Differential and Integral Calculus (MATH-205)

MT Exam/Semester III (2022-23) Time Allowed: 2 Hours

Date: Sunday, May 14, 2023 **Maximum Marks:** 30

Note: Attempt all SIX questions and give detailed solutions. Read statements of the questions carefully and make sure you have answered each question completely.

Question 1: (4°) Determine whether the following sequence converges or diverges. What is its limiting value as $n \to \infty$?

$$\left\{\frac{n^2}{\sqrt{2n}-1} - \frac{n^2}{\sqrt{2n}+1}\right\}_{n=1}^{\infty}.$$

Question 2: (4°) Determine whether the series $\sum\limits_{n=1}^{\infty}\left(3 \times 2^{-n} - 2^{-3n}\right)$ converges or diverges. Find its sum, if it converges.

Question 3: (5°) Using integral test, determine whether the series $\sum\limits_{n=3}^{\infty} \frac{1}{n(2n-5)}$ converges or diverges.

Question 4: (6°) Using alternating series test, show that the alternating series $\sum\limits_{n=1}^{\infty}(-1)^n \frac{\ln n}{1.5^n}$ converges. Determine whether it converges absolutely or conditionally.

Question 5: (5°) Use the first 5 terms of an infinite series to find the approximate value of the following integral upto to 4 decimal points.

$$\int_0^1 \frac{1-e^{-x}}{x}\,dx$$

Question 6: (6°) Find the volume of a box whose coterminous edges are determined by the points $P(-3,0,5)$, $Q(2,-1,-3)$, $R(4,1,-1)$ and $S(2,-4,0)$. Find also the distance of the vertex P of the box from the line passing through points Q and R.

--- Good Luck ---

Sol. Q. 1 (4)

n^{th} term of the sequence, $a_n = \frac{n^2}{\sqrt{2n-1}} - \frac{n^2}{\sqrt{2n+1}}$, $n \geq 1$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{n^2}{\sqrt{2n-1}} - \frac{n^2}{\sqrt{2n+1}}\right), \quad \infty - \infty$$

$$= \lim_{n\to\infty} \frac{\cancel{\sqrt{2}n^{5/2}} + n^2 - \cancel{\sqrt{2}n^{5/2}} + n^2}{2n-1} = \lim_{n\to\infty} \frac{2n^2}{2n-1}, \quad \frac{\infty}{\infty}$$

(3)

$$= \lim_{n\to\infty} \frac{2n}{2-\frac{1}{n}} = \frac{\infty}{2} = \infty$$

∴ The given sequence is a divergent sequence. It has no limiting value. (1)

Sol Q.2 (4) Given infinite series, $\sum_{n=1}^{\infty}\left(3\cdot 2^{-n} - 2^{-3n}\right)$

Let $a_n = \frac{3}{2^n}$, $n \geq 1$ & $b_n = \frac{1}{2^{3n}}$, $n \geq 1$

then $\sum a_n = \sum_{n=1}^{\infty} \frac{3}{2^n} = 3 \cdot \sum_{n=1} \frac{1}{2^n}$ is a convergent G.S.

with $r = \frac{1}{2} < 1$, and $\sum b_n = \sum_{n=1}^{\infty} \frac{1}{2^{3n}} = \sum_{n=1}^{\infty} \frac{1}{8^n}$

is also a convergent G.S. with $r = \frac{1}{8} < 1$.

∴ $\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty}\left(3\cdot 2^{-n} - 2^{-3n}\right)$ is also a convergent series

(3)

Sum of $\sum_{n=1}^{\infty} 3\cdot 2^{-n}$ is $S_A = 3 \cdot \frac{1/2}{1-1/2} = 3$

& Sum of $\sum_{n=1}^{\infty} 2^{-3n}$ is $S_B = \frac{1/8}{1-1/8} = \frac{1}{8}\cdot\frac{8}{7} = \frac{1}{7}$.

∴ Sum of the given series is,

$$S = S_A - S_B = 3 - \frac{1}{7} = \frac{20}{7}$$ (1)

Sol. Q.3 / 5

Given $\sum_{n=3}^{\infty} \frac{1}{n(2n-5)}$, Here $a_n = \frac{1}{n(2n-5)} > 0,\ \forall\ n \geq 3$

$\therefore \sum_{n=3}^{\infty} a_n$ is a +ve term series.

Let $f(n) = a_n = \frac{1}{n(2n-5)}$, $n \geq 3$, and replace n by 'x', then

$$f(x) = \frac{1}{x(2x-5)},\ x \in [3, \infty)$$

c-I Clearly, $f(x) = \frac{1}{x(2x-5)} > 0,\ \forall\ x \in [3, \infty)$

c-ii Clearly, $f \in C[3, \infty)$

c-iii $f'(x) = \frac{-1}{x^2(2x-5)} - \frac{2}{x(2x-5)^2} = \frac{5-4x}{x^2(2x-5)^2} < 0,\ \forall\ x \in [3, \infty)$ 1

$\therefore f(x)$ is $\downarrow$ on $[3, \infty)$.

Consider,

$$I = \int_3^{\infty} \frac{dx}{x(2x-5)} = \lim_{T \to \infty} \int_3^T \frac{dx}{x(2x-5)} \quad \text{——— (I)}$$

$$\int \frac{dx}{x(2x-5)} \overset{PFs}{=} \int \left(\frac{-1/5}{x} + \frac{2/5}{2x-5} \right) dx = -\frac{1}{5} \cdot \ln x + \frac{1}{5} \ln|2x-5|$$

$$= \frac{1}{5} \cdot \ln\left(\frac{2x-5}{x}\right) = \frac{1}{5} \cdot \ln\left(2 - \frac{5}{x}\right)$$

$\therefore$ From (I),

$$I = \frac{1}{5} \cdot \lim_{T \to \infty} \left[\ln\left(2 - \frac{5}{x}\right) \right]_3^T = \frac{1}{5} \cdot \lim_{T \to \infty} \left[\ln\left(2 - \frac{5}{T}\right) - \ln\left(2 - \frac{5}{3}\right) \right]$$ 3

$$= \frac{1}{5} \cdot \left[\ln 2 - \ln\left(\frac{1}{3}\right) \right] = \frac{1}{5} \cdot \ln 6$$

$\Rightarrow$ The definite integral $\int_3^{\infty} \frac{dx}{x(2x-5)}$ Converges. Hence, the given series $\sum_{n=3}^{\infty} \frac{1}{n(2n-5)}$ Converges by integral test. 1

Sol. Q.5 5

Given $I = \int_0^1 \frac{1-e^{-x}}{x} \cdot dx$, Here $f(x) = \frac{1-e^{-x}}{x}$.

we know that

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots + (-1)^{n} \cdot \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} (-1)^{n} \cdot \frac{x^n}{n!}$$

$$\Rightarrow -e^{-x} = \sum_{n=0}^{\infty} (-1)^{n+1} \cdot \frac{x^n}{n!} \quad \Rightarrow 1 - e^{-x} = \cancel{1} - \cancel{1} + \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{x^n}{n!}$$

$$\Rightarrow \frac{1-e^{-x}}{x} = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{x^{n-1}}{n!}$$

3

$$= 1 - \frac{x}{2!} + \frac{x^2}{3!} - \frac{x^3}{4!} + \frac{x^4}{5!}$$, 1st five terms

$$\therefore I \approx \int_0^1 \frac{1-e^{-x}}{x} dx = \int_0^1 \left(1 - \frac{x}{2!} + \frac{x^2}{3!} - \frac{x^3}{4!} + \frac{x^4}{5!}\right) dx$$

$$\approx \left[x - \frac{x^2}{2.2!} + \frac{x^3}{3.3!} - \frac{x^4}{4.4!} + \frac{x^5}{5.5!}\right]_0^1$$

$$\approx 1 - \frac{1}{4} + \frac{1}{18} - \frac{1}{96} + \frac{1}{600} = 1 - 0.25 + 0.0556 - 0.0104 + 0.0017$$

$$\approx 0.7969$$ 2

Sol Q.6 / 6

Let the 3 coterminus edges of the box be $\vec{PQ}$, $\vec{PR}$ & $\vec{PS}$.

$\vec{PQ} = [5, -1, -8]$, $\vec{PR} = [7, 1, -6]$, and $\vec{PS} = [5, -4, -5]$

Volume of the box $= |\vec{PQ} \cdot \vec{PR} \times \vec{PS}|$ 3

$$\because \vec{PQ} \cdot \vec{PR} \times \vec{PS} = \begin{vmatrix} 5 & -1 & -8 \\ 7 & 1 & -6 \\ 5 & -4 & -5 \end{vmatrix} = 114$$, $\therefore V = 114$ cubic units

The distance of vertex/Corner P from the line through Q & R is

$$d = \frac{\|\vec{QP} \times \vec{QR}\|}{\|\vec{QP}\|}, \quad \because \vec{QP} \times \vec{QR} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -5 & 1 & 8 \\ 2 & 2 & 2 \end{vmatrix} = -14\hat{i} + 26\hat{j} - 12\hat{k}$$

$$\therefore d = \sqrt{\frac{1016}{90}} = \sqrt{\frac{508}{45}} = 3.36$$ units 3

$$\Rightarrow \|\vec{QP} \times \vec{QR}\| = \sqrt{196 + 676 + 144} = \sqrt{1016}$$

Also $\|\vec{QP}\| = \sqrt{25 + 1 + 64} = \sqrt{90}$

Sol Q.4 (6)

Given A.S. is $\sum_{n=1}^{\infty} \frac{(-1)^n \cdot \ln n}{1.5^n}$, Here $a_n = \frac{\ln n}{1.5^n}$, $n \geq 1$

AST:

C-I: let $f(n) = a_n = \frac{\ln n}{1.5^n}$, $n \geq 1$, then

$$f'(n) = \frac{1}{n \cdot 1.5^n} - \frac{(\ln n) \cdot \ln(1.5)}{1.5^n} = \frac{1 - (\ln 1.5) \cdot n \cdot \ln n}{n \cdot 1.5^n} < 0, \ \forall n \geq 2$$

$\therefore$ $a_{n+1} \leq a_n$, $\forall n \geq 2$, C-I is satisfied.

C-II $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln n}{1.5^n} \left(\frac{\infty}{\infty}\right) = \lim_{n\to\infty} \frac{1}{n \cdot 1.5^n \cdot \ln 1.5}$ L'Hospital's Rule

$= 0$

$\therefore$ C-II is satisfied. Hence the given A.S. Converges by A.S.T.

(3)

Absolute Convergence:

The Corresponding absolute ~~value~~ term series is

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n \cdot \ln n}{1.5^n} \right| = \sum_{n=1}^{\infty} \frac{\ln n}{1.5^n}$$

$\because$

$$\frac{a_{n+1}}{a_n} = \frac{\ln(n+1)}{1.5^{n+1}} \cdot \frac{1.5^n}{\ln n} = \frac{1}{1.5} \cdot \frac{\ln(n+1)}{\ln n}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{1.5} \cdot \lim_{n\to\infty} \frac{\ln(n+1)}{\ln n} \left(\frac{\infty}{\infty}\right) = \frac{1}{1.5} \cdot \lim_{n\to\infty} \frac{n}{n+1} = \frac{1}{1.5} < 1$$

$\therefore$ By ratio test the Series $\sum_{n=1}^{\infty} \left| \frac{(-1)^n \cdot \ln n}{1.5^n} \right|$ Converges.

Hence, the given Series Converges absolutely. (3)