Differential and Integral Calculus (MATH-205)

Final Exam/Sem II (2022-23) Time Allowed: 3 Hours

Date: Tuesday, February 21, 2023 **Maximum Marks:** 40

Note: Solve all **9** questions and give **DETAILED** solutions. Make sure your solutions are clearly written and contain all necessary details.

Question 1: (5°) Show that the following infinite series converges and find its sum.

$$\sum_{n=0}^{\infty}\left(\frac{9}{(3n+1)(3n+4)}-\frac{2^n}{3^{n+1}}\right)$$

Question 2: (5°) Find a power series representation of $f(x)=\ln(8+x^3)$. Specify the radius and interval of convergence of the series.

Question 3: (3°) A constant force of magnitude 4 lb has the same direction as the vector $\mathbf{a}=\hat{\imath}+\hat{\jmath}+\hat{k}$. If distance is measured in feet, find the work done if the point of application moves along the $y-$axis from $(0,2,0)$ to $(0,-1,0)$. Find the component of the force along the direction of displacement and interpret why the work done is negative.

Question 4: (4°) Let the lines l_1 and l_2 have the respective parametrizations given by

$$l_1: \quad x=4+5t, \quad y=3+2t, \quad z=3t, \quad t\in\mathbb{R}$$
$$l_2: \quad x=-5+2v, \quad y=4-v, \quad z=1, \quad v\in\mathbb{R}$$

Determine whether l_1 and l_2 are parallel, intersecting, or skew lines.

Question 5: (5°) Find the extrema of $f(x,y,z)=2x^2+y^2+3z^2$ subject to the constraint $2x-3y-4z=49$. Is it a minimum or maximum?

Question 6: (3°) Let $w=2xy$, where $x=s^2+t^2$ and $y=\frac{s}{t}$. Find w_s, w_t and w_{st}. Give your answers in terms of s and t in simplified form.

— PTO —

Question 7: (5°) Let $z = f(x, y)$ be defined implicitly as a function of x and y by the equation

$$x^2 - 2y^2 - z^2 = 0.$$

Find the directional derivative of f at $(-\frac{3}{4}, 0)$ in the direction of maximum increase in f.

Question 8: (5°) Evaluate the double integral $\iint\limits_R (x^2 + y^2)\, dA$, where R is the region bounded between the graphs of $x - y + 1 = 0$ and $x + y + 1 = 0$ and $x = 0$. Sketch the region R.

Question 9: (5°) Find the volume V of the solid that lies under the graph of the equation $z = x^2 + 4$ and over the region R in the xy-plane bounded by the graphs of $x = 4 - y^2$ and $x + y = 2$. Sketch the region.

— Good Luck —

sol. Q-I 5

let $a_n = \dfrac{9}{(3n+1)(3n+4)}$, $b_n = \dfrac{2^n}{3^n} = \left(\dfrac{2}{3}\right)^n$, $n \geq 0$

using PFs, we have

$$a_n = \frac{1}{3}\left[\frac{1}{3n+1} - \frac{1}{3n+4}\right].$$

The nth partial sum of $\sum_{n=0}^{\infty} a_n$ is given by

$$S_n = \frac{1}{3}\left[\left(1-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{7}\right)+\left(\frac{1}{7}-\frac{1}{10}\right)+\cdots+\left(\frac{1}{3n-2}-\frac{1}{3n+1}\right) + \left(\frac{1}{3n+1}-\frac{1}{3n+4}\right)\right]$$

$$= \frac{1}{3}\left(1 - \frac{1}{3n+4}\right).$$

$$\lim_{n\to\infty} S_n = \frac{1}{3}\lim_{n\to\infty}\left(1-\frac{1}{3n+4}\right) = \frac{1}{3}.$$

$\therefore \{S_n\}_{n=1}^{\infty}$ is a convergent sequence and its limiting value is $\frac{1}{3}$.

$\Rightarrow \sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{1}{3}\left[\frac{1}{3n+1} - \frac{1}{3n+4}\right]$ converges ~~to~~ and its sum is $1/3$. 3

Moreover, $\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n$ is a G.S. with $r = \frac{2}{3} < 1$. 1

Hence it is a convergent G.S. & its sum is $S = \dfrac{1}{1-2/3} = 3$.

$\therefore \sum_{n=0}^{\infty}\left[\dfrac{9}{(3n+1)(3n+4)} - \dfrac{2^n}{3^{n+1}}\right]$ converges and its sum is

$$S = \frac{1}{3} - \frac{1}{3}\cdot 3 = \frac{1}{3} - 1 = -\frac{2}{3}. \quad 1$$

Sol. Q.2

Consider,

$$\frac{1}{8+x} = \frac{1}{8}\cdot\frac{1}{1+\frac{x}{8}} = \frac{1}{8}\left[1-\frac{x}{8}+\left(\frac{x}{8}\right)^2-\left(\frac{x}{8}\right)^3+\left(\frac{x}{8}\right)^4-\cdots\right] = \frac{1}{8}\sum_{n=0}^{\infty}(-1)^n\cdot\left(\frac{x}{8}\right)^n$$

1 $\quad -1<\frac{x}{8}<1$

Integrating both sides, we get

$$\ln(8+x) = \frac{1}{8}\cdot\sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^{n+1}}{8^n(n+1)}$$

Replacing x with x^3, we get 1

$$\ln(8+x^3) = \frac{1}{8}\cdot\sum_{n=0}^{\infty}(-1)^n\cdot\frac{x^{3n+3}}{2^{3n}(n+1)}$$

which is the required series representation of the given function

$$\therefore \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}\cdot x^{3n+6}}{2^{3n+6}\cdot(n+2)}\cdot\frac{2^{3n+3}\cdot(n+1)}{(-1)^n\cdot x^{3n+3}}\right|$$

$$= \lim_{n\to\infty}\frac{|x|^3}{2^3}\cdot\left(\frac{n+1}{n+2}\right) = \frac{|x|^3}{2^3}$$

using Ratio test for absolute convergence, the series will converge if $\frac{|x|^3}{2^3} < 1 \Rightarrow -2<x<2$. 1

At $x=2$, ~~the given~~ we have, $\frac{1}{8}\sum_{n=0}^{\infty}(-1)^n\cdot\frac{2^{3n+3}}{2^{3n}\cdot(n+1)} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}$, which is a convergent Alternating series. 1

At $x=-2$; we have $\frac{1}{8}\cdot\sum_{n=0}^{\infty}\frac{(-1)^n(-2)^{3n+3}}{2^{3n}\cdot(n+1)} = \sum_{n=0}^{\infty}\frac{-1}{n+1} = -\sum_{n=0}^{\infty}\frac{1}{n+1}$

which is a divergent shifted harmonic series.

$\therefore$ Interval of convergence is $(-2,2]$ 1

& radius of convergence is 2.

Sol. Q.3 / (3)

Let $\vec{F}$ be the force acting in the direction of $\vec{a} = \hat{i} + \hat{j} + \hat{k}$. A unit vector in the direction of $\vec{a}$ is $\hat{a} = \frac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})$. Therefore $\vec{F}$ is given by

$$\vec{F} = 4\hat{a} = \frac{4}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k}). \quad (1)$$

$\vec{F}$ moves the point of application from $A(0,2,0)$ to $B(0,-1,0)$. The displacement vector is

$$\vec{d} = \vec{AB} = [0,-3,0].$$

$$\therefore W = \vec{F} \cdot \vec{d} = \frac{-4 \times 3}{\sqrt{3}} = -4\sqrt{3} \text{ ft-lb} \quad (1)$$

$$\text{comp}_{\vec{d}} \vec{F} = \frac{\vec{F} \cdot \vec{d}}{\|\vec{d}\|} = \frac{-4\sqrt{3}}{3} = -\frac{4}{\sqrt{3}} \quad (1/2)$$

Interpretation of -ive sign: The work done is negative means that the force $\vec{F}$ is not acting in the same direction the displacement $\vec{d}$ is taking place. (1/2)

Sol. Q.4 / (4)

The d'vectors of l_1 & l_2 are

$$\vec{a} = [5,2,3] \quad \& \quad \vec{b} = [2,-1,0]$$

$$\because \frac{5}{2} \neq \frac{2}{-1} \neq \frac{3}{0} \Rightarrow l_1 \& l_2 \text{ are not parallel.} \quad (1)$$

For point of intersection of l_1 & l_2; we proceed as follows.

$$\left.\begin{aligned} 4+5t &= -5+2v \\ 3+2t &= 4-v \\ 3t &= 1 \end{aligned}\right\} \Rightarrow \begin{aligned} 5t-2v &= -9 \quad \text{—(i)} \\ 2t+v &= 1 \quad \text{—(ii)} \\ t &= 1/3 \quad \text{—(iii)} \end{aligned} \quad (3)$$

using (iii) in (i), $v = \frac{1}{2}\left(\frac{5}{3}+9\right) = \frac{16}{3}$

using $t = 1/3$, $v = \frac{16}{3}$, (ii) $\Rightarrow$ L.H.S $= \frac{2}{3} + \frac{16}{3} = 6 \neq 1 = $ R.H.S.

$\therefore$ The linear system (i), (ii) & (iii) has no solution. $\Rightarrow l_1$ & l_2 do not intersect.

Hence l_1 & l_2 are skew lines.

Sol. Q.5/ 5

Given $f(x,y,z) = 2x^2 + y^2 + 3z^2$ — (i)

& $g(x,y,z) = 2x - 3y - 4z - 49 = 0$ — (ii)

The relative extrema points of (i) s.t. (ii) are included among the pts (x,y,z) determined by the 1st 3 coords of the solutions (x,y,z,λ) of the following system of equations,

$$\left.\begin{array}{l} f_x = \lambda g_x \\ f_y = \lambda g_y \\ f_z = \lambda g_z \\ g(x,y,z) = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} 4x = 2\lambda \\ 2y = -3\lambda \\ 6z = -4\lambda \\ 2x - 3y - 4z - 49 = 0 \end{array}\right\} \Rightarrow \begin{array}{ll} x = \lambda/2 & \text{— (iii)} \\ y = -\frac{3}{2}\lambda & \text{— (iv)} \\ z = -\frac{2}{3}\lambda & \text{— (v)} \\ 2x - 3y - 4z - 49 = 0 & \text{— (vi)} \end{array}$$

using (iii), (iv) & (v) in (vi), we get

$$2 \times \frac{\lambda}{2} - 3 \times -\frac{3}{2}\lambda - 4 \times -\frac{2}{3}\lambda - 49 = 0 \Rightarrow \lambda = 6$$ 3

$\therefore x = 3,\ y = -9,\ z = -4,\ \lambda = 6$ is the only R solution of (iii) — (vii).

Therefore, extrema of (i) will be at $(3,-9,-4)$, and extreme value is

$$f(3,-9,-4) = 18 + 81 + 48 = 147.$$

Any other pt. on the plane (ii) is $(0, 0, -\frac{49}{4})$,

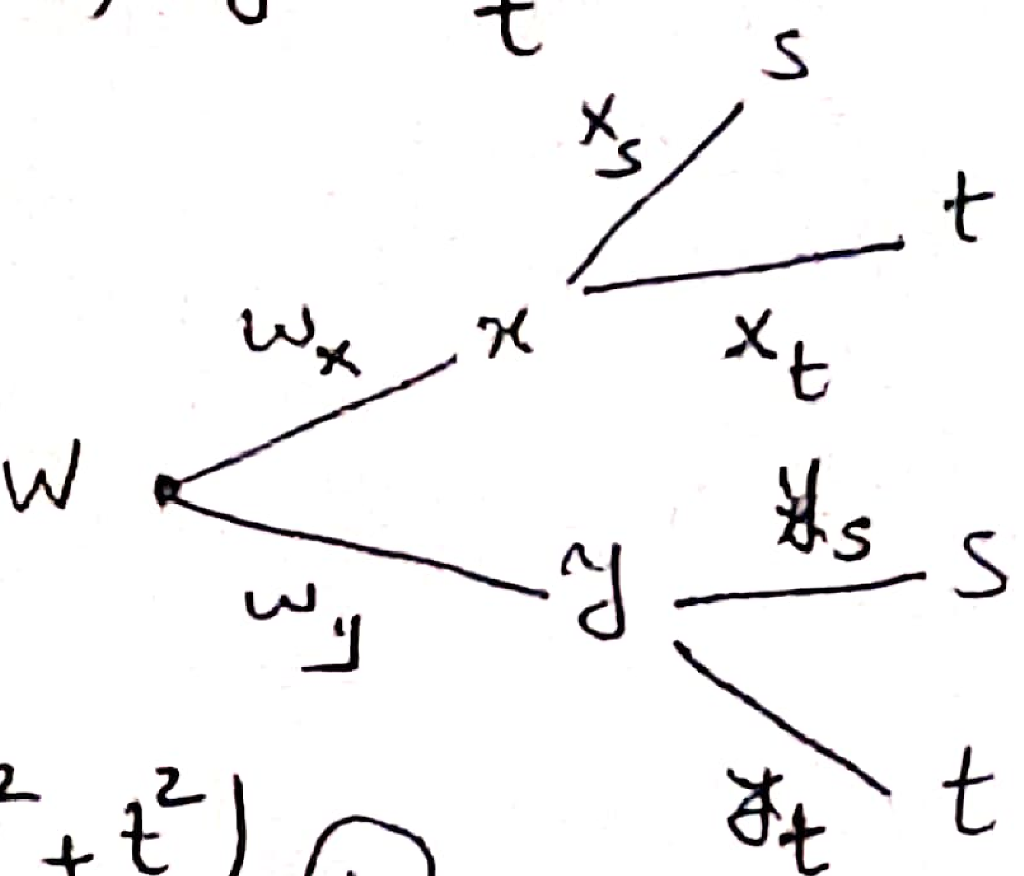
$$\because f(0,0,-\tfrac{49}{4}) = 0 + 0 + 3 \times \left(-\tfrac{49}{4}\right)^2 = 450.18 > 147.$$ 2

$\therefore f(x,y,z)$ has a minimum value at $(3,-9,-4)$.

Sol. Q.6 / 3

Given $w = 2xy$, where $x = s^2+t^2$, $y = \frac{s}{t}$.

using chain rule,



$w_s = w_x \cdot x_s + w_y \cdot y_s$

$= 2y \cdot 2s + 2x \cdot \frac{1}{t}$

$= 4\cdot\frac{s}{t}\cdot s + 2\cdot(s^2+t^2)\cdot\frac{1}{t} = \frac{2}{t}(3s^2+t^2)$ 1

&

$w_t = w_x \cdot x_t + w_y \cdot y_t = 2y\cdot 2t + 2x\cdot\left(-\frac{s}{t^2}\right) = 4\cdot\frac{s}{\cancel{t}}\cdot\cancel{t} - \frac{2s}{t^2}(s^2+t^2)$

$= \frac{2s(t^2-s^2)}{t^2}$ 1

Diff. w_s partially w.r.t. 't', we get

$w_{st} = 2\cdot\frac{\partial}{\partial t}\left(\frac{3s^2+t^2}{t}\right) = 2\cdot\frac{t\cdot(2t)-(3s^2+t^2)\cdot 1}{t^2}$ 1

$= \frac{2}{t^2}\left[2t^2-3s^2-t^2\right] = \frac{2}{t^2}(t^2-3s^2)$

Sol. Q.7 5 Given $F(x,y,z) = x^2-2y^2-z^2 = 0$ — (i)

∵ grad $f(x,y) = f_x\,\hat{i} + f_y\,\hat{j}$

where

$f_x(x,y) = z_x = -\frac{F_x}{F_z} = -\frac{2x}{-2z} = \frac{x}{z}$; $f_y(x,y) = z_y = -\frac{F_y}{F_z} = +\frac{4y}{-2z} = -\frac{2y}{z}$

At $\left(-\frac{3}{4},0\right) \overset{(i)}{\Rightarrow} \left(-\frac{3}{4}\right)^2 - 2\cdot 0 - z^2 = 0 \Rightarrow z^2 = \left(-\frac{3}{4}\right)^2 \Rightarrow z = -\frac{3}{4}$

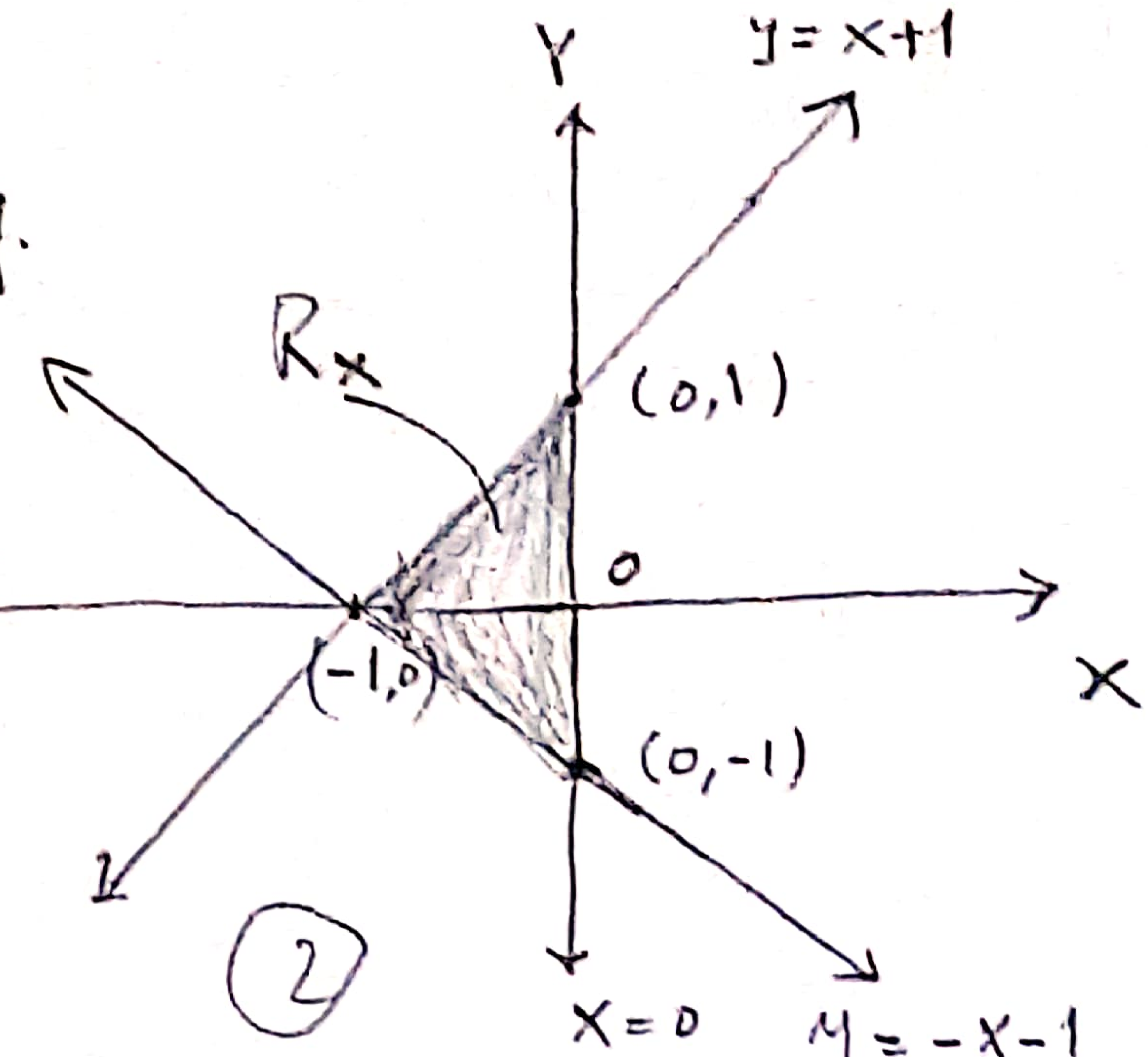
∴ $f_x\left(-\frac{3}{4},0\right) = \frac{-3/4}{-3/4} = 1$, $f_y\left(-\frac{3}{4},0\right) = 0$

⇒ grad $f\left(-\frac{3}{4},0\right) = \hat{i} + 0\cdot\hat{j} = \hat{i}$ 3

Max. ↑ in f is in the direction of grad $f\left(-\frac{3}{4},0\right) = \hat{i} = \hat{u}$ 1

∴ $D_u f\left(-\frac{3}{4},0\right) =$ grad $f\left(-\frac{3}{4},0\right)\cdot\hat{u} = 1$ 1

Sol. Q. 8



The region R is sketched in fig.

it is an R_x region given by

$$R_x = \{(x,y) : -1 \le x \le 0,\ -x-1 \le y \le x+1\}$$

$$\therefore \iint_R (x^2+y^2).\,dA$$

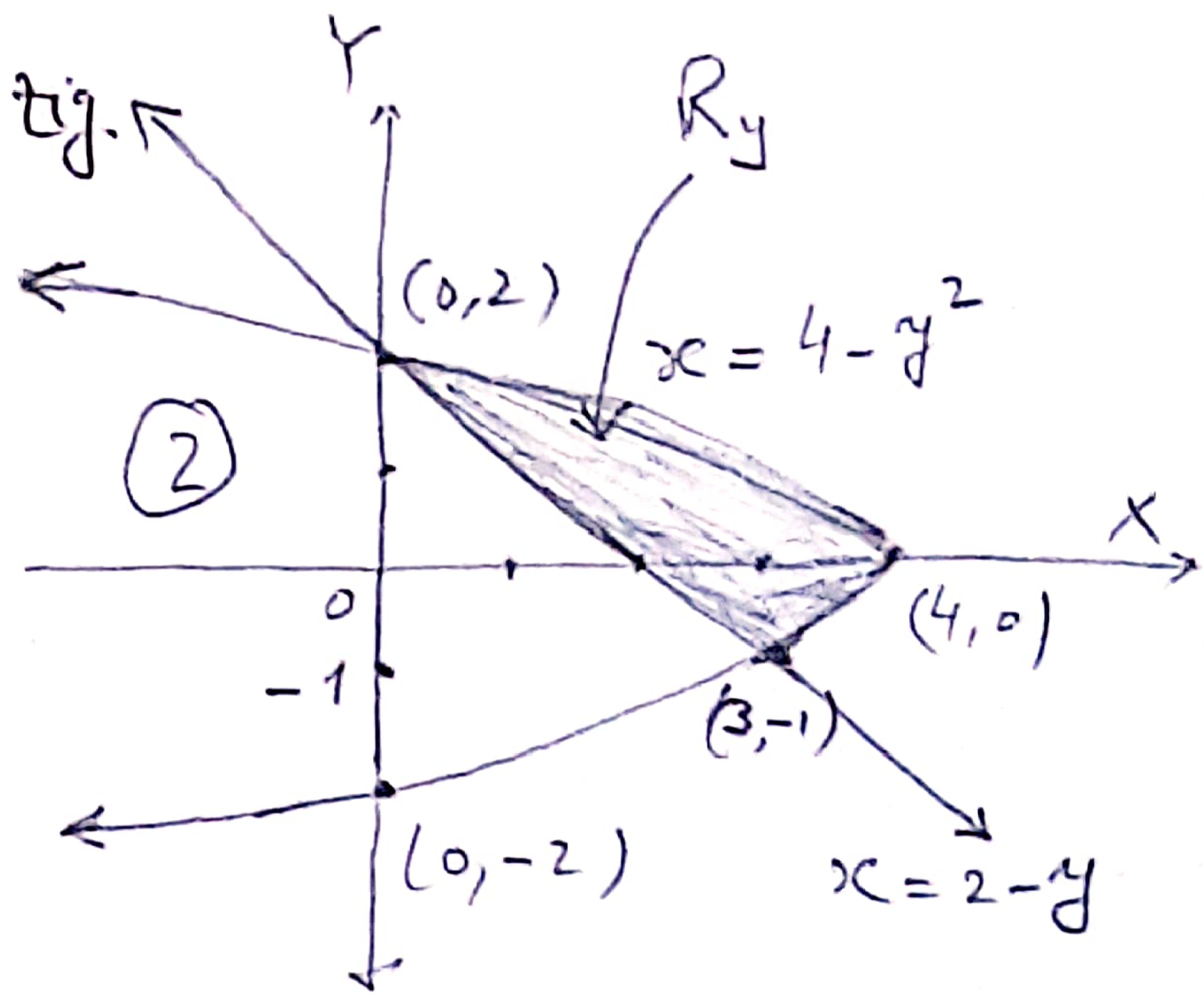
$$= \int_{-1}^{0}\int_{-x-1}^{x+1} (x^2+y^2).\,dy.\,dx \quad (1)$$

$$= \int_{-1}^{0} \left| x^2 y + \frac{y^3}{3} \right|_{-x-1}^{x+1} .dx = \int_{-1}^{0} \left[x^2(x+1) + \frac{(x+1)^3}{3} + x^2(x+1) + \frac{(x+1)^3}{3} \right] dx$$

$$= 2.\int_{-1}^{0} \left[x^2(x+1) + \frac{(x+1)^3}{3} \right] dx = 2 \left| \frac{x^4}{4} + \frac{x^3}{3} + \frac{(x+1)^4}{12} \right|_{-1}^{0}$$

$$= 2.\left[0 + 0 + \frac{1}{12} - \frac{1}{4} + \frac{1}{3} \right] = 2 \times \frac{1-3+4}{12} = \frac{1}{3} \quad (2)$$

Sol. Q.9 (5) The region R is shown in fig.



It is an R_y-region.

Left B'dary: $x = 2-y$

Right B'dary: $x = 4-y^2$

Limits of y: $-1 \le y \le 2$

$$\therefore V = \int_{-1}^{2}\int_{2-y}^{4-y^2} (x^2+4).\,dx.\,dy \quad (1)$$

$$= \int_{-1}^{2} \left| \frac{x^3}{3} + 4x \right|_{2-y}^{4-y^2} dy = \int_{-1}^{2} \left[\frac{(4-y^2)^3}{3} + 4(4-y^2) - \frac{(2-y)^3}{3} - 4(2-y) \right] dy$$

$$= \int_{-1}^{2} \frac{1}{3}(64 - y^6 - 48y^2 + 12y^4)\,dy + 4\left| 4y - \frac{y^3}{3} \right|_{-1}^{2} + \frac{1}{3} \left| \frac{(2-y)^4}{4} \right|_{-1}^{2} - 4\left| 2y - \frac{y^2}{2} \right|_{-1}^{2} \quad (2)$$

$$= \frac{1}{3} \left| 64y - \frac{y^7}{7} - 16y^3 + \frac{12y^5}{5} \right|_{-1}^{2} + 36 - \frac{27}{4} - 18 = \frac{1269}{35} - \frac{27}{4} + 18 = \frac{6651}{140} \text{ c.units}$$

$$= 47.51 \text{ c.units.}$$