Differential and Integral Calculus (MATH-205)

MT-I Exam/Fall 2023 Time Allowed: 1.5 Hours

Date: Wednesday, October 4, 2023 **Maximum Marks:** 25

Note: Attempt all FIVE questions and give detailed solutions. Read statements of the questions carefully and make sure you have answered each question completely.

Question 1: (4°) Find the nth term of the sequence $-\frac{1}{2}, \frac{4}{5}, -\frac{9}{10}, \frac{16}{17}, -\frac{25}{26}, \cdots$. Determine whether it converges or diverges, and if it converges, find its limit.

Question 2: (5°) Determine the nth partial sum of the series $\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$. Hence, determine whether it converges or diverges. Find its sum, if it converges.

Question 3: (4°) Determine whether the series $\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{1+n^2}$ converges or diverges.

Question 4: (6°) Determine whether the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{\ln(n+1)}$ absolutely convergent, conditionally convergent, or divergent.

Question 5: (6°) Find the radius and interval of convergence of the power series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2x-1)^n}{n\, 10^n}$.

— Good Luck —

Sol. I (4)

Given sequence is $-\frac{1}{2}, \frac{4}{5}, -\frac{9}{10}, \frac{16}{17}, -\frac{25}{26}, \ldots$

The nth term of the sequence is (2)

$$a_n = (-1)^n \cdot \frac{n^2}{n^2+1}, \quad n \geq 1$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n \cdot \frac{n^2}{1+n^2} = \lim_{n\to\infty} (-1)^n \cdot \frac{1}{1+\frac{1}{n^2}} = \begin{cases} -1, & n \text{ odd} \\ +1, & n \text{ even} \end{cases}$$

$\Rightarrow \lim_{n\to\infty} a_n$ DNE. Hence the given sequence is a divergent sequence. It has no sum. (2)

Sol. II (5) Given series is $\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$

using PFs, we get

$$a_n = \frac{1}{4n^2-1} = \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right), \quad n \geq 1$$

Now $S_1 = a_1 = \frac{1}{2}\left(1 - \frac{1}{3}\right)$

$S_2 = a_1 + a_2 = \frac{1}{2}\left(1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{5}\right) = \frac{1}{2}\left(1 - \frac{1}{5}\right)$

$S_3 = a_1 + a_2 + a_3 = \frac{1}{2}\left(1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \frac{1}{5} - \frac{1}{7}\right) = \frac{1}{2}\left(1 - \frac{1}{7}\right), \ldots$ and so on

$$S_n = \frac{1}{2}\left(1 - \frac{1}{2n+1}\right) \quad (3)$$

$$\lim_{n\to\infty} S_n = \frac{1}{2}\lim_{n\to\infty}\left(1 - \frac{1}{2n+1}\right) = \frac{1}{2}$$

$\Rightarrow \{S_n\}_{n=1}^{\infty}$ Converges. Hence, $\sum_{n=1}^{\infty} \frac{1}{4n^2-1}$ Converges and its sum is $1/2$. (2)

S.P. III : (4)

Given infinite series is $\sum_{n=1}^{\infty} \frac{\tan^{-1} n}{1+n^2}$. It is a +ve term series.

Here $a_n = \frac{\tan^{-1} n}{1+n^2}$, $n \geq 1$. Let $f(n) = a_n$, then

$f(x) = \frac{\tan^{-1} x}{1+x^2}$, $x \geq 1$. clearly $f(x) > 0, \forall x \in [1, \infty)$

& $f \in C[1, \infty)$.

$$f'(x) = \frac{1}{(1+x^2)^2} + \tan^{-1} x \cdot \frac{-2x}{(1+x^2)^2} = \frac{1 - 2x \cdot \tan^{-1} x}{(1+x^2)^2} < 0, \forall x \in [1, \infty)$$ (1)

$\therefore f(x) \downarrow$ on $[1, \infty)$.

Consider, $$I = \int_1^{\infty} \frac{\tan^{-1} x}{1+x^2} \, dx = \lim_{T \to \infty} \int_1^T \frac{\tan^{-1} x}{1+x^2} \, dx \quad \text{--- (1)}$$

$$\because \int \frac{\tan^{-1} x}{1+x^2} \, dx = \frac{(\tan^{-1} x)^2}{2}$$

$$\therefore I = \lim_{T \to \infty} \left| \frac{(\tan^{-1} x)^2}{2} \right|_1^T = \frac{1}{2} \cdot \lim_{T \to \infty} \left((\tan^{-1}(T))^2 - (\tan^{-1} 1)^2 \right)$$

$$= \frac{1}{2} \left[(\tan^{-1}(\infty))^2 - (\tan^{-1}(1))^2 \right] = \frac{1}{2} \left[\left(\frac{\pi}{2}\right)^2 - (\pi/4)^2 \right] = \frac{1}{2} \left(\frac{\pi^2}{4} - \frac{\pi^2}{16} \right) = \frac{3\pi^2}{16}$$

$\Rightarrow \int_1^{\infty} \frac{\tan^{-1} x}{1+x^2} \, dx$ Converges. Hence, the given series is also Convergent. (3)

Sd. IV (6)

Given Infinite series: $\sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{2}{\ln(n+1)}$, AS

The corresponding absolute term series is $\sum_{n=1}^{\infty} \frac{2}{\ln(n+1)}$

Here $a_n = \frac{2}{\ln(n+1)} > 0, \forall n \geq 1$. Let $b_n = \frac{1}{n}, n \geq 1$ then

$\sum b_n = \sum \frac{1}{n}$ is a divergent harmonic series.

Ratio: $\frac{a_n}{b_n} = \frac{2}{\ln(n+1)} \cdot \frac{n}{1} = \frac{2n}{\ln(n+1)}$

$\Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = 2 \cdot \lim_{n\to\infty} \frac{n}{\ln(n+1)} \left(\frac{\infty}{\infty}\right) = 2 \cdot \lim_{n\to\infty} \frac{1}{1/(n+1)}$, L'Hospital Rule

$= 2 \cdot \lim_{n\to\infty} (n+1) = \infty$

$\because \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges, Hence by LCT, the ~~given~~ series $\sum_{n=1}^{\infty} \frac{2}{\ln(n+1)}$

also diverges. Therefore, given AS is not AC. (3)

Lets use AST:

C-I Consider $\frac{a_{k+1}}{a_k} = \frac{1}{\ln(k+1)} \cdot \ln k = \frac{\ln k}{\ln(k+1)} < 1, \forall k \geq 1$

$\Rightarrow a_{k+1} < a_k, \forall k \geq 1$, i.e, $\{a_n\}_{n=1}^{\infty}$ is monotonically ↓.

C-II $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{\ln(n+1)} = \frac{1}{\ln \infty} = \frac{1}{\infty} = 0$

$\therefore$ C-II is satisfied

Hence, the given AS is Convergent by AST. Hence, it is CC. (3)

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Sol. V (6) P7

Given Power Series is $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} (2x-1)^n}{n \times 10^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot 2^n (x-1/2)^n}{n \times 10^n}$

It is a generalized power series with $c = 1/2$.

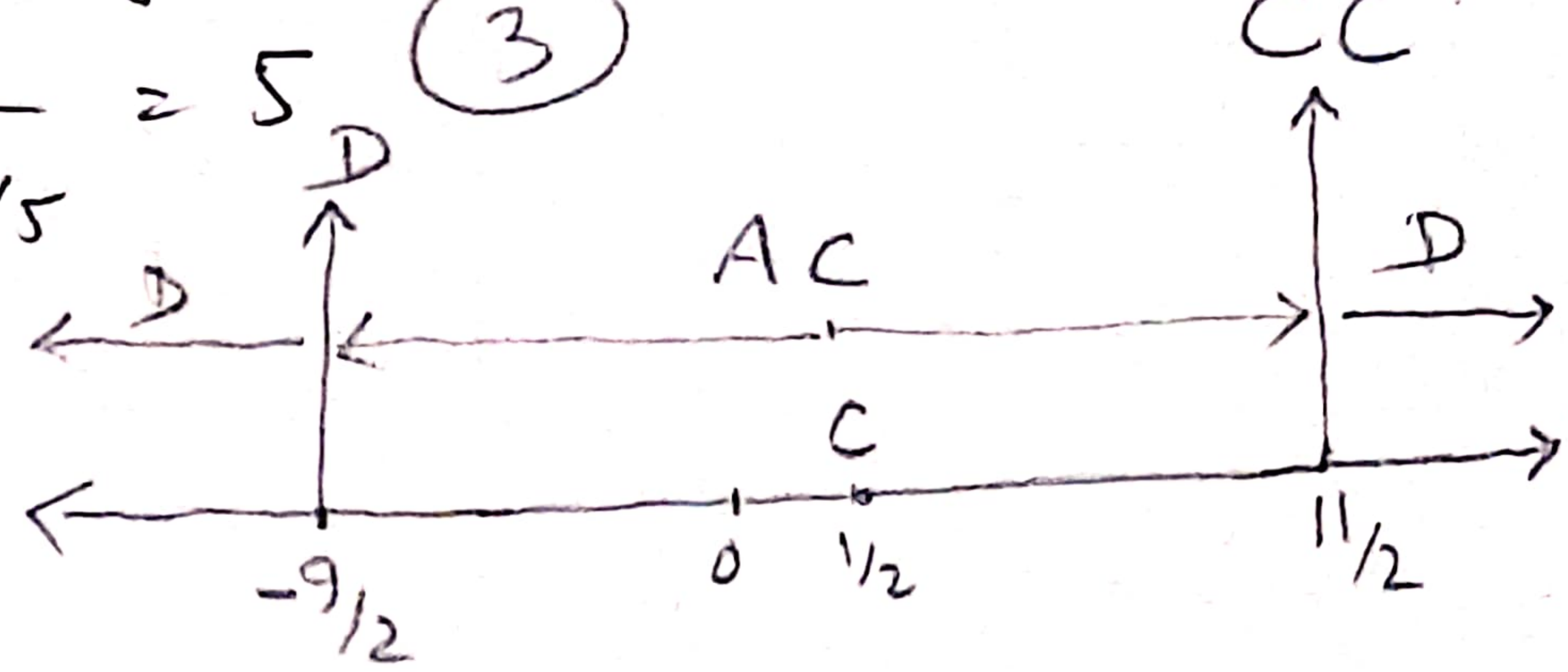
Here $a_n = \frac{(-1)^{n+1} \cdot 2^n}{n \times 10^n}$, $n \geq 1$

Ratio: $\frac{a_{n+1}}{a_n} = \frac{(-1)^{n+1} \cdot 2^{n+1}}{(n+1) \times 10^{n+1}} \cdot \frac{n \times 10^n}{(-1)^{n+1} \cdot 2^n} = -\frac{2}{10} \cdot \frac{1}{(1+\frac{1}{n})}$

$\Rightarrow \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1}{5} \cdot \frac{1}{1+\frac{1}{n}} = \frac{1}{5}$

$\therefore$ Radius of Convergence of the given power series is (3)

$R = \frac{1}{\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|} = \frac{1}{1/5} = 5$



$\therefore$ The given power series is AC on $(-5+\frac{1}{2}, 5+\frac{1}{2}) = (-\frac{9}{2}, \frac{11}{2})$

and it is divergent on $(-\infty, -9/2)$ & $(11/2, \infty)$.

At $x = -9/2$: From the given series,

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot (-10)^n}{n \times 10^n} = \sum_{n=1}^{\infty} \frac{(-1)^{2n+1} \cdot 10^n}{n \times 10^n} = -\sum_{n=1}^{\infty} \frac{1}{n}$ which is a divergent harmonic series.

At $x = 11/2$: From the given series,

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} \cdot 10^n}{n \times 10^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$, It is an AHS which is CC by AST.

Hence interval of Convergence of the given power series is

$(-\frac{9}{2}, \frac{11}{2}]$ (3)