b} \Rightarrow \vec{b} = 0.\hat{i} + 0.\hat{j} + 0.\hat{k} = \vec{0} \quad 1\frac{1}{2}$

$\therefore$ (iv) $\Rightarrow \frac{d}{dt}(\vec{r}(t)) = t\hat{i} + t^2\hat{j} + t^3\hat{k}$, Integrating again w.r.t. 't',

we get $\vec{r}(t) = \frac{t^2}{2}\hat{i} + \frac{t^3}{3}\hat{j} + \frac{t^4}{4}\hat{k} + \vec{c}$ —(v), '$\vec{c}$' is constt of integration.

At t=0, $\vec{r}(0) = 0.\hat{i} + 0.\hat{j} + 0.\hat{k} + \vec{c}$

$\Rightarrow \vec{c} = 2\hat{i} + \hat{j} + 3\hat{k}$, using (iii). —(vi) 1

$\therefore$ (v) $\Rightarrow \vec{r}(t) = \left(\frac{t^2}{2} + 2\right)\hat{i} + \left(\frac{t^3}{3} + 1\right)\hat{j} + \left(\frac{t^4}{4} + 3\right)\hat{k}$

$\therefore$ location of object at t=3 is, $\vec{r}(3) = \frac{13}{2}\hat{i} + 10\hat{j} + \frac{93}{4}\hat{k}$ $\frac{1}{2}$

Q.5 5 P#

Parametric eqs. of line through $A(1,3,0)$ & $B(0,4,5)$ are

$l_1: x = 1-t, \; y = 3+t, \; z = 5t, \; t \in \mathbb{R}$ — (i)

Parametric eqs. of line through $C(-2,-1,2)$ & $D(5,1,0)$ are

$l_2: x = -2+7s, \; y = -1+2s, \; z = 2-2s, \; s \in \mathbb{R}$ — (ii)

1

D'vectors of l_1 & l_2 are

$\vec{n}_1 = [-1, 1, 5], \; \vec{n}_2 = [7, 2, -2]$

$\because \frac{-1}{7} \neq \frac{1}{2} \neq \frac{5}{-2} \Rightarrow \vec{n}_1 \nparallel \vec{n}_2 \Rightarrow l_1 \nparallel l_2$ ½

For intersection, we solve the following system of eqs.

For x: $1-t = -2+7s \Rightarrow 7s+t = 3$ — (iii)

" y: $3+t = -1+2s \Rightarrow 2s-t = 4$ — (iv)

$5t = 2-2s \Rightarrow 2s+5t = 2$ — (v)

(iii) + (iv) $\Rightarrow 9s = 7 \Rightarrow \boxed{s = 7/9}$, (iv) $\Rightarrow t = \frac{14}{9} - 4 = -\frac{22}{9} \Rightarrow \boxed{t = -\frac{22}{9}}$

From (v), L.H.S. $= \frac{14}{9} - \frac{110}{9} = -\frac{96}{9} = -\frac{32}{3} \neq 2 =$ R.H.S.

$\Rightarrow$ The linear system (iii), (iv) & (v) is inconsistent. Hence, l_1 & l_2 do not intersect. therefore, l_1 & l_2 are skew lines.

2½

The distance b/w l_1 & l_2 is given by

$d = \frac{|\vec{AB} \times \vec{CD} \cdot \vec{AC}|}{\|\vec{AB} \times \vec{CD}\|}$ — (vi)

$\vec{AB} = [-1, 1, 5], \; \vec{CD} = [7, 2, -2]$

$\vec{AC} = [-3, -4, 2]$

$\vec{AB} \times \vec{CD} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 5 \\ 7 & 2 & -2 \end{vmatrix} = -12\hat{i} + 33\hat{j} - 9\hat{k}$

$\Rightarrow \|\vec{AB} \times \vec{CD}\| = \sqrt{144 + 1089 + 81} = \sqrt{1314}$

& $\vec{AB} \times \vec{CD} \cdot \vec{AC} = 36 - 132 - 18 = -114$

$\therefore$ From (vi),

$d = \frac{|-114|}{\sqrt{1314}} = 3.14$ units.

1

Q.6 (4)

Given $z = f(x,y) = \frac{x^3}{3} + 4xy - 9x - y^2$ —— (i), $D_f = \mathbb{R}^2$

$z_x = x^2 + 4y - 9$, $z_y = 4x - 2y$

For critical pts: $z_x = 0$ & $z_y = 0$ $\Rightarrow$ $\left.\begin{array}{l} x^2+4y-9=0 \\ 2x-y=0 \end{array}\right\}$ $\Rightarrow$ (1,2) & (−9,−18) are critical pts.

$1\frac{1}{2}$

For Discriminant: $f_{xx} = 2x$, $z_{yy} = -2$, $f_{xy} = 4 = z_{yx}$

$\therefore D(x,y) = \begin{vmatrix} 2x & 4 \\ 4 & -2 \end{vmatrix} = -4x - 16$

For (1,2): $D(1,2) = -4 - 16 < 0 \Rightarrow (1,2)$ is a saddle pt.

For (−9,−18): $D(-9,-18) = 36 - 16 = 20 > 0$, $\Rightarrow f$ has an extrema at (−9,−18).

$\therefore z_{xx}(-9,-18) = 2x - 9 = -18 < 0$, $\Rightarrow$ z has a local max. at (−9,−18).

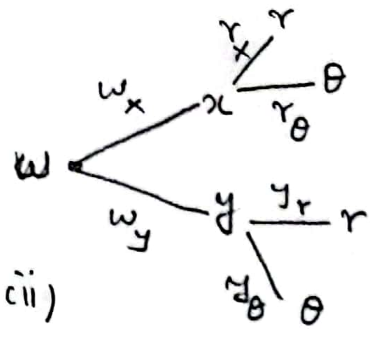
Max. value is $z(-9,-18) = \frac{-729}{3} + 648 + 81 - 324 = \frac{486}{3} = 162$

$2\frac{1}{2}$

CamScanner

Q. 7 (3)

Here, $w = f(x, y)$, where $x = r\cos\theta$, $y = r\sin\theta$



$$w_r = w_x \cdot x_r + w_y \cdot y_r$$

$$= \cos\theta \cdot w_x + \sin\theta \cdot w_y \quad \text{—(i)}$$

$$w_\theta = w_x \cdot x_\theta + w_y \cdot y_\theta = (-r\sin\theta).w_x + (r\cos\theta).w_y \quad \text{—(ii)}$$

1½

Consider,

$$w_r^2 + \frac{1}{r^2}\cdot w_\theta^2 = (\cos\theta \cdot w_x + \sin\theta \cdot w_y)^2 + \frac{1}{r^2}(-r\sin\theta . w_x + r\cos\theta . w_y)^2$$

$$= \cos^2\theta \, w_x^2 + \sin^2\theta . w_y^2 + 2\sin\theta . \cos\theta . w_x . w_y$$

$$+ \cancel{\frac{1}{r^2}} \cdot \cancel{r^2}(\sin^2\theta . w_x^2 + \cos^2\theta . w_y^2 - 2\sin\theta . \cos\theta . w_x . w_y)$$

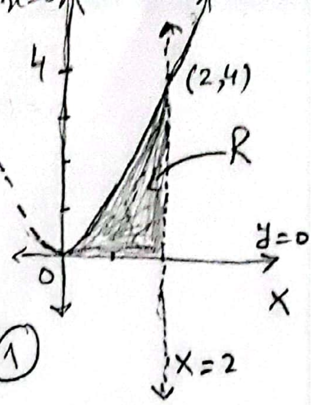
$$= (\sin^2\theta + \cos^2\theta) w_x^2 + (\sin^2\theta + \cos^2\theta) w_y^2 = w_x^2 + w_y^2$$

$\Rightarrow \; w_x^2 + w_y^2 = w_r^2 + \frac{1}{r^2} \cdot w_\theta^2$, which is the required result.

1½

Q.9 (5)

The region R is an R_x-region as shown in fig.



$$\therefore \iint_R x^3 \cdot \cos xy \cdot dA = \int_0^2 \int_0^{x^2} x^3 \cdot \cos xy \cdot dy \cdot dx \quad (1)$$

$$= \int_0^2 x^3 \cdot \left| \frac{\sin xy}{x} \right|_0^{x^2} \cdot dx = \int_0^2 \sin x^3 \cdot x^2 \cdot dx$$

$$= \left| -\frac{\cos x^3}{3} \right|_0^2 = \frac{1}{3}(1-\cos 8) = 0.38 \quad (1)$$

(3)

Q.8 (4) P7

Here $z = f(x,y) = 300 - 2x^2 - 3y^2$ (Depth of the lake at any pt. $P(x,y)$)

$$\nabla f(x,y) = -4x\hat{i} - 6y\hat{j}$$

$$\Rightarrow \nabla f(x,y)\Big|_{(4,9)} = -16\hat{i} - 54\hat{j}$$

$\therefore$ The depth of lake at $P(4,9)$ ↓ most rapidly in the direction of $-\nabla f\Big|_{(4,9)} = 16\hat{i} + 54\hat{j}$. (2)

Let $\vec{w} = w_1\hat{i} + w_2\hat{j}$ be the vector along which depth of the lake is constt, then

$$D_w f(x,y)\Big|_{(4,9)} = \nabla f\Big|_{(4,9)} \cdot \vec{w} = 0 \Rightarrow -16w_1 - 54w_2 = 0, \Rightarrow w_1 = \frac{27}{8}w_2$$

$$\therefore \vec{w} = \frac{27}{8}w_2\hat{i} + w_2\hat{j} = w_2 \cdot \left(\frac{27}{8}\hat{i} + \hat{j}\right), \; w_2 \in \mathbb{R} \setminus \{0\}$$

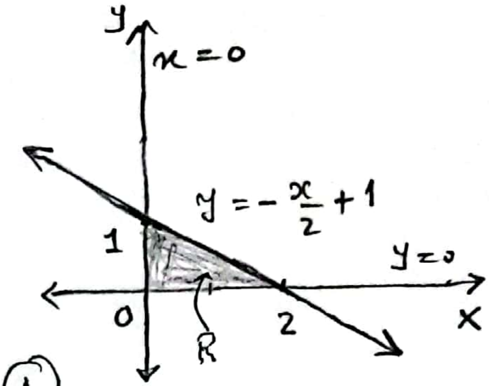
$\Rightarrow$ The depth of the lake will remain constt along $\vec{w} = \frac{27}{8}\hat{i} + \hat{j}$.

A unit vector along which depth remains constt is given by

$$\hat{w} = \frac{\vec{w}}{\|\vec{w}\|} = ✓ \quad (2)$$

Q.10 (5)

The region R is an R_x-region as shown in fig.



$$\therefore V = \iint_R (3xy - x^2 - y^2 + 6)\, dA \quad (1)$$

$$= \int_0^2 \int_0^{-\frac{x}{2}+1} (3xy - x^2 - y^2 + 6).dy.dx$$

$$= \int_0^2 \left| \frac{3x y^2}{2} - x^2 y - \frac{y^3}{3} + 6y \right|_0^{-\frac{x}{2}+1} dx \quad (1)$$

$$= \int_0^2 \left[\frac{3}{2}x\left(1-\frac{x}{2}\right)^2 - x^2\left(1-\frac{x}{2}\right) - \frac{1}{3}\left(1-\frac{x}{2}\right)^3 + 6\left(1-\frac{x}{2}\right) - 0\right] dx \quad \text{—(i)}$$

$$\because \left(1-\frac{x}{2}\right)^2 = 1 + \frac{x^2}{4} - x, \quad \left(1-\frac{x}{2}\right)^3 = 1 - \frac{x^3}{8} - 3\cdot\frac{x}{2}\left(1-\frac{x}{2}\right) = 1 - \frac{x^3}{8} - \frac{3}{2}x + \frac{3x^2}{4}$$

$$\therefore \frac{3}{2}x\left(1-\frac{x}{2}\right)^2 = \frac{3}{2}x\left(1+\frac{x^2}{4}-x\right) = \frac{3}{2}x + \frac{3}{8}x^3 - \frac{3}{2}x^2$$

7 mins

$$\Rightarrow \frac{3}{2}x\left(1-\frac{x}{2}\right)^2 - x^2\left(1-\frac{x}{2}\right) - \frac{1}{3}\left(1-\frac{x}{2}\right)^3 + 6\left(1-\frac{x}{2}\right)$$

$\frac{17}{3}$ ✓

$$= \frac{3}{2}x + \frac{3}{8}x^3 - \frac{3}{2}x^2 - x^2 + \frac{x^3}{2} - \frac{1}{3} + \frac{x^3}{24} + \frac{x}{2} - \frac{x^2}{4} + 6 - 3x$$

$$= \frac{11}{12}x^3 - \frac{11}{4}x^2 - x + \frac{17}{3}$$

∴ From (i)

$$V = \int_0^2 \left(\frac{11}{12}x^3 - \frac{11}{4}x^2 - x + \frac{17}{3}\right) dx = \left|\frac{11}{48}x^4 - \frac{11}{12}x^3 - \frac{x^2}{2} + \frac{17}{3}\right|_0^2$$

$$= \frac{11}{3} - \frac{22}{3} - 2 - \frac{34}{3} = \frac{17}{3} \text{ cubic units.} \quad (3)$$