PHYS 502

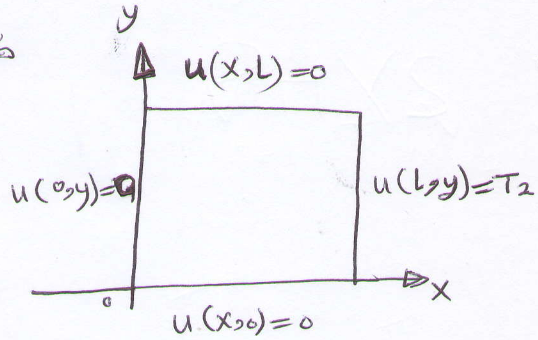
Homework 4

Solution of problem 2

of homework

another method for Problem(4)

first



$$\frac{X''}{X}+\frac{Y''}{Y}=0 \Rightarrow \frac{Y''}{Y}=-\frac{X''}{X}=-\lambda \quad (\lambda>0)$$

$$\Rightarrow \left.\begin{aligned} X(x) &= X_1 \cosh\sqrt{\lambda}x + X_2 \sinh\sqrt{\lambda}x \\ Y(y) &= Y_1 \cos\sqrt{\lambda}y + Y_2 \sin\sqrt{\lambda}y \end{aligned}\right\}$$

$\because u(x,0)=0 \Rightarrow X(x)\,Y(0)=0 \Rightarrow Y(0)=0$

$$\Rightarrow \boxed{Y_1=0}$$

$u(x,L)=0 \Rightarrow X(x)\,Y(L)=0 \Rightarrow Y(L)=0$

$$\Rightarrow \boxed{\lambda=\frac{n^2\pi^2}{L^2}} \quad n=1,2\ldots$$

$$\Rightarrow X_n = X_{1n}\cosh\left(\frac{n\pi x}{L}\right)+X_{2n}\sinh\left(\frac{n\pi x}{L}\right)$$

$$Y_n = Y_{2n}\sin\left(\frac{n\pi y}{L}\right)$$

$$\Rightarrow u_{1n}(x,y)=\left\{a_n\cosh\left(\frac{n\pi x}{L}\right)+b_n\sinh\left(\frac{n\pi x}{L}\right)\right\}\sin\left(\frac{n\pi y}{L}\right)$$

$$\Rightarrow u_1(x,y)=\sum_{n=1}^{\infty}\left\{a_n\cosh\left(\frac{n\pi x}{L}\right)+b_n\sinh\left(\frac{n\pi x}{L}\right)\right\}\sin\left(\frac{n\pi y}{L}\right)$$

$$\because u(L,y)=T_2=\sum_{n=1}^{\infty}\left\{\underbrace{a_n\cosh(n\pi)+b_n\sinh(n\pi)}_{*}\right\}\sin\left(\frac{n\pi y}{L}\right)$$

$$\Rightarrow a_n\cosh(n\pi)+b_n\sinh(n\pi)=\frac{2}{L}\int T_2\sin\left(\frac{n\pi y}{L}\right)dy$$

$$\Rightarrow a_n \cosh(n\pi) + b_n \sinh(n\pi) = \frac{2T_2}{n\pi}(1-\cos n\pi)$$

$$\because u(0,y)=0 \Rightarrow 0 = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi y}{L}\right)$$

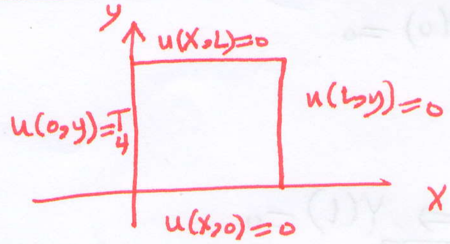
$$\Rightarrow \boxed{a_n = 0}$$

$$\Rightarrow 0 + b_n \sinh(n\pi) = \frac{2T_2}{n\pi}(1-\cos n\pi)$$

$$\Rightarrow \boxed{b_n = \frac{2T_2(1-\cos n\pi)}{n\pi \sinh(n\pi)}} = \frac{4T_2}{n\pi \sinh(n\pi)} \quad n: \text{odd}$$

$$\Rightarrow \boxed{u_1(x,y) = \sum_{n=1}^{\infty} \left\{\frac{2T_2}{n\pi \sinh n\pi}(1-\cos n\pi) \sinh\left(\frac{n\pi x}{L}\right)\right\} \sin\left(\frac{n\pi y}{L}\right)} \longrightarrow (1)$$

second



$$\frac{X''}{X} + \frac{Y''}{Y} = 0 \Rightarrow \frac{Y''}{Y} = -\frac{X''}{X} = -\lambda \;(\lambda > 0)$$

$$\Rightarrow \left.\begin{aligned} X &= X_1 \cosh\sqrt{\lambda}x + X_2 \sinh\sqrt{\lambda}x \\ Y &= Y_1 \cos\sqrt{\lambda}y + Y_2 \sin\sqrt{\lambda}y \end{aligned}\right\}$$

$u(x,0)=0 \Rightarrow X(x)Y(0)=0 \Rightarrow \boxed{Y_1 = 0}$

$u(x,L)=0 \Rightarrow Y(L)=0$

$$\Rightarrow \boxed{\lambda = \frac{n^2\pi^2}{L^2}}$$

$$\Rightarrow X_n = X_{1n}\cosh\left(\frac{n\pi x}{L}\right) + X_{2n}\sinh\left(\frac{n\pi x}{L}\right)$$

$$Y_n = Y_{2n}\sin\left(\frac{n\pi y}{L}\right)$$

$$\Rightarrow u_{2n} = \left(a_n \cosh\left(\frac{n\pi x}{L}\right) + b_n \sinh\left(\frac{n\pi x}{L}\right) \right\} \sin\left(\frac{n\pi y}{L}\right)$$

$$\Rightarrow u_2(x,y) = \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sinh\left(\frac{n\pi x}{L}\right) \right\} \sin\left(\frac{n\pi y}{L}\right)$$

$$\because u(0,y) = T_4 = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi y}{L}\right)$$

$$\Rightarrow \boxed{a_n = \frac{2T_4}{n\pi}(1-\cos n\pi)}$$

$$\because u(L,y) = 0 = \sum_{n=1}^{\infty} \left(a_n \cosh(n\pi) + b_n \sinh(n\pi) \right) \sin\left(\frac{n\pi y}{L}\right)$$

$$\Rightarrow a_n \cosh(n\pi) + b_n \sinh(n\pi) = 0$$

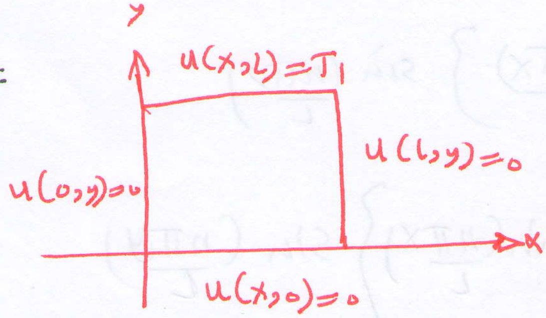
$$\Rightarrow b_n \sinh(n\pi) = - a_n \cosh n\pi$$

$$\Rightarrow \boxed{b_n = \frac{-2T_4}{n\pi \sinh n\pi}(1-\cos n\pi)\cosh(n\pi)}$$

$$\Rightarrow \boxed{u_2(x,y) = \sum_{n=1}^{\infty} \left\{ \frac{2T_4}{n\pi}(1-\cos n\pi)\cosh\left(\frac{n\pi x}{L}\right) - \frac{2T_4(1-\cos n\pi)}{n\pi \sinh(n\pi)} \cosh n\pi \sinh\left(\frac{n\pi x}{L}\right) \right\} \sin\left(\frac{n\pi y}{L}\right)}$$

2

3rd



$$\frac{X''}{X} + \frac{Y''}{Y} = 0 \quad \Rightarrow \quad \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda \; (\lambda > 0)$$

$$\Rightarrow \left.\begin{aligned} X &= X_1 \cos\sqrt{\lambda}x + X_2 \sin\sqrt{\lambda}x \\ Y &= Y_1 \cosh\sqrt{\lambda}y + Y_2 \sinh\sqrt{\lambda}y \end{aligned}\right\}$$

$$u(0,y) = 0 \Rightarrow X(0) = 0 \Rightarrow \boxed{X_1 = 0}$$

$$u(L,y) = 0 \Rightarrow X(L) = 0 \Rightarrow \boxed{\lambda = \frac{n^2\pi^2}{L^2}}, \quad n = 1, 2 \ldots$$

$$\Rightarrow X_n = X_{2n} \sin\left(\frac{n\pi x}{L}\right)$$

$$Y_n = Y_{1n} \cosh\left(\frac{n\pi y}{L}\right) + b_n \sinh\left(\frac{n\pi y}{L}\right)$$

$$u_{3n} = \left\{a_n \cosh\left(\frac{n\pi y}{L}\right) + b_n \sinh\left(\frac{n\pi y}{L}\right)\right\} \sin\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow u_3(x,y) = \sum_{n=1}^{\infty} \left\{a_n \cosh\left(\frac{n\pi y}{L}\right) + b_n \sinh\left(\frac{n\pi y}{L}\right)\right\} \sin\left(\frac{n\pi x}{L}\right)$$

$$\because u(x,L) = T_1 = \sum_{n=1}^{\infty} \left(a_n \cosh n\pi + b_n \sinh n\pi\right) \sin\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow a_n \cosh(n\pi) + b_n \sinh n\pi = \frac{2}{L}\int_0^L T_1 \sin\left(\frac{n\pi x}{L}\right) dx$$

$$\Rightarrow a_n \cosh n\pi + b_n \sinh n\pi = \frac{2T_1}{n\pi}(1-\cos n\pi)$$

$$\because u(x,0)=0 = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)$$

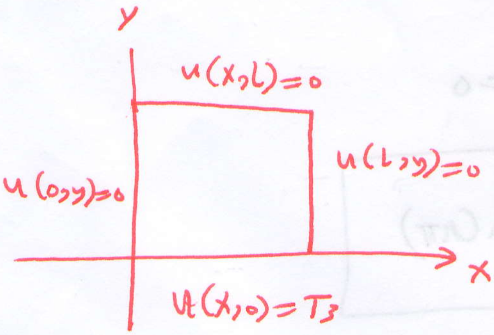
$$\Rightarrow \boxed{a_n = 0}$$

$$\Rightarrow b_n \sinh(n\pi) = \frac{2T_1}{n\pi}(1-\cos n\pi)$$

$$\Rightarrow \boxed{b_n = \frac{2T_1}{n\pi \sinh n\pi}(1-\cos n\pi)}$$

$$\Rightarrow \boxed{u_3(x,y) = \sum_{n=1}^{\infty}\left\{\frac{2T_1(1-\cos n\pi)}{n\pi \sinh n\pi}\sinh\left(\frac{n\pi y}{L}\right)\right\}\sin\left(\frac{n\pi x}{L}\right)} \rightarrow 3$$

4Th



$$\frac{X''}{X} + \frac{Y''}{Y} = 0 \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda \ (\lambda > 0)$$

$$\Rightarrow \left.\begin{aligned} X &= X_1 \cos\sqrt{\lambda}x + X_2 \sin\sqrt{\lambda}x \\ Y &= Y_1 \cosh\sqrt{\lambda}y + Y_2 \sinh\sqrt{\lambda}y \end{aligned}\right\}$$

$$\because u(0,y)=0 \Rightarrow X(0)=0 \Rightarrow \boxed{X_1=0}$$

$$u(L,y)=0 \Rightarrow X(L)=0 \Rightarrow \boxed{\lambda=\frac{n^2\pi^2}{L^2}}$$

$$\Rightarrow X_n = X_{2n}\sin\left(\frac{n\pi x}{L}\right)$$

$$Y_n = Y_{1n}\cosh\left(\frac{n\pi y}{L}\right)+b_n\sinh\left(\frac{n\pi y}{L}\right)$$

$$\Rightarrow u_4(x,y)=\sum_{n=1}^{\infty}\left\{\left(a_n\cosh\left(\frac{n\pi y}{L}\right)+b_n\sinh\left(\frac{n\pi y}{L}\right)\right\}\sin\left(\frac{n\pi x}{L}\right)\right.$$

$$\because u(x,0)=T_3=\sum_{n=1}^{\infty}a_n\sin\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow a_n=\frac{2}{L}\int_0^L T_3\sin\left(\frac{n\pi x}{L}\right)dx \Rightarrow \boxed{a_n=\frac{2T_3}{n\pi}(1-\cos n\pi)}$$

$$\because u(x,L)=0=\sum_{n=1}^{\infty}\left(a_n\cosh n\pi+b_n\sinh n\pi\right)\sin\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow a_n\cosh(n\pi)+b_n\sinh(n\pi)=0$$

$$\Rightarrow \boxed{b_n=\frac{-2T_3}{n\pi\sinh(n\pi)}(1-\cos n\pi)\cosh(n\pi)}$$

$$\Rightarrow \boxed{u_4(x,y)=\sum_{n=1}^{\infty}\left\{\frac{2T_3}{n\pi}(1-\cos n\pi)\cosh\left(\frac{n\pi y}{L}\right)-\frac{2T_3}{n\pi\sinh n\pi}(1-\cos n\pi)\cosh(n\pi)\sinh\left(\frac{n\pi y}{L}\right)\right\}\sin\left(\frac{n\pi x}{L}\right)}$$

4

Now The general solution is

$$u(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y) + u_4(x,y)$$

$$\Rightarrow \boxed{\begin{aligned} u(x,y) = &\sum_{n=1}^{\infty} \left\{ \frac{2}{n\pi}(1-\cos n\pi) \left[T_3 \cosh\left(\frac{n\pi y}{L}\right) + \frac{(T_1 - T_3 \cosh n\pi)}{\sinh n\pi} \sinh\left(\frac{n\pi y}{L}\right) \right] \right\} \sin\left(\frac{n\pi x}{L}\right) \\ &+ \sum_{n=1}^{\infty} \left\{ \frac{2}{n\pi}(1-\cos n\pi) \left[T_4 \cosh\left(\frac{n\pi x}{L}\right) + \frac{(T_2 - T_4 \cosh n\pi)}{\sinh n\pi} \sinh\left(\frac{n\pi x}{L}\right) \right] \right\} \sin\left(\frac{n\pi y}{L}\right) \end{aligned}}$$