No. $1^{st}$ homework
Date لطاف السليمي

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①

$$L^2 = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]$$

$$L_z = -i\hbar\frac{\partial}{\partial\varphi}$$

The function $Y_2^2(\theta,\varphi) = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\,\sin^2\theta\, e^{2i\Phi}$

$$L^2 Y_2^2(\theta,\varphi) = -\hbar^2\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]\cdot\left(\frac{1}{4}\right)\sqrt{\frac{15}{2\pi}}\,\sin^2\theta\, e^{2i\Phi}$$

$$= -\frac{\hbar^2}{4}\sqrt{\frac{15}{2\pi}}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\cdot\frac{\partial^2}{\partial\varphi^2}\right]\sin^2\theta\, e^{2i\Phi}$$

$$= -\frac{\hbar^2}{4}\sqrt{\frac{15}{2\pi}}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)\sin^2\theta\, e^{2i\Phi} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\sin^2\theta\, e^{2i\Phi}\right]$$

$$= -\frac{\hbar^2}{4}\sqrt{\frac{15}{2\pi}}\left[\frac{e^{2i\Phi}}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\sin^2\theta\right) + \frac{\cancel{\sin^2\theta}}{\cancel{\sin^2\theta}}\frac{\partial^2}{\partial\varphi^2}e^{2i\Phi}\right]$$

$*\ \frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\sin^2\theta\right)$

$$\sin^2\theta = \tfrac{1}{2}(1-\cos 2\theta)$$

$$\frac{\partial}{\partial\theta}\left(\sin\theta\cdot\left(\tfrac{1}{2}\right)\frac{\partial}{\partial\theta}(1-\cos 2\theta)\right)$$

$$\frac{d}{d\theta}\left(\sin\theta\cdot\left(\tfrac{1}{2}\right)(2\sin 2\theta)\right) = \frac{d}{d\theta}(\sin\theta\cdot\sin 2\theta)$$

$$= \sin\theta\cdot(2\cos 2\theta) + \cos\theta\cdot\sin 2\theta)$$
$$= 2\sin\theta\cos 2\theta + \cos\theta\sin 2\theta$$

$$L^2 Y_2^2(\theta,\varphi) = -\frac{\hbar^2}{4}\sqrt{\frac{15}{2\pi}}\left[\frac{e^{2i\Phi}}{\sin\theta}(2\sin\theta\cos 2\theta + \cos\theta\sin 2\theta) + (2i)^2 e^{2i\Phi}\right]$$

$$= -\frac{\hbar^2}{4}\sqrt{\frac{15}{2\pi}}\left[\frac{2\sin\theta\cos 2\theta}{\sin\theta} + \frac{\cos\theta\sin 2\theta}{\sin\theta} - 4\right]e^{2i\Phi}$$

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$$L^2 Y_2^2(\theta,\varphi) = -\frac{\hbar^2}{4}\sqrt{\frac{15}{2\pi}}\left[2\cos 2\theta + \cot\theta \sin 2\theta - 4\right] e^{2i\Phi}$$

$\cos 2\theta = (1 - 2\sin^2\theta)$ , $\sin 2\theta = 2\sin\theta\cos\theta$

$$L^2 Y_2^2(\theta,\varphi) = -\frac{\hbar^2}{4}\sqrt{\frac{15}{2\pi}}\left[2(1-2\sin^2\theta) + \frac{\cos\theta}{\sin\theta}(2\sin\theta\cos\theta) - 4\right] e^{2i\Phi}$$

$$= -\frac{\hbar^2}{4}\sqrt{\frac{15}{2\pi}}\left[2 - 4\sin^2\theta + 2\cos^2\theta - 4\right] e^{2i\Phi}$$

$\cos^2\theta = 1 - \sin^2\theta$

$$L^2 Y_2^2(\theta,\varphi) = -\frac{\hbar^2}{4}\sqrt{\frac{15}{2\pi}}\left[2 - 4\sin^2\theta + 2(1-\sin^2\theta) - 4\right] e^{2i\Phi}$$

$$= -\frac{\hbar^2}{4}\sqrt{\frac{15}{2\pi}}\left[\cancel{2} - 4\sin^2\theta + \cancel{2} - 2\sin^2\theta \cancel{-4}\right] e^{2i\Phi}$$

$$= -\frac{\hbar^2}{4}\sqrt{\frac{15}{2\pi}}\left(-6\sin^2\theta\right) e^{2i\Phi}$$

$$L^2 Y_2^2(\theta,\varphi) = 6\hbar^2\left(\frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\Phi}\right)$$

$$L^2 Y_2^2(\theta,\varphi) = \boxed{6\hbar^2} Y_2^2(\theta,\varphi)$$

eigenvalue

$L^2 |l,m\rangle = l(l+1)\hbar^2 |l,m\rangle$ , $l=2$

$L^2 Y_2^2 = 2(2+1)\hbar^2 Y_2^2 = 6\hbar^2 Y_2^2$

* $L_z = -i\hbar\frac{\partial}{\partial\varphi}$

$$L_z Y_2^2(\theta,\varphi) = -i\hbar\frac{\partial}{\partial\varphi}\left[\frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\Phi}\right]$$

$$= -i\hbar\left(\frac{1}{4}\right)\sqrt{\frac{15}{2\pi}}\sin^2\theta\frac{\partial}{\partial\varphi}e^{2i\Phi}$$

$$= -i\hbar\left(\frac{1}{4}\right)\sqrt{\frac{15}{2\pi}}\sin^2\theta(2i)e^{2i\Phi} = 2\hbar\left(\frac{1}{4}\right)\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\Phi}$$

$$L_z Y_2^2(\theta,\varphi) = \boxed{2\hbar} Y_2^2(\theta,\varphi)$$

eigenvalue

$L_z Y_l^m = m\hbar Y_l^m$

$L_z Y_2^2 = 2\hbar Y_2^2$

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(2)

$L_- Y_\ell^{-\ell} = 0$

$(L_x - iL_y) Y_\ell^{-\ell} = 0$

$L_x = i\hbar\left(\sin\varphi \frac{\partial}{\partial\theta} + \frac{\cos\varphi}{\tan\theta}\frac{\partial}{\partial\varphi}\right)$ ; $L_y = i\hbar\left(-\cos\varphi\frac{\partial}{\partial\theta} + \frac{\sin\varphi}{\tan\theta}\frac{\partial}{\partial\varphi}\right)$

$\left[i\hbar\left(\sin\varphi\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{\tan\theta}\frac{\partial}{\partial\varphi}\right) - i(i\hbar)\left(-\cos\varphi\frac{\partial}{\partial\theta} + \frac{\sin\varphi}{\tan\theta}\frac{\partial}{\partial\varphi}\right)\right] Y_\ell^{-\ell} = 0$

$\left[i\sin\varphi\frac{\partial}{\partial\theta} + i\frac{\cos\varphi}{\tan\theta}\frac{\partial}{\partial\varphi} - \cos\varphi\frac{\partial}{\partial\theta} + \frac{\sin\varphi}{\tan\theta}\frac{\partial}{\partial\varphi}\right] Y_\ell^{-\ell} = 0$

$(i\sin\varphi - \cos\varphi)\frac{\partial}{\partial\theta} Y_\ell^{-\ell} + \left(\frac{\sin\varphi + i\cos\varphi}{\tan\theta}\right)\frac{\partial}{\partial\varphi} Y_\ell^{-\ell} = 0$

$i(\sin\varphi + i\cos\varphi)\frac{\partial}{\partial\theta} Y_\ell^{-\ell} + \left(\frac{\sin\varphi + i\cos\varphi}{\tan\theta}\right)\frac{\partial}{\partial\varphi} Y_\ell^{-\ell} = 0$

$i\frac{\partial}{\partial\theta} Y_\ell^{-\ell} + \frac{1}{\tan\theta}\frac{\partial}{\partial\varphi} Y_\ell^{-\ell} = 0$

* let: $Y_\ell^{-\ell} = \Theta(\theta)\Phi(\varphi)$

$i\frac{\partial}{\partial\theta}\Theta(\theta)\Phi(\varphi) + \frac{1}{\tan\theta}\frac{\partial}{\partial\varphi}\Theta(\theta)\Phi(\varphi) = 0$

$i\Phi(\varphi)\frac{\partial\Theta}{\partial\theta} + \frac{\Theta(\theta)}{\tan\theta}\frac{\partial\Phi(\varphi)}{\partial\varphi} = 0$ * $\frac{\tan\theta}{\Phi\cdot\Theta}$

$\frac{i\tan\theta}{\Theta(\theta)}\frac{\partial\Theta}{\partial\theta} + \frac{1}{\Phi(\varphi)}\frac{\partial\Phi}{\partial\varphi} = 0$

$\frac{i\tan\theta}{\Theta(\theta)}\frac{\partial\Theta}{\partial\theta} = -\frac{1}{\Phi}\frac{\partial\Phi}{\partial\varphi} = \lambda$

$\frac{1}{\Phi}\frac{\partial\Phi}{\partial\varphi} = \lambda \Rightarrow \frac{\partial\Phi}{\Phi} = -\lambda\varphi \Rightarrow \ln\Phi = -\lambda\varphi$

$\Rightarrow \Phi(\varphi) = e^{-\lambda\Phi}$

$\Phi(\varphi) = \Phi(\varphi + 2\pi)$

$e^{-\lambda\Phi} = e^{-\lambda(\varphi+2\pi)}$

$e^{-\lambda\Phi} = e^{-\lambda\Phi} \; e^{-\lambda 2\pi}$

$e^{-\lambda 2\pi} = 1$

$\lambda = il$

$\Phi(\varphi) = e^{-il\Phi}$

$\frac{i\tan\theta}{\Theta(\theta)} \; \frac{\partial\Theta}{\partial\theta} = il$

$\frac{\partial\Theta}{\Theta} = \frac{l}{\tan\theta} \; \partial\theta$

$\ln\Theta = l\int \frac{d\theta}{\tan\theta} = l\ln\sin\theta \Rightarrow \ln\Theta = \ln(\sin\theta)^l$

$\Theta(\theta) = \sin^l\theta$

$Y_l^{-l} = N_{ll} \sin^l\theta \, e^{-il\Phi}$

$N_{ll} = ?$

$\int_{\theta=0}^{\pi}\int_{\varphi=0}^{2\pi} |Y_l^{-l}|^2 \sin\theta \, d\theta \, d\varphi = 1$

$|N_{ll}|^2 \int_{\theta=0}^{\pi} \sin^{2l+1}\theta \, d\theta \int_{\varphi=0}^{2\pi} d\varphi = 1$

$|N_{ll}|^2 (2\pi) \boxed{\int_{\theta=0}^{\pi} \sin^{2l+1}\theta \, d\theta} = 1 \rightarrow \frac{2^{2l+1}(l!)^2}{(2l+1)!}$

$|N_{ll}|^2 = \frac{(2l+1)!}{4\pi(2l)^2(l!)^2} \Rightarrow N_{ll} = \frac{\sqrt{(2l+1)!}}{\sqrt{4\pi}\cdot 2^l \cdot l!}$

$Y_l^{-l}(\theta,\varphi) = \frac{1}{2^l l!}\sqrt{\frac{(2l+1)!}{4\pi}} \sin^l \, e^{-il\Phi}$

③

$$L_x = i\hbar\left(\sin\varphi\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{\tan\theta}\frac{\partial}{\partial\varphi}\right) \quad ; \quad L_y = i\hbar\left(-\cos\varphi\frac{\partial}{\partial\theta} + \frac{\sin\varphi}{\tan\theta}\frac{\partial}{\partial\varphi}\right)$$

we know that:

$$Y_1^0(\theta,\varphi) = \sqrt{\frac{3}{4\pi}}\cos\theta$$

$$\ell_-|Y_1^0\rangle = \hbar\sqrt{1(1+1)-0}\;|Y_1^{-1}\rangle$$

$$\ell_-|Y_1^0\rangle = \sqrt{2}\hbar|Y_1^{-1}\rangle$$

$$|Y_1^{-1}\rangle = \frac{1}{\sqrt{2}\hbar}\ell_-|Y_1^0\rangle \qquad (1)$$

$$\ell_- = \ell_x - i\ell_y \qquad (2)$$

(2) in (1)

$$|Y_1^{-1}\rangle \frac{1}{\sqrt{2}\hbar}\left\{\ell_x|Y_1^0\rangle - i\ell_y|Y_1^0\rangle\right\} \qquad (3)$$

* $$L_x|Y_1^0\rangle = i\hbar\left\{\sin\varphi\frac{\partial}{\partial\theta}Y_1^0 + \frac{\cos\varphi}{\tan\theta}\frac{\partial}{\partial\varphi}Y_1^0\right\}$$

zero

$$= i\hbar\sqrt{\frac{3}{4\pi}}\left\{\sin\varphi\frac{\partial}{\partial\theta}\cos\theta + \frac{\cos\varphi}{\tan\theta}\frac{\partial}{\partial\varphi}\cos\theta\right\}$$

$$= i\hbar\sqrt{\frac{3}{4\pi}}\left\{-\sin\varphi\sin\theta\right\} \qquad = -i\hbar\sqrt{\frac{3}{4\pi}}\sin\varphi\sin\theta \qquad (4)$$

* $$L_y|Y_1^0\rangle = i\hbar\left\{-\cos\varphi\frac{\partial}{\partial\theta}Y_1^0 + \frac{\sin\varphi}{\tan\theta}\frac{\partial}{\partial\varphi}Y_1^0\right\}$$

zero

$$= i\hbar\sqrt{\frac{3}{4\pi}}\left[-\cos\varphi\frac{\partial}{\partial\theta}\cos\theta + \frac{\sin\varphi}{\tan\theta}\frac{\partial}{\partial\varphi}\cos\theta\right]$$

$$= i\hbar\sqrt{\frac{3}{4\pi}}\cos\varphi\sin\theta \qquad (5)$$

(4) , (5) $\rightarrow$ (3)

$$|\psi_1^{-1}\rangle = \frac{1}{\sqrt{2}\,\hbar}\left[-i\hbar\sqrt{\frac{3}{4\pi}}\sin\varphi\sin\theta - i(i\hbar)\sqrt{\frac{3}{4\pi}}\cos\varphi\sin\theta\right]$$

$$= \frac{\hbar}{\sqrt{2}\,\hbar}\sqrt{\frac{3}{4\pi}}\left[-i\sin\varphi\sin\theta + \cos\varphi\sin\theta\right]$$

$$= \sqrt{\frac{3}{8\pi}}\sin\theta\,(\underbrace{\cos\varphi - i\sin\varphi}_{\downarrow})$$

$$|\psi_1^{-1}\rangle = \sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\Phi}$$