433203564 Phys(505) Afaf Alsulaimi No. 2nd homework

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(1)

The base vector for $s=1$:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$S_x \begin{pmatrix} a \\ b \\ c \end{pmatrix} = +\hbar \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$S_x = \frac{\sqrt{2}}{2}\hbar \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$S_x \alpha_x = +\hbar \alpha_x$

$$\frac{\sqrt{2}}{2}\cancel{\hbar} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = +\cancel{\hbar} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$\frac{\sqrt{2}}{2} \begin{pmatrix} b \\ a+c \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$\frac{\sqrt{2}}{2} b = a$

$\frac{\sqrt{2}}{2} a + \frac{\sqrt{2}}{2} c = b$

$\frac{\sqrt{2}}{2} b = c$

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$\Rightarrow$

$a = c$

$b = \sqrt{2}\,a$

The Probability:

$|a|^2 + |b|^2 + |c|^2 = 1$

$a^2 + 2a^2 + a^2 = 1$

$4a^2 = 1$

$\Rightarrow a^2 = \frac{1}{4} \Longrightarrow a = \frac{1}{2}$

$b = \sqrt{2}\,a = \frac{\sqrt{2}}{2}$

$c = \frac{1}{2}$

. The base vector for S_x with the eigenvalue $(+\hbar)$:

$$\alpha = \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{2}}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$P(+\hbar) = |\langle x_+ | \alpha \rangle|^2 = \left| (1 \quad 0 \quad 0) \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{2}}{2} \\ \frac{1}{2} \end{pmatrix} \right|^2 = \left|\tfrac{1}{2}\right|^2 = \frac{1}{4}$$

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$$P(-\hbar) = |\langle x_- | \alpha \rangle|^2 = \left| (0 \quad 0 \quad 1) \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{2}}{2} \\ \frac{1}{2} \end{pmatrix} \right|^2 = \left|\frac{1}{2}\right|^2 = \frac{1}{4}$$

$$P(0) = |\langle x_0 | \alpha \rangle|^2 = \left| (0 \quad 1 \quad 0) \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{2}}{2} \\ \frac{1}{2} \end{pmatrix} \right|^2 = \left| \frac{\sqrt{2}}{2} \right|^2 = \frac{2}{4} = \frac{1}{2}$$

$* \ \Delta S_z = ?$

$$S_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$\Delta S_z = \sqrt{\langle S_z^2 \rangle - \langle S_z \rangle^2}$$

$$\langle S_z^2 \rangle = \langle \alpha | \, S_z^2 \, | \alpha \rangle$$

$$S_z^2 = \hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\langle S_z^2 \rangle = \hbar^2 \left(\tfrac{1}{2} \quad \tfrac{\sqrt{2}}{2} \quad \tfrac{1}{2} \right) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \tfrac{1}{2} \\ \tfrac{\sqrt{2}}{2} \\ \tfrac{1}{2} \end{pmatrix}$$

$$\langle S_z^2 \rangle = \hbar^2 \left(\tfrac{1}{2} \quad 0 \quad \tfrac{1}{2} \right) \begin{pmatrix} \tfrac{1}{2} \\ \tfrac{\sqrt{2}}{2} \\ \tfrac{1}{2} \end{pmatrix} = \hbar^2 \left(\tfrac{1}{4} + \tfrac{1}{4} \right) = \frac{\hbar^2}{2}$$

$$\langle S_z \rangle = \langle \alpha | S_z | \alpha \rangle$$

$$= \hbar \left(\tfrac{1}{2} \; \tfrac{\sqrt{2}}{2} \; \tfrac{1}{2} \right) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \tfrac{1}{2} \\ \tfrac{\sqrt{2}}{2} \\ \tfrac{1}{2} \end{pmatrix} = \hbar \left(\tfrac{1}{2} - \tfrac{1}{2} \right) = 0$$

$$\Delta S_z = \sqrt{\langle S_z^2 \rangle - \langle S_z \rangle^2}$$

$$= \sqrt{\frac{\hbar^2}{2} - 0}$$

$$\Delta S_z = \frac{\hbar}{\sqrt{2}}$$

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(2)

$S = 1/2 \qquad \chi = \frac{1}{\sqrt{6}}\begin{pmatrix} 1+i \\ 2 \end{pmatrix}$

$$U_R = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2)\, e^{-i\varphi} \\ \sin(\theta/2)\, e^{i\varphi} & \cos(\theta/2) \end{pmatrix}$$

* for rotation around the z-axis $\varphi = 0$, along z-axis $\theta = 0$

$$U_R = \begin{pmatrix} \cos(0) & -\sin(0)\, e^{(0)} \\ \sin(0)\, e^{(0)} & \cos(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$|+\rangle_z = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |-\rangle_z = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The probability along z-axis:

$$P_\pm = |C_\pm|^2 = |{}_z\langle \pm | \chi \rangle|^2$$

$${}_z\langle + | \chi \rangle = \frac{1}{\sqrt{6}} (1 \quad 0) \begin{pmatrix} 1+i \\ 2 \end{pmatrix} = \frac{1}{\sqrt{6}}(1+i)$$

$$P_+ = |{}_z\langle + | \chi \rangle|^2 = \frac{1}{6}(1+i)(1-i) = \frac{1}{6}(1+1) = \frac{2}{6} = \frac{1}{3}$$

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$$P_- = |{}_z\langle -|x\rangle|^2 = \frac{1}{6}\left|\begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}1+i \\ 2\end{pmatrix}\right|^2 = \frac{1}{6}|2|^2 = \frac{4}{6} = \frac{2}{3}$$

<u>or</u>: $P_+ + P_- = 1 \Rightarrow P_- = 1 - P_+ \Rightarrow P_- = 1 - \frac{1}{3} \Rightarrow P_- = \frac{2}{3}$

* To go from +z-axis to +x-axis:

$\varphi = 0 \quad , \quad \theta = \frac{\pi}{2}$

$$U_R = \begin{pmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

The spin states along z-axis $|+\rangle_z = \begin{pmatrix}1 \\ 0\end{pmatrix}$; $|-\rangle_z = \begin{pmatrix}0 \\ 1\end{pmatrix}$

$$|+\rangle = U_R|+\rangle_z = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix}1 \\ 0\end{pmatrix}$$

$$\Rightarrow |+\rangle_x = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix}1 \\ 1\end{pmatrix}$$

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* To go from +z-axis to -x-axis:

$\varphi = \pi \qquad \theta = \frac{\pi}{2}$

$$U_R = \begin{pmatrix} \cos(\frac{\pi}{4}) & -\sin(\frac{\pi}{4})e^{-i\pi} \\ \sin(\frac{\pi}{4})e^{i\pi} & \cos(\frac{\pi}{4}) \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

$$|-\rangle_x = U_R|+\rangle_z$$

$$|-\rangle_x = \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$|-\rangle_x = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The probability along the x-axis:

$$P_{\pm} = |c_{\pm}|^2 = |{}_x\langle \pm | x\rangle|^2$$

$${}_x\langle + | x\rangle = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & 1 \end{pmatrix} \cdot \frac{1}{\sqrt{6}}\begin{pmatrix} 1+i \\ 2 \end{pmatrix}$$

$$= \frac{\sqrt{2}}{2\sqrt{6}}(1+i+2) = \frac{\sqrt{2}}{2\sqrt{6}}(3+i)$$

$$P_+ = |{}_x\langle + | x\rangle|^2 = \frac{\cancel{2}}{\cancel{4}_2 \times 6}(3+i)(3-i) = \frac{1}{12}(9+1) = \frac{10}{12} = \frac{5}{6}$$

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$$P_- = |\langle -|x\rangle|^2 = \frac{2}{4\times 6}\left|\begin{pmatrix}1 & -1\end{pmatrix}\begin{pmatrix}1+i\\2\end{pmatrix}\right|^2 = \frac{2}{4\times 6}\left|(1+i-2)\right|^2 = \frac{2}{4\times 6}|(-1+i)|^2$$

$$P_- = \frac{2}{4\times 6}(1+1) = \frac{1}{6}$$

or:

$$P_+ + P_- = 1$$

$$P_- = 1 - P_+ \Rightarrow P_- = 1 - \frac{5}{6} \Rightarrow P_- = \frac{1}{6}$$

(3)

$$X = A\begin{pmatrix}3i\\4\end{pmatrix}$$

(a) $A = ?$

The normalize x :

$$\langle x|x\rangle = 1$$

$$|A|^2\begin{pmatrix}-3i & 4\end{pmatrix}\begin{pmatrix}3i\\4\end{pmatrix} = 1$$

$$|A|^2(9+16) = 1$$

$$|A|^2 = \frac{1}{25} \Rightarrow A = \frac{1}{5}$$

(b) $\langle S_x \rangle$, $\langle S_y \rangle$, $\langle S_z \rangle$ ..?

* $\langle S_x \rangle$:

$$\langle S_x \rangle = \langle x | S_x | x \rangle = \frac{1}{5} (-3i \quad 4) \cdot \overbrace{\frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}^{S_x} \cdot \frac{1}{5} \begin{pmatrix} 3i \\ 4 \end{pmatrix}$$

$$= \frac{\hbar}{2 \times 5 \times 5} (-3i \quad 4) \begin{pmatrix} 4 \\ 3i \end{pmatrix}$$

$$\langle S_x \rangle = \frac{\hbar}{50} (-12i + 12i) = 0$$

* $\langle S_y \rangle$:

$$\langle S_y \rangle = \langle x | S_y | x \rangle = \frac{1}{5} (-3i \quad 4) \cdot \overbrace{\frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}^{S_y} \cdot \frac{1}{5} \begin{pmatrix} 3i \\ 4 \end{pmatrix}$$

$$\langle S_y \rangle = \frac{\hbar}{50} (-3i \quad 4) \begin{pmatrix} -4i \\ -3 \end{pmatrix} = \frac{\hbar}{50} (-12 - 12) = \frac{-24}{50} \hbar = \frac{-12}{25} \hbar$$

* $\langle S_z \rangle$:

$$\langle S_z \rangle = \langle x | S_z | x \rangle = \frac{1}{5} (-3i \quad 4) \cdot \overbrace{\frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}^{S_z} \cdot \frac{1}{5} \begin{pmatrix} 3i \\ 4 \end{pmatrix}$$

$$\langle S_z \rangle = \frac{\hbar}{50} (-3i \quad 4) \begin{pmatrix} 3i \\ -4 \end{pmatrix} = \frac{\hbar}{50} (9 - 16) = -\frac{7}{50} \hbar$$

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(c)

$\Delta S_x, \Delta S_y, \Delta S_z$ .. ?

* ΔS_x :

$$\Delta S_x = \sqrt{\langle S_x^2 \rangle - \langle S_x \rangle^2}$$

$$S_x^2 = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\hbar^2}{4} \mathbb{I}$$

$$\langle S_x^2 \rangle = \langle x | S_x^2 | x \rangle = \frac{\hbar^2}{4} \langle x | \mathbb{I} | x \rangle = \frac{\hbar^2}{4} \langle x | x \rangle = \frac{\hbar^2}{4}$$

$$\Delta S_x = \sqrt{\frac{\hbar^2}{4} - 0} = \frac{\hbar}{2}$$

* ΔS_y :

$$\Delta S_y = \sqrt{\langle S_y^2 \rangle - \langle S_y \rangle^2}$$

$$S_y^2 = \frac{i\hbar}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \cdot \frac{i\hbar}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\hbar^2}{4} \mathbb{I}$$

$$\langle S_y^2 \rangle = \langle x | S_y^2 | x \rangle = \frac{\hbar^2}{4} \langle x | \mathbb{I} | x \rangle = \frac{\hbar^2}{4} \langle x | x \rangle = \frac{\hbar^2}{4}$$

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$$\Delta S_y = \sqrt{\frac{\hbar^2}{4} - \left(\frac{12}{25}\right)^2 \hbar^2} = \hbar \sqrt{\frac{1}{4} - \frac{144}{625}} = \hbar \sqrt{\frac{625-576}{2500}}$$

$$\Delta S_y = \hbar \sqrt{\frac{49}{2500}} = \frac{7}{50} \hbar$$

* ΔS_z:

$$\Delta S_z = \sqrt{\langle S_z^2 \rangle - \langle S_z \rangle^2}$$

$$S_z^2 = \frac{\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\hbar^2}{4} \mathbb{I}$$

$$\langle S_z^2 \rangle = \langle x | S_z^2 | x \rangle = \frac{\hbar^2}{4} \langle x | \mathbb{I} | x \rangle = \frac{\hbar^2}{4} \langle x | x \rangle = \frac{\hbar^2}{4}$$

$$\Delta S_z = \sqrt{\frac{\hbar^2}{4} - \left(\frac{7}{50}\right)^2 \hbar^2} = \hbar \sqrt{\frac{1}{4} - \frac{49}{2500}} = \hbar \sqrt{\frac{625-49}{2500}}$$

$$\Delta S_z = \hbar \sqrt{\frac{576}{2500}} = \frac{24}{50} \hbar = \frac{12}{25} \hbar$$