Differential and Integral Calculus (MATH-205)

MT-II Exam/Fall 2023 Time Allowed: 1.5 Hours

Date: Wed., November 8, 2023 **Maximum Marks:** 25

Note: Attempt all FIVE questions and give detailed solutions. Read statements of the questions carefully and make sure you have answered each question completely.

Question 1: (4°) Given $\mathbf{a} = 5\,\hat{\imath} - 6\,\hat{\jmath} - \hat{k}$, $\mathbf{b} = -2\,\hat{\imath} + 3\,\hat{\jmath} + \hat{k}$, and $\mathbf{c} = 3\,\hat{\imath} + \hat{k}$. Show that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$. Hence, find the volume of the parallelepiped (box) whose coterminous edges are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

Question 2: (6°) Show that p_1 and p_2, given below, are not parallel planes.

$$p_1 : \ 2x - y + 4z = 4, \qquad p_2 : \ x + 3y - 2z = 1$$

Find the line of intersection of these planes in symmetric form. Also, find the angle between these planes.

Question 3: (5°) Identify and describe the surface: $x^2 + 16\,y + 4\,z^2 = 0$. Find, sketch, and describe its traces in xy-, yz-, xz-, and $y = -2$ planes.

Question 4: (5°) Let C be the curve with parametric equations

$$C : \ x = t, \ y = t^2, \ z = t^3, \ t \geq 0$$

Find parametric equations for the tangent line to C at the point corresponding to $t = \sqrt{2}$.

Question 5: (5°) If the acceleration of an object is given by

$$\mathbf{a}(t) = (t+1)^{-\frac{3}{2}}\,\hat{\imath} + 2\,\hat{\jmath} + 6\ln(t+1)\,\hat{k}, \quad t \geq 0$$

Find the object's velocity and position functions given that the initial velocity is $\mathbf{v}(0) = \hat{\jmath} - \hat{k}$ and the initial position is $\mathbf{r}(0) = \hat{\imath} - 2\hat{\jmath} + 3\,\hat{k}$.

—- Good Luck —-

Sol. Q.1 (4)

Given $\vec{a} = [5, -6, -1]$, $\vec{b} = [-2, 3, 1]$, $\vec{c} = [3, 0, 1]$

$$\vec{b} \times \vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2 & 3 & 1 \\ 3 & 0 & 1 \end{vmatrix} = 3\hat{i} + 5\hat{j} - 9\hat{k}$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & -6 & -1 \\ 3 & 5 & -9 \end{vmatrix} = 59\hat{i} + 42\hat{j} + 43\hat{k} \quad \text{—(i)}$$

$$\vec{a} \cdot \vec{c} = 15 - 1 = 14, \quad \vec{a} \cdot \vec{b} = -10 - 18 - 1 = -29$$

$$\therefore (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c} = 14\vec{b} + 29\vec{c} = 59\hat{i} + 42\hat{j} + 43\hat{k} \quad \text{—(ii)}$$

From (i) & (ii), we get (2)

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

This completes the proof. Also $\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} 5 & -6 & -1 \\ -2 & 3 & 1 \\ 3 & 0 & 1 \end{vmatrix} = -6$ (2)

$\therefore$ volume of box $= |\vec{a} \cdot (\vec{b} \times \vec{c})| = 6$ cubic units

Sol. Q.2 (6)

Given $P_1 : 2x - y + 4z - 4 = 0$ —(i), $P_2 : x + 3y - 2z - 1 = 0$ —(ii)

Normal vectors of P_1 & P_2 are

$$\vec{n}_1 = 2\hat{i} - \hat{j} + 4\hat{k}, \quad \vec{n}_2 = \hat{i} + 3\hat{j} - 2\hat{k}$$ (1)

$\because \frac{2}{1} \neq \frac{-1}{3} \neq \frac{4}{-2}$, i.e, components of $\vec{n}_1$ & $\vec{n}_2$ are not proportional, $\nexists$ any $c \in \mathbb{R}$ s.t. $\vec{n}_1 = c\vec{n}_2$, i.e, $\vec{n}_1 \nparallel \vec{n}_2$.

For intersection of P_1 & P_2, we proceed as follows.

(i) $\Rightarrow$ $2x - y = 4 - 4z$ —(iii)

(ii) $\Rightarrow$ $x + 3y = 1 + 2z$ —(iv)

Solving (iii) & (iv), we get $x = \frac{13}{7} - \frac{10}{7}z$, $y = -\frac{2}{7} + \frac{8}{7}z$. Let $z = t$,

then we have

$l : x = \frac{13}{7} - \frac{10}{7}t, \; y = -\frac{2}{7} + \frac{8}{7}t, \; z = t, \; t \in \mathbb{R}$ —(v)

This is the parametric equation of the line of intersection of P_1 & P_2. The symmetric form of l is 3

$$l: \frac{x-13/7}{-\frac{10}{7}} = \frac{y+2/7}{8/7} = \frac{z-0}{1} \quad \text{or} \quad \frac{x-13/7}{-10} = \frac{y+2/7}{8} = \frac{z}{7}$$

The angle θ b/w P_1 & P_2 is the angle b/w $\vec{n}_1$ & $\vec{n}_2$.

$$\therefore \theta = \cos^{-1}\left(\frac{\vec{n}_1 \cdot \vec{n}_2}{\|\vec{n}_1\|\,\|\vec{n}_2\|}\right) = \cos^{-1}\left(\frac{-9}{\sqrt{21}\cdot\sqrt{14}}\right) = \cos^{-1}(-0.52)$$

$= 121^\circ.33$ or 2.12 rad 2

Sol. Q.3 5

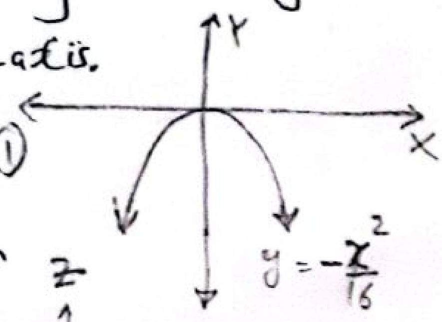
Given $S: x^2+16y+4z^2=0$ or $\frac{x^2}{16}+\frac{z^2}{4}=-y$ — (i)

Identify & Describe: It is a paraboloid having vertex at $(0,0,0)$ and axis along y-axis, opening along $-$ve y-axis. 1

✓ XY-Trace: Put $z=0$,

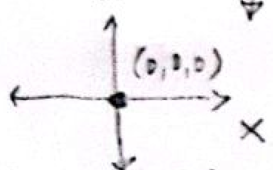
(i) $\Rightarrow$ $y=-\frac{1}{16}x^2$, it is a parabola having axis along y-axis and facing in the direction of $-$ve y-axis.

1



✓ XZ-Trace: Put $y=0$,

(i) $\Rightarrow$ $x^2+16z^2=0$, which is true only when $x=0$, $z=0$, i.e, XZ-trace is $(0,0,0)$.



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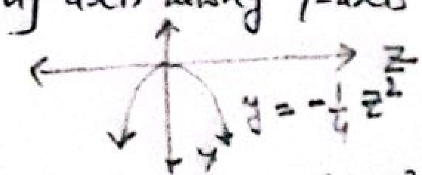
✓ YZ-Trace: Put $x=0$,

(i) $\Rightarrow$ $y=-\frac{1}{4}z^2$, it is a parabola having axis along y-axis and facing in the $-$ve y-direction.



✓ Trace in $y=-2$:

$x^2+4z^2=+32 \Rightarrow \frac{x^2}{32}+\frac{z^2}{8}=1$ (Ellipse in the plane $y=-2$,

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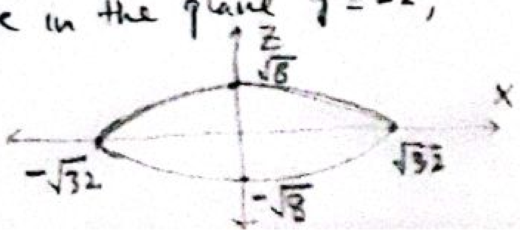

Sol. Q.4) 5

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Given space curve, $C: x=t,\ y=+t^2,\ z=t^3,\ t\geq 0$ —(i)

It is determined by the vector function,

$$\vec{r}(t) = t\hat{i} + t^2\hat{j} + t^3\hat{k},\ t\geq 0 \quad \text{—(ii)}$$

At $t=\sqrt{2}$, $\vec{r}(t) = \sqrt{2}\hat{i} + 2\hat{j} + 2\sqrt{2}.\hat{k}$, it is the position vector of pt. of tangency, $P(\sqrt{2}, 2, 2\sqrt{2})$. 2

Director of the tangent line at any pt. 't' is

$$\dot{\vec{r}}(t) = \hat{i} + 2t\hat{j} + 3t^2\hat{k},\ t\geq 0$$

$\Rightarrow \dot{\vec{r}}(\sqrt{2}) = \hat{i} + 2\sqrt{2}\hat{j} + 6\hat{k}$, is the director of the tangent line to C at pt $t=\sqrt{2}$. Therefore, parametric eqs. of the tangent line to C at pt. $t=\sqrt{2}$ are, 3

$$l:\ x=\sqrt{2}+s,\ y = 2+2\sqrt{2}s,\ z=2\sqrt{2}+6s,\ s\in\mathbb{R}$$

Sol. Q.5 5

Acceleration of the object is

$$\vec{a}(t) = \frac{d^2\vec{r}}{dt} = (t+1)^{-3/2}\hat{i} + 2\hat{j} + 6.\ln(t+1)\hat{k},\ t\geq 0$$

Integrating both sides w.r.t. 't', we get

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = -2(t+1)^{-1/2}\hat{i} + 2t\hat{j} + 6\int \ln(t+1)\,dt.\hat{k}, \text{ using integration by parts}$$

$$\Rightarrow \vec{v}(t) = -2(t+1)^{-1/2}\hat{i} + 2t\hat{j} + 6\left[(t+1).\ln(t+1) - t\right]\hat{k} + \underset{\text{Const. of integration}}{\vec{C}}$$

$\because \vec{v}(0) = \hat{j} - \hat{k}$,

$\therefore \hat{j} - \hat{k} = -2\hat{i} + \vec{c} \Rightarrow \vec{c} = 2\hat{i} + \hat{j} - \hat{k}$ 2

$$\therefore \vec{v}(t) = \left[-2(t+1)^{-1/2} + 2\right]\hat{i} + \left[2t+1\right]\hat{j} + \left[6\{(t+1)\ln(t+1) - t\} - 1\right]\hat{k}$$

Integrating again w.r.t. 't', we get

$$\vec{r}(t) = \left[-4(t+1)^{1/2} + 2t\right]\hat{i} + \left[t^2+t\right]\hat{j} + \left[6\left\{\frac{(t+1)^2}{2}.\ln(t+1) - \frac{1}{4}(t+1)^2 - \frac{t^2}{2}\right\} - t\right]\hat{k} + \vec{d}$$

where $\vec{d}$ is constt. of integration. Since $\vec{r}(0) = \hat{i} - 2\hat{j} + 3\hat{k}$, therefore,

$$\hat{i} - 2\hat{j} + 3\hat{k} = -4\hat{i} + \hat{j} - \frac{3}{2}\hat{k} + \vec{d} \Rightarrow \vec{d} = 5\hat{i} - 3\hat{j} + \frac{9}{2}\hat{k}$$

$$\therefore \vec{r}(t) = \left[-4(t+1)^{1/2} + 2t + 5\right]\hat{i} + \left[t^2+t-3\right]\hat{j} + \left[6\left\{\frac{(t+1)^2}{2}.\ln(t+1) - \frac{1}{4}(t+1)^2 - \frac{t^2}{2}\right\} - t + \frac{9}{2}\right]\hat{k}$$

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