Solution MT-SP22

MATH-205

Q.1 (4) Given Sequence $\left\{ \frac{(-1)^n \cdot \sqrt{n} \cdot \sin\sqrt{n}}{n+1} \right\}_{n=1}^{\infty}$

nth term: $a_n = \frac{(-1)^n \cdot \sqrt{n} \cdot \sin\sqrt{n}}{n+1}$, $n \geq 1$

1st Term: $a_1 = -\frac{\sin 1}{2}$; 2nd Term: $a_2 = \frac{\sqrt{2} \cdot \sin\sqrt{2}}{3}$

3rd Term: $a_3 = -\frac{\sqrt{3} \cdot \sin\sqrt{3}}{4}$; 4th Term: $a_4 = \frac{2\sin 2}{5}$

5th Term: $a_5 = -\frac{\sqrt{5} \cdot \sin\sqrt{5}}{6}$ (1)

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \left| \frac{(-1)^n \cdot \sqrt{n} \cdot \sin\sqrt{n}}{n+1} \right| = \lim_{n\to\infty} \frac{\sqrt{n} \cdot \sin\sqrt{n}}{n+1}$$

$$= \lim_{n\to\infty} \frac{n}{n+1} \cdot \lim_{n\to\infty} \frac{\sin\sqrt{n}}{\sqrt{n}} \quad \text{---} (1)$$

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$$

$\because 0 \leq \sin\sqrt{n} \leq 1 \Rightarrow 0 \leq \frac{\sin\sqrt{n}}{\sqrt{n}} \leq \frac{1}{\sqrt{n}}$

$$\Rightarrow 0 \leq \lim_{n\to\infty} \frac{\sin\sqrt{n}}{\sqrt{n}} \leq \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$$

$\Rightarrow$ By Sandwich theorem, $\lim_{n\to\infty} \frac{\sin\sqrt{n}}{\sqrt{n}} = 0$

$\therefore (1) \Rightarrow \lim_{n\to\infty} |a_n| = 1 \times 0 = 0$

$\Rightarrow \lim_{n\to\infty} a_n = 0$ $\quad \because \lim_{n\to\infty} |a_n| = 0 \Rightarrow \lim_{n\to\infty} a_n = 0$

$\therefore$ The given sequence Converges to 0, Its limiting value is 0. (3)

Q.2 Given Power Series $\sum_{n=0}^{\infty} \frac{(-1)^n . x^{n+1}}{n+1}$ —(1)

(6)

n^{th} term: $u_n = \frac{(-1)^n . x^{n+1}}{n+1}$,

Ratio: $\frac{u_{n+1}}{u_n} = -\frac{n+1}{n+2} . x \Rightarrow \left|\frac{u_{n+1}}{u_n}\right| = \frac{n+1}{n+2} . |x|$

$\Rightarrow \lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = |x|$

By ratio test for AC, the given power series is AC if $|x| < 1$ & it is divergent if $|x| > 1$.

For $|x| = 1$, i.e, for $x = \pm 1$, we check separately.

(1)

For $x = 1$:

(1) $\Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}$ It is an alternating harmonic series.

The corresponding absolute term series is $\sum_{n=0}^{\infty} \left|\frac{(-1)^n}{n+1}\right| = \sum_{n=0}^{\infty} \frac{1}{n+1}$

It is a divergent harmonic series. Now, use the AST.

C-I. $n+1 < n+2$, $\forall n \geq 0$

$\Rightarrow \frac{1}{n+1} > \frac{1}{n+2}$, $\forall n \geq 0 \Rightarrow a_n > a_{n+1}$, $\forall n \geq 2$

$\therefore$ C-I is satisfied.

C-II $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n+1} = 0 \Rightarrow$ C-II is satisfied.

$\therefore$ Series (1) converges conditionally (CC) for $x = 1$. (2)

For $x = -1$:

(1) $\Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^{2n+1}}{n+1} = \sum_{n=0}^{\infty} \frac{-1}{n+1} = -\sum_{n=0}^{\infty} \frac{1}{n+1}$

$\because \sum_{n=0}^{\infty} \frac{1}{n+1}$ diverges $\therefore -\sum_{n=0}^{\infty} \frac{1}{n+1}$ also diverges.

$\therefore$ IC $= (-1, 1]$ (1)

Given power series is P+2

$$\sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

we know that

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots, \quad |x| < 1$$

$$\Rightarrow \int \frac{dx}{1+x} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$\because$ I.C. = $(-1, 1]$.

$$\Rightarrow \ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{n+1}}{n+1}, \quad -1 < x \le 1$$

(2)

$\therefore$ Given power series represents $\ln(1+x)$.

Q3 (5) $\int_0^{1/2} x \cdot \cos x^3 \cdot dx \approx ?$

we know that Maclaurin's series of $\cos x$ is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\Rightarrow \cos x^3 = 1 - \frac{x^6}{2!} + \frac{x^{12}}{4!} - \frac{x^{18}}{6!} + \cdots$$

$$\Rightarrow x \cdot \cos x^3 = x - \frac{x^7}{2!} + \frac{x^{13}}{4!} - \frac{x^{19}}{6!} + \cdots$$

(2)

$$\Rightarrow x \cdot \cos x^3 \approx x - \frac{x^7}{2!} + \frac{x^{13}}{4!}$$

(Taking only 1st 3 non-zero terms).

$$\Rightarrow \int_0^{1/2} x \cdot \cos x^3 \cdot dx \approx \int_0^{1/2} \left(x - \frac{x^7}{2!} + \frac{x^{13}}{4!}\right) \cdot dx = \left[\frac{x^2}{2} - \frac{x^8}{2! \, 8} + \frac{x^{14}}{4! \, 14}\right]_0^{1/2}$$

$$\Rightarrow \int_0^{1/2} x \cdot \cos x^3 \cdot dx \approx \frac{1}{8} - \frac{1}{4096} + \frac{1}{5505024} = ~~0.5~~ \; 0.125 - ~~0.06250~~ \; 0.00024 + ~~0.00248~~ \; 0.00000$$

$$\Rightarrow \int_0^{1/2} x \cdot \cos x^3 \cdot dx \approx ~~0.44048~~ \; ~~0.49776~~ \; 0.12476$$

(3)

Q.4 $\vec{a} = [2, 0, -1]$, $\vec{b} = [-3, 1, 0]$, $\vec{c} = [1, -2, 4]$

(5)

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 0 & -1 \\ -3 & 1 & 0 \end{vmatrix} = \hat{i} + 3\hat{j} + 2\hat{k}$$

let θ be the angle b/w the vectors $\vec{a} \times \vec{b}$ & $\vec{c}$.

$$\therefore \cos\theta = \frac{(\vec{a} \times \vec{b}) \cdot \vec{c}}{\|\vec{a} \times \vec{b}\| \, \|\vec{c}\|} = \frac{[1, +3, 2] \cdot [1, -2, 4]}{\sqrt{14} \cdot \sqrt{21}} = \frac{3}{\sqrt{294}} = 0.175$$

$$\Rightarrow \theta = \cos^{-1}(0.175) = 79.92^\circ \approx 80^\circ$$ (3)

$$\text{Comp}_{\vec{c}} \, \vec{a} \times \vec{b} = \|\vec{a} \times \vec{b}\| \cdot \cos\theta = \sqrt{14} \times 0.175 = 0.655$$ (2)

Q.5 Given planes:

(6)

$P_1: x - 5y + 3z = 11$ —(I), $P_2: -3x + 2y - 2z = -7$ —(II)

Normal vector of P_1: $\vec{n}_1 = [1, -5, 3]$

" " " P_2: $\vec{n}_2 = [-3, 2, -2]$

$\because \frac{1}{-3} \neq \frac{-5}{2} \neq \frac{3}{-2}$, i.e., components of $\vec{n}_1$ & $\vec{n}_2$ are not proportional. $\therefore \vec{n}_1$ & $\vec{n}_2$ are not $\parallel$. $\Rightarrow P_1 \nparallel P_2$. (2)

For line of intersection of P_1 & P_2:

(I) $\Rightarrow$ $x - 5y = 11 - 3z$ —(III)

(II) $\Rightarrow$ $3x - 2y = 7 - 2z$ —(IV)

3(III) − (IV) $\Rightarrow$ $-13y = 26 - 7z$ $\Rightarrow$ $y = -2 + \frac{7}{13}z$ —(V)

2(III) − 5(IV) $\Rightarrow$ $-13x = -13 + 4z$ $\Rightarrow$ $x = 1 - \frac{4}{13}z$ —(VI)

Let $z = t$, then we have,

$l: x = 1 - \frac{4}{13}t$, $y = -2 + \frac{7}{13}t$, $z = t$, $t \in \mathbb{R}$ —(VII)

This is the line of intersection of P_1 & P_2. (4)

Q.6 Given

(4) Vector differential Equation: $\ddot{\vec{r}}(t) = \frac{t}{(1+t^2)^2}\hat{i} + \frac{1}{(1+t)^2}\hat{j} + 2^{3t}\cdot\hat{k}$ —(I)

I^n Cds: $\vec{r}(0) = \frac{1}{2}\hat{i} + 2\hat{j} - \frac{2}{3\cdot\ln 2}\hat{k}$ —(II)

& $\dot{\vec{r}}(0) = 6\hat{i} + \hat{j}$ —(III)

Integrating (I) on both sides w.r.t. 't'

$$\int \ddot{\vec{r}}(t)\cdot dt = \int \frac{t}{(1+t^2)^2}\cdot dt\ \hat{i} + \int \frac{dt}{(1+t)^2}\ \hat{j} + \int 2^{3t}\cdot dt\ \hat{k}$$

$$\Rightarrow \dot{\vec{r}}(t) = \hat{i}\int (1+t^2)^{-2}\cdot t\cdot dt + \hat{j}\int (1+t)^{-2}\cdot dt + \hat{k}\int 2^{3t}\cdot dt$$

$$= -\frac{1}{2(1+t^2)}\cdot\hat{i} - \frac{1}{1+t}\cdot\hat{j} + \frac{2^{3t}}{3\ln 2}\cdot\hat{k} + \vec{C} \quad \text{—(IV)}$$

(1)

where $\vec{C}$ is arbitrary constt of integration

At $t = 0$; $\dot{\vec{r}}(0) = -\frac{1}{2}\hat{i} - \hat{j} + \frac{1}{3\cdot\ln 2}\hat{k} + \vec{C}$

$$\Rightarrow \frac{1}{2}\hat{i} + 2\hat{j} - \frac{2}{3\ln 2}\hat{k} = -\frac{\hat{i}}{2} - \hat{j} + \frac{1}{3\ln 2}\hat{k} + \vec{C}$$

using (II)

$$\Rightarrow \vec{C} = \hat{i} + 3\hat{j} - \frac{1}{\ln 2}\hat{k} \quad (1)$$

$$\therefore (IV) \Rightarrow \dot{\vec{r}}(t) = \left(-\frac{1}{2(1+t^2)} + 1\right)\hat{i} + \left(\frac{-1}{1+t} + 3\right)\hat{j} + \left(\frac{2^{3t}}{3\ln 2} - \frac{1}{\ln 2}\right)\hat{k}$$

Integrating again w.r.t. 't'

$$\int \dot{\vec{r}}(t)\cdot dt = \hat{i}\int \left[1 - \frac{1}{2(1+t^2)}\right]dt + \hat{j}\left[\int \left(3 - \frac{1}{1+t}\right)dt\right] + \hat{k}\int\left(\frac{2^{3t}}{3\cdot\ln 2} - \frac{1}{\ln 2}\right)dt$$

$$\Rightarrow \vec{r}(t) = \hat{i}\left(t - \frac{\tan^{-1} t}{2}\right) + \hat{j}\left(3t - \ln(1+t)\right) + \hat{k}\left(\frac{2^{3t}}{9\cdot(\ln 2)^2} - \frac{t}{\ln 2}\right) + \vec{d} \quad (1)$$

where $\vec{d}$ is arbitrary constt of integration.

At $t = 0$,

$$\vec{r}(0) = 0\cdot\hat{i} + 0\cdot\hat{j} + \frac{1}{9\cdot(\ln 2)^2}\hat{k} + \vec{d}$$

(iii) $\Rightarrow 6\hat{i} + \hat{j} = \frac{\hat{k}}{9(\ln 2)^2} + \vec{d} \Rightarrow \vec{d} = 6\hat{i} + \hat{j} - \frac{\hat{k}}{9(\ln 2)^2}$

$$\therefore \vec{r}(t) = \hat{i}\left[6 + t - \frac{\tan^{-1} t}{2}\right] + \hat{j}\left[1 + 3t - \ln(1+t)\right] + \hat{k}\left[-\frac{1}{9(\ln 2)^2} - \frac{t}{\ln 2} + \frac{2^{3t}}{9(\ln 2)^2}\right] \quad (1)$$