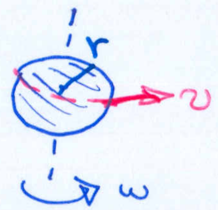
HANDOUT 3

1 If spin were a classical rotation around the electron's axis passing through its center of mass then we would have the following picture:



$v = \omega \cdot r$

In this case we would have: $I\omega = S_z \Rightarrow I\omega = \hbar/2$ (1)

where I is the moment of inertia. This is equal (for a sphere of radius r) to $I = \frac{2mr^2}{5}$ (2). But what is r for an electron?

According to high energy experiments this is smaller than 10^{-19} m (3)

From (1), (2) we get $\omega = \frac{\hbar}{2I} \Rightarrow \omega = \frac{\hbar 5}{2 \cdot 2mr^2} \Rightarrow \omega = \frac{5\hbar}{4mr^2}$

With $\hbar = h/2\pi = \frac{6.63 \times 10^{-34}}{6.28}$ $m = 9.1 \times 10^{-31}$ kg $r = 10^{-19}$ we get $\omega = 0.14 \times 10^{38}$ r/s

Thus the speed at a point in the equator $v = \omega \cdot r = 0.14 \times 10^{19}$ m/s $= 0.46 \cdot 10^{10} c$!

HANDOUT-3

2. First you must note that the vector X is normalized

- For $\langle S_z \rangle = \langle x | S_z | X \rangle = \underbrace{\frac{1}{\sqrt{5}}(1 \quad 2)}_{\langle x |} \underbrace{\frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{S_z} \underbrace{\frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{| x \rangle}$

$= \frac{\hbar}{10}(1 \quad -2)\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{\hbar}{10}(1-4) = -\frac{3\hbar}{10}$

- For $\langle S_x \rangle = \langle x | S_x | X \rangle = \frac{1}{\sqrt{5}}(1 \quad 2) \frac{\hbar}{2}\underbrace{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{S_x} \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{2}{5}\hbar$

- For $\langle S_y \rangle = \langle x | S_y | X \rangle = \frac{1}{\sqrt{5}}(1 \quad 2) \underbrace{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\frac{\hbar}{2}}_{S_y} \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 0$

Handout-3

(3) The average value along z is given by:

$$\left.\begin{aligned} \langle S_z \rangle &= P_+\left(\frac{\hbar}{2}\right) + P_-\left(-\frac{\hbar}{2}\right) \quad (3a) \\ \text{and} \quad P_+ + P_- &= 1 \quad (3b) \end{aligned}\right\} \Rightarrow$$

sz

$$\Rightarrow \left.\begin{aligned} -\frac{\hbar}{6} &= P_+\frac{\hbar}{2} - P_-\frac{\hbar}{2} \\ P_+ + P_- &= 1 \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{aligned} -\frac{1}{6} &= \frac{P_+}{2} - \frac{P_-}{2} \\ P_+ + P_- &= 1 \end{aligned}\right\} \Rightarrow$$

$$\Rightarrow \left.\begin{aligned} -\frac{1}{3} &= P_+ - P_- \\ 1 &= P_+ + P_- \end{aligned}\right\} \Rightarrow \begin{aligned} P_+ &= 1/3 \\ P_- &= 2/3 \end{aligned}$$

(4) Let a state $X = \begin{pmatrix} \alpha \\ b \end{pmatrix}$ (we know $|\alpha|^2 + |b|^2 = 1$)

$$\langle S_x \rangle = \langle X | S_x | X \rangle = \frac{\hbar}{2} \underbrace{(\alpha^*, b^*)}_{\langle X|} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \underbrace{\begin{pmatrix} \alpha \\ b \end{pmatrix}}_{|X\rangle} = \frac{\hbar}{2} (\alpha^* b + b^* \alpha)$$

$$\langle S_y \rangle = \langle X | S_y | X \rangle = \frac{i\hbar}{2} (\alpha^*, b^*) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \alpha \\ b \end{pmatrix} = \frac{i\hbar}{2} (\alpha b^* - \alpha^* b)$$

If $X = X_+ = \begin{pmatrix} 1 = \alpha \\ 0 = b \end{pmatrix}$ $\quad \langle S_x \rangle = \langle S_y \rangle = 0$

$(\alpha = 1 \quad b = 0)$

We cannot measure simultaneously z, x, y spin components!

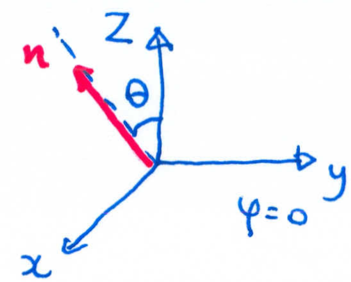
$$S_x^2 = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\hbar^2}{4} \mathbb{I}$$

$$S_y^2 = \frac{i\hbar}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{(i\hbar)}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = +\frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = +\frac{\hbar^2}{4} \mathbb{I}$$

$$\langle S_x^2 \rangle = \langle X | S_x^2 | X \rangle = \frac{\hbar^2}{4} \langle X | \mathbb{I} | X \rangle = \frac{\hbar^2}{4} \langle X | X \rangle = \frac{\hbar^2}{4}$$

$$\Delta S_x = \sqrt{\langle S_x^2 \rangle - \langle S_x \rangle^2} = \sqrt{\hbar^2/4 - 0} = \frac{\hbar}{2}$$

(5) For a rotation around z axis we have the corresponding rotation matrix



$$U_R = \begin{pmatrix} \cos\vartheta/2 & -\sin\vartheta/2 \\ \sin\vartheta/2 & \cos\vartheta/2 \end{pmatrix}$$

The state with spin up is $X_+ = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

In the new system

$$X_R = U_R X_+ = \begin{pmatrix} \cos(\vartheta/2) & -\sin(\vartheta/2) \\ \sin(\vartheta/2) & \cos(\vartheta/2) \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow X_R = \begin{pmatrix} \cos(\vartheta/2) \\ \sin(\vartheta/2) \end{pmatrix}$$

The probability to find it with spin-up is $P_+ = \cos^2\vartheta/2$

$$P_+ = |\langle + | X_R \rangle|^2 = \left| \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \cos\vartheta/2 \\ \sin\vartheta/2 \end{pmatrix} \right|^2 = \cos^2\vartheta/2$$

6 Let $X = \begin{pmatrix} \alpha \\ b \end{pmatrix}$ this must satisfy the relation

$$\underset{\sim}{S_x} X = \pm \frac{\hbar}{2} X$$

$$\underbrace{\frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}_{S_x} \begin{pmatrix} \alpha \\ b \end{pmatrix} = \pm \frac{\hbar}{2} \begin{pmatrix} \alpha \\ b \end{pmatrix}$$

$$\begin{pmatrix} b \\ \alpha \end{pmatrix} = \pm \begin{pmatrix} \alpha \\ b \end{pmatrix}$$

$\Rightarrow$ $b = \pm \alpha$ $\Rightarrow$ $\begin{cases} b = +\alpha & \text{(for } \uparrow) \\ \text{or} & \\ b = -\alpha & \text{(for } \downarrow) \end{cases}$

$\alpha = \pm b$

but $\alpha^2 + b^2 = 1$ $\Big<$ $2b^2 = 1$, $2b^2 = 1$

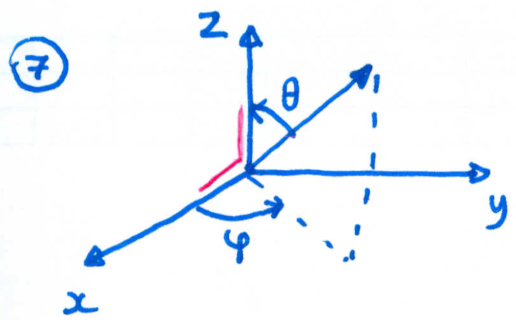
$\Rightarrow$ $b^2 = 1/2$ $\Rightarrow$ $b = \pm \frac{1}{\sqrt{2}}$

$$X_+ = \frac{\hbar}{2} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1/\sqrt{2} \end{pmatrix}$$

$$X_- = \frac{\hbar}{2} \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

HANDOUT 3

(7)



The rotation matrix is given by

$$\mathcal{U}_R = \begin{pmatrix} \cos(\vartheta/2) & -\sin(\theta/2)e^{-i\phi} \\ \sin(\vartheta/2)e^{i\phi} & \cos(\vartheta/2) \end{pmatrix}$$

To go from (+z)-axis to (+x)-axis
$\varphi = 0$ and $\vartheta = \pi/2$

$$\mathcal{U}_R = \begin{pmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{pmatrix} \Rightarrow$$

$$\mathcal{U}_R = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}$$

If we have the spin states along z-axis $|+\rangle_z = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $|-\rangle_z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Then the states along x-axis

$$|+\rangle_x = \mathcal{U}_R\,|+\rangle_z = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow |+\rangle_x = \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (1)$$

To go from (+z)-axis to (−x)-axis
$\varphi = \pi$ and $\vartheta = +\pi/2$, so

$$\mathcal{U}_R = \begin{pmatrix} \cos(\pi/4) & -e^{-i\pi}\sin(\pi/4) \\ \sin(\pi/4)e^{i\pi} & \cos(\pi/4) \end{pmatrix} = \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & +\sqrt{2}/2 \end{pmatrix}$$

Thus $|-\rangle_x = U_R \, |+\rangle_z \Rightarrow$

$$|-\rangle_x = \begin{pmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Rightarrow |-\rangle_x = \begin{pmatrix} \sqrt{2}/2 \\ -\sqrt{2}/2 \end{pmatrix} = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (2)$$

To find the propabilities to be with spin-up or spin-down along the x-axis

$$P_{\pm} = |c_{\pm}|^2 = |{}_x\langle \pm | X \rangle|^2 \quad (3)$$

$${}_x\langle + | X \rangle = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & 1 \end{pmatrix} \frac{1}{3} \begin{pmatrix} 1+2i \\ 2 \end{pmatrix} =$$

$$= \frac{\sqrt{2}}{6} (1 + 2i + 2) = \frac{\sqrt{2}}{6} (3 + 2i)$$

$$P_+ = |c_+|^2 = \frac{2}{36} (3+2i)(3-2i)$$

$$= \frac{2}{36} (3^2 + 2^2) = \frac{13}{18}$$

Since $P_+ + P_- = 1 \Rightarrow$

$P_- = 1 - P_+ \Rightarrow P_- = 1 - 13/18$

$\Rightarrow P_- = \frac{18-13}{18} \Rightarrow P_- = \frac{5}{18}$

8 If we choose the z-axis as our quantization axis this means that the corresponding spin matrix will have the form

$$S_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad (1)$$

To construct the matrices S_x, S_y we need to apply the rule we have in the mathematical supplement in Lecture-3 ppt notes: We put as columns the vectors which are taken by the action of the operators on the basis vectors.

The basis vectors are: $X_+ = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \; X_0 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \; X_- = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad (2)$

We also know that $S_x = \dfrac{S_+ + S_-}{2} \qquad S_y = \dfrac{S_+ - S_-}{2i} \qquad (3)$

For the action of raising and lowering operators we have:

$$S_\pm \psi_j^m = \hbar\sqrt{j(j+1) - m(m\pm 1)}\ \psi_j^{m\pm 1}$$

Here $j = 1 \quad m = 1, 0, -1$

$$S_+ \psi_1^1 = S_+ X_+ = 0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$S_+ \psi_1^0 = S_+ X_0 = \hbar\sqrt{1(1+1) - 0(0+1)}\ \psi_1^1 = \sqrt{2}\hbar\, \psi_1^1 = \sqrt{2}\hbar\, X_+ = \sqrt{2}\hbar \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$S_+ \psi_1^{-1} = S_+ X_- = \hbar\sqrt{1(1+1) - (-1)(-1+1)}\ \psi_1^0 = \sqrt{2}\hbar\, \psi_1^0 = \sqrt{2}\hbar \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$S_- \psi_1^1 = \hbar\sqrt{1(1+1) - 1(1-1)}\ \psi_1^0 = \sqrt{2}\hbar\, \psi_1^0 = \sqrt{2}\hbar \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$S_- \psi_1^0 = \hbar\sqrt{1(1+1) - 0(0-1)}\ \psi_1^{-1} = \sqrt{2}\hbar\, \psi_1^{-1} = \sqrt{2}\hbar \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$S_- \psi_1^{-1} = 0$$

- $$S_x \psi_1^1 = \left(\frac{S_+ + S_-}{2}\right)\psi_1^1 = \frac{1}{2} S_+ \psi_1^1 + \frac{1}{2} S_- \psi_1^1 = \frac{1}{2}\begin{pmatrix}0\\0\\0\end{pmatrix} + \frac{1}{2}\sqrt{2}\hbar\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\Rightarrow S_x \psi_1^1 = \frac{\sqrt{2}}{2}\hbar \begin{pmatrix}0\\1\\0\end{pmatrix}$$

- $$S_x \psi_1^0 = \left(\frac{S_+ + S_-}{2}\right)\psi_1^0 = \frac{1}{2} S_+ \psi_1^0 + \frac{1}{2} S_- \psi_1^0 = \frac{1}{2}\sqrt{2}\hbar\begin{pmatrix}1\\0\\0\end{pmatrix} + \frac{1}{2}\sqrt{2}\hbar\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$= \frac{\sqrt{2}\hbar}{2}\begin{pmatrix}1\\0\\1\end{pmatrix}$$

- $$S_x \psi_1^{-1} = \frac{(S_+ + S_-)\psi_1^{-1}}{2} = \frac{1}{2} S_+ \psi_1^{-1} + \frac{1}{2} S_- \psi_1^{-1} = \frac{\sqrt{2}\hbar}{2}\begin{pmatrix}0\\1\\0\end{pmatrix} + 0 = \frac{\sqrt{2}}{2}\hbar\begin{pmatrix}0\\1\\0\end{pmatrix}$$

Thus $$S_+ = \frac{\sqrt{2}}{2}\hbar \begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}$$

Similarly you can prove that

$$S_y = \frac{\sqrt{2}}{2}\hbar \begin{pmatrix}0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{pmatrix}$$