Handout 2

PHYS 505/551 HANDOUT-2

3) $l_- l_+ = l^2 - l_z(l_z+\hbar)$

$l_- l_+ = (l_x - i l_y)(l_x + i l_y) =$

$l_x^2 + i l_x l_y - i l_y l_x + l_y^2$

$= l_x^2 + l_y^2 + i(l_x l_y - l_y l_x)$

$= l^2 - l_z^2 + i[l_x, l_y]$

$= l^2 - l_z^2 + i\, l_z (i\hbar)$

$= l^2 - l_z(l_z + \cancel{i}\hbar)$

$l_+ l_- = (l_x + i l_y)(l_x - i l_y) =$

$= l_x^2 - i l_x l_y + i l_y l_x + l_y^2 =$

$= l_x^2 + l_y^2 - i(l_x l_y - l_y l_x)$

$= l^2 - l_z^2 - i[l_x, l_y] =$

$= l^2 - l_z^2 - i(i\hbar l_z) =$

$= l^2 - l_z^2 + \hbar l_z = l^2 - l_z(l_z - \hbar)$

$[l_+, l_-] = l_+ l_- - l_- l_+ = \cancel{l^2} - l_z^2 + \hbar l_z - \cancel{l^2} + l_z^2 + \hbar l_z$

$= 2\hbar l_z$

) From (2.12)

$$l_x = \frac{(l_+ + l_-)}{2}$$

$$l_x^2 = \frac{1}{4}(l_+ + l_-)^2 = \frac{1}{4}(l_+ - l_-)(l_+ - l_-)$$

$$= \frac{1}{4}\left(l_+^2 + l_+ l_- + l_- l_+ + l_-^2\right)$$

But $\langle l_+^2 \rangle = \langle l_-^2 \rangle = 0$

$$\langle l_+^2 \rangle = \langle Y_l^m | l_+^2 | Y_l^m \rangle = \hbar\sqrt{l(l+1)-m(m+1)}\ \langle Y_l^m | l_+ | Y_l^{m+1} \rangle$$

$$= \hbar\sqrt{l(l+1)-m(m+1)} \cdot \hbar\sqrt{l(l+1)-(m+1)(m+2)}\ \overset{0}{\langle Y_l^m | Y_l^{m+2} \rangle} = 0$$

Similarly $\langle l_-^2 \rangle = 0$

$$\langle l_+ l_- \rangle = \langle Y_l^m | l_+ l_- | Y_l^m \rangle = \hbar\sqrt{l(l+1)-m(m-1)}\ \langle Y_l^m | l_+ | Y_l^{m-1} \rangle$$

$$= \hbar^2\sqrt{l(l+1)-m(m-1)} \cdot \sqrt{l(l+1)-m(m-1)}\ \underbrace{\langle Y_l^m | Y_l^m \rangle}_{=1}$$

$$= \hbar^2\left[l(l+1)-m(m-1)\right].$$

$$\langle l_- l_+ \rangle = \langle Y_l^m | l_- l_+ | Y_l^m \rangle =$$

$$= \hbar \sqrt{l(l+1) - m(m+1)} \cdot \langle Y_l^m | l_- | Y_l^{m+1} \rangle$$

$$= \hbar \sqrt{l(l+1) - m(m+1)}\, \hbar \sqrt{l(l+1) - (m+1)m}\, \underbrace{\langle Y_l^m | Y_l^m \rangle}_{1}$$

$$= \hbar^2 [\, l(l+1) - m(m+1) \,]$$

Thus $\langle l_x^2 \rangle = \frac{\hbar^2}{4} \{\, l(l+1) - m(m-1) + l(l+1) - m(m+1) \}$

$$\Rightarrow \langle l_x^2 \rangle = \frac{\hbar^2}{4} \{\, 2\, l(l+1) - m\, [\, m - \cancel{1} + m \cancel{+1} \,] \}$$

$$\Rightarrow \langle l_x^2 \rangle = \frac{\hbar^2}{4} \{\, 2 l(l+1) - 2m^2 \} =$$

$$= \frac{\hbar^2}{2} \{\, l(l+1) - m^2 \}$$

5) a) $l_x^2 + l_y^2 + l_z^2 = l^2 \Rightarrow$

$$\Rightarrow l_x^2 + l_y^2 = l^2 - l_z^2 \Rightarrow$$

$$\langle Y_l^m | l_x^2 + l_y^2 | Y_l^m \rangle =$$

$$= \langle Y_l^m | (l^2 - l_z^2) | Y_l^m \rangle =$$

$$= \langle Y_l^m | l^2 | Y_l^m \rangle - \langle Y_l^m | l_z^2 | Y_l^m \rangle$$

$$= \hbar^2 l(l+1) - m^2\hbar^2 = \hbar^2 \left[l(l+1) - m^2 \right]$$

b) $l_x^2 + l_y^2 - l_z^4 = l^2 - l_z^2 - l_z^4$

$$\langle Y_l^m | l_x^2 + l_y^2 - l_z^4 | Y_l^m \rangle = \langle Y_l^m | l^2 - l_z^2 - l_z^4 | Y_l^m \rangle$$

$$= \langle Y_l^m | l^2 | Y_l^m \rangle - \langle Y_l^m | l_z^2 | Y_l^m \rangle - \langle Y_l^m | l_z^4 | Y_l^m \rangle$$

$$= \hbar^2 (l+1) l - m^2 \hbar^2 - \hbar^4 m^4$$

c) $l_- l_+^2 l_-$

It contains same number of l_+, l_-. So when they act on Y_l^m ~~they~~ it will bring it back to Y_l^m again

$$l_- l_+^2 l_- = l_- l_+ \, l_+ l_- =$$

$$= (l_- l_+)(l_+ l_-)$$

but $l_- l_+ = (l_x - i l_y)(l_x + i l_y)$

$$= l_x^2 + l_y^2 - i l_y l_x + i l_y l_x$$

$$= l^2 - l_z^2 - \hbar l_z$$

Similarly

$$(l_+ l_-) = l^2 - l_z^2 + l_z$$

Thus $\langle Y_\ell^m | \ell_- \ell_+^2 \ell_- | Y_\ell^m \rangle =$

$$= \langle Y_\ell^m | (\ell^2 - \ell_z^2 - \hbar \ell_z)(\ell^2 - \ell_z^2 + \hbar \ell_z) | Y_\ell^m \rangle$$

$$= \left[\left(\ell(\ell+1) - m^2 - m \right) \hbar^2 \right] \left[\ell(\ell+1) - m^2 + m \right] \hbar^2$$

$$= \left\{ \left[\ell(\ell+1) - m^2 \right]^2 - m^2 \right\} \hbar^4$$

Similarly if you act with ℓ_-, ℓ_+ successively.

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$$) \; \langle l_x^4 \rangle = \langle Y_l^l \,|\, l_x^4 \,|\, Y_l^l \rangle = \langle Y_l^l \,|\, l_x^2\, l_x^2 \,|\, Y_l^l \rangle$$

$$= \| l_x^2 Y_l^l \|^2$$

But $l_x Y_l^l = \frac{1}{2}(l_+ + l_-) Y_l^l = \frac{1}{2}(l_+ Y_l^l) + \frac{1}{2}(l_- Y_l^l)$

$$= 0 + \frac{\hbar}{2}\sqrt{l(l+1) - l(l-1)}\, Y_l^{l-1} = \hbar \sqrt{\frac{l}{2}}\, Y_l^{l-1}$$

$$l_x^2 Y_l^l = l_x \left(\hbar \sqrt{\frac{l}{2}}\, Y_l^{l-1}\right) = \hbar \sqrt{\frac{l}{2}}\, l_x \left(Y_l^{l-1}\right)$$

$$= \hbar \sqrt{\frac{l}{2}} \; \frac{1}{2}(l_+ + l_-) Y_l^{l-1} =$$

$$= \frac{\hbar}{2}\sqrt{\frac{l}{2}} \left\{ l_+ Y_l^{l-1} + l_- Y_l^{l-1} \right\} =$$

$$= \frac{\hbar^2}{2}\sqrt{\frac{l}{2}} \left\{ \sqrt{l(l+1) - l(l-1)}\, Y_l^l + \sqrt{l(l+1) - (l-1)(l-2)}\, Y_l^{l-2} \right\}$$

$$= \frac{\hbar^2}{2}\sqrt{\frac{l}{2}} \left\{ \sqrt{2l}\, Y_l^l + \sqrt{2(2l-1)}\, Y_l^{l-2} \right\}$$

continue

$$\langle Y_l^l | l_x^2\, l_x^2 | Y_l^l \rangle = \left(\frac{\hbar^2}{2}\sqrt{\frac{l}{2}}\right)^2 \left\{ \sqrt{2l}\,\langle Y_l^l | + \sqrt{2(2l-1)}\,\langle Y_l^{l-2} | \right\} \cdot \left\{ \sqrt{2l}\,| Y_l^l \rangle + | Y_l^{l-2} \rangle \sqrt{2(2l-1)} \right\}$$

$$= \frac{\hbar^4}{4}\left(\frac{l}{2}\right) \Big\{ 2l \langle Y_l^l | Y_l^l \rangle + \sqrt{2l}\sqrt{2(2l-1)}\,\langle Y_l^l | Y_l^{l-2} \rangle +$$

$$\sqrt{2l}\cdot\sqrt{2(2l-1)}\,\langle Y_l^{l-2} | Y_l^l \rangle + 2(2l-1)\,\langle Y_l^{l-2} | Y_l^{l-2} \rangle \Big\}$$

$$= \frac{\hbar^4}{4}\left(\frac{l}{2}\right)\left\{ 2l + 2(2l-1) \right\} = \frac{\hbar^4}{4}\left(\frac{l}{2}\right)\left\{ 6l - 2 \right\} = \frac{\hbar^4 l}{4}\left\{ 3l - 1 \right\}$$

For the classical limit:

7. $Y_2^{+1}(\theta,\varphi) = -\sqrt{\frac{15}{8\pi}}\,\sin\vartheta\cos\vartheta\, e^{i\Phi}$

We know that [(2.11α)]

$$l_+|Y_2^1\rangle = \hbar\sqrt{2(2+1)-1\cdot(1+1)}\,|Y_2^2\rangle$$

$$\Rightarrow l_+|Y_2^1\rangle = 2\hbar\,|Y_2^2\rangle \Rightarrow$$

$$\Rightarrow |Y_2^2\rangle = \frac{1}{2\hbar}\,\underbrace{l_+|Y_2^1\rangle} \quad ①$$

But $l_+ = l_x + i\, l_y$ ②

② → ①

$$|Y_2^2\rangle = \frac{1}{2\hbar}\left\{ l_x|Y_2^1\rangle + i\, l_y|Y_2^1\rangle \right\} \quad ③$$

$$l_x|Y_2^1\rangle = i\hbar\left\{ \sin\varphi\,\frac{\partial}{\partial\theta}Y_2^1 + \frac{\cos\varphi}{\tan\vartheta}\,\frac{\partial}{\partial\Phi}Y_2^1 \right\}$$

$$= -i\hbar\sqrt{\frac{15}{8\pi}}\left\{ \sin\varphi\, e^{i\Phi}\underbrace{\frac{\partial}{\partial\theta}(\sin\vartheta\cos\vartheta)}_{\cos2\theta} + \frac{\cos\varphi}{\tan\theta}\,\sin\vartheta\cos\vartheta\,\underbrace{\frac{\partial}{\partial\Phi}e^{i\Phi}}_{ie^{i\Phi}} \right\}$$

$$= -i\hbar\sqrt{\frac{15}{8\pi}}\left\{ \sin\varphi\, e^{i\phi}\cos2\theta + i e^{i\Phi}\cos\varphi\cos^{?2}\theta \right\}$$

$$= -i\hbar\sqrt{\frac{15}{8\pi}}\, e^{i\Phi}\left\{ \sin\varphi\cos2\theta + i\cos\varphi\cos^2\theta \right\} \quad ④$$

Similarley:

$$l_y|Y_2^1\rangle = i\hbar\left(-\cos\varphi\,\frac{\partial}{\partial\theta}(Y_2^1) + \frac{\sin\varphi}{\tan\theta}\,\frac{\partial}{\partial\varphi}Y_2^1 \right)$$

$$= -i\hbar\sqrt{\frac{15}{8\pi}}\left(-\cos\varphi\, e^{i\Phi}\underbrace{\frac{\partial}{\partial\theta}(\sin\vartheta\cos\vartheta)}_{\cos2\theta} + \frac{\sin\varphi}{\tan\theta}\,\sin\vartheta\cos\vartheta\,\underbrace{\frac{\partial}{\partial\varphi}e^{i\phi}}_{ie^{i\phi}} \right)$$

$$= -i\hbar\sqrt{\frac{15}{8\pi}}\left(-\cos\varphi\cos2\theta\, e^{i\Phi} + \frac{i\sin\varphi}{\tan\theta}\,\sin\vartheta\cos\vartheta\, e^{i\Phi} \right)$$

$$= -i\hbar \sqrt{\frac{15}{8\pi}}\, e^{i\Phi} \left(-\cos\varphi \cos 2\theta + i \sin\varphi \cos^2\theta\right) \quad (5)$$

(5), (4) $\rightarrow$ (3)

$$|Y_2^2\rangle = \frac{1}{2\cancel{\hbar}} (-i\cancel{\hbar}) \sqrt{\frac{15}{8\pi}}\, e^{i\Phi} \left(\sin\varphi \cos 2\theta + i \cos\varphi \cos^2\theta - i \cos\varphi \cos 2\theta - \sin\varphi \cos^2\theta\right)$$

$$|Y_2^2\rangle = \frac{1}{2} \sqrt{\frac{15}{8\pi}}\, e^{i\Phi} \left(-i \sin\varphi \cos 2\theta + \cos\varphi \cos^2\theta - \cos\varphi \cos 2\theta + i \sin\varphi \cos^2\theta\right)$$

$$= \frac{1}{2} \sqrt{\frac{15}{8\pi}}\, e^{i\Phi} \left(-\cos 2\theta (\cos\varphi + i \sin\varphi) + \cos^2\theta (\cos\varphi + i \sin\phi)\right)$$

$$= \frac{1}{2} \sqrt{\frac{15}{8\pi}}\, e^{i\Phi} \underbrace{(\cos\varphi + i \sin\varphi)}_{e^{i\phi}} \left(-\cos 2\theta + \cos^2\theta\right)$$

$$= \frac{1}{2} \sqrt{\frac{15}{8\pi}}\, e^{2i\Phi} \left(-2\cos^2\theta + 1 + \cos^2\theta\right)$$

$$= \frac{1}{2} \sqrt{\frac{15}{8\pi}}\, e^{2i\Phi} \left(1 - \cos^2\theta\right) = \frac{1}{2} \sqrt{\frac{15}{8\pi}} \sin^2\theta\, e^{2i\Phi} = \sqrt{\frac{15}{32\pi}} \sin^2\theta\, e^{2i\Phi}$$

8. $l_+ Y_l^l = 0 \Rightarrow (l_x + i l_y) Y_l^l = 0$

problem 7

$$\Rightarrow \left(i\hbar \sin\varphi \frac{\partial}{\partial\theta} + \frac{i\hbar\cos\varphi}{\tan\theta}\frac{\partial}{\partial\phi} + \hbar\cos\varphi\frac{\partial}{\partial\theta} - \frac{\hbar\sin\varphi}{\tan\theta}\frac{\partial}{\partial\varphi} \right) Y_l^l = 0$$

$$\Rightarrow (i\sin\varphi + \cos\varphi)\frac{\partial}{\partial\theta} Y_l^l + i\left(\frac{\cos\varphi + i\sin\varphi}{\tan\theta}\right)\frac{\partial}{\partial\varphi} Y_l^l = 0$$

Let $Y_l^l = \Theta(\theta)\Phi(\varphi)$

$$(\cancel{i\sin\varphi + \cos\varphi})\,\Phi\frac{\partial\Theta}{\partial\theta} + i\left(\frac{\cancel{\cos\varphi + i\sin\varphi}}{\tan\theta}\right)\Theta\frac{\partial\Phi}{\partial\varphi} = 0$$

$$\Rightarrow \Phi\frac{\partial\Theta}{\partial\theta} + \frac{i\Theta}{\tan\theta}\frac{\partial\Phi}{\partial\varphi} = 0 \Rightarrow$$

$$\Rightarrow \frac{i\tan\theta}{\Theta}\frac{\partial\Theta}{\partial\theta} + \frac{1}{\Phi}\frac{\partial\Phi}{\partial\varphi} = 0$$

$$\Rightarrow \frac{i\tan\theta}{\Theta}\frac{\partial\Theta}{\partial\theta} = -\frac{1}{\Phi}\frac{\partial\Phi}{\partial\varphi} = \lambda$$

$$-\frac{1}{\Phi}\frac{\partial\Phi}{\partial\varphi} = \lambda \Rightarrow \frac{\partial\Phi}{\partial\varphi} = -\lambda\Phi$$

$$\Rightarrow \frac{\partial\Phi}{\Phi} = -\lambda\varphi \Rightarrow \ln\Phi = -\lambda\varphi$$

$$\Rightarrow \boxed{\Phi = e^{-\lambda\Phi}}$$

But $\Phi(\varphi) = \Phi(\varphi+2\pi) \Rightarrow$

$$\Rightarrow e^{-\lambda\Phi} = e^{-\lambda(\varphi+2\pi)} \Rightarrow$$

$$\Rightarrow e^{-2\pi\lambda} = 1 < \begin{matrix}\lambda = 0 \\ \text{or } \boxed{\lambda = il}\end{matrix}$$

Also $\frac{\cancel{i}\tan\vartheta}{\Theta}\frac{\partial\Theta}{\partial\theta} = \cancel{i}l \Rightarrow$

$$\frac{\partial\Theta}{\cancel{\partial\theta}\,\Theta} = \frac{\cancel{i}l}{\tan\vartheta}\partial\theta \Rightarrow$$

$$\ln\Theta = \cancel{i}l\int\frac{d\theta}{\tan\theta}$$

$$\Rightarrow \ln\Theta = \cancel{i}l\ln\sin\theta$$

$$\Rightarrow \ln\Theta = \ln(\sin\theta)^{\cancel{i}l}$$

$$\Rightarrow \Theta = (\sin\theta)^{il} = \cancel{\frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)^{il}}$$

$$\Rightarrow \Theta = \sin^l\theta$$

Thus $Y_l^l = N_{ll}\sin^l\theta\cdot e^{i\Phi l}$

$$\int_{\vartheta=0}^{\pi}\int_{\varphi=0}^{2\pi}|Y_l^l|^2\sin\vartheta\, d\vartheta\, d\Phi = 1 \Rightarrow$$

$$\int_{\vartheta=0}^{\pi}\int_{\varphi=0}^{2\pi}|N_{ll}|^2\sin^{2l}\theta\,\sin\vartheta\, d\vartheta\, d\phi = 1$$

$$\Rightarrow |N_{ll}|^2\int_{\vartheta=0}^{\pi}\sin^{2l+1}\vartheta\, d\vartheta\int_{\varphi=0}^{2\pi}d\phi = 1$$

$$\Rightarrow |N_{ll}|^2\, 2\pi\int_{\vartheta=0}^{\pi}\sin^{2l+1}d\Theta = 1$$

$$\Rightarrow \cancel{|N_{ll}|^2\, 2\pi\int_{\vartheta=0}^{\pi}(\sin^2\vartheta)^l\, d(-\cos\vartheta) = 1}$$

You may use the relation

$$\int_{\vartheta=0}^{\pi} \sin^{2\ell+1}\theta \, d\theta = \frac{2^{2\ell+1}(\ell!)^2}{(2\ell+1)!}$$

and you get:

$$|N_{\ell\ell}|^2 \cdot 2\pi \cdot \frac{2^{2\ell+1}(\ell!)^2}{(2\ell+1)!} = 1$$

$$|N_{\ell\ell}|^2 = \frac{4\pi \cdot (2^\ell)^2 (\ell!)^2}{(2\ell+1)!}$$

$$|N_{\ell\ell}|^2 = \frac{(2\ell+1)!}{4\pi (2^\ell)^2 \cdot (\ell!)^2}$$

$$N_{\ell\ell} = \frac{\sqrt{2\ell+1}}{\sqrt{4\pi}\, 2^\ell \cdot \ell!}$$