Handout 1

(2-1)

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8. a) The function $Y_1^{-1}(\theta,\varphi)$ is given by $Y_1^{-1}(\theta,\varphi) = \sqrt{\frac{3}{8\pi}} \sin\theta\, e^{-i\phi}$

$$L^2 Y_{-1}^{-1} = -\hbar^2 \left[\frac{1}{\sin\vartheta}\frac{\partial}{\partial\theta}\left(\sin\vartheta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}\right] Y_1^{-1}$$

$$= -\hbar^2\sqrt{\frac{3}{8\pi}}\left[\frac{1}{\sin\vartheta}\frac{\partial}{\partial\theta}\left(\sin\vartheta\frac{\partial}{\partial\theta}\left(\sin\theta\, e^{-i\phi}\right)\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}\left(\sin\vartheta\, e^{-i\phi}\right)\right]$$

$$= -\hbar^2\sqrt{\frac{3}{8\pi}}\left[\frac{e^{-i\phi}}{\sin\vartheta}\frac{\partial}{\partial\theta}\left(\sin\vartheta\underbrace{\frac{\partial}{\partial\theta}(\sin\vartheta)}\right) + \frac{\cancel{\sin\vartheta}}{\sin^{\cancel{2}}\vartheta}\frac{\partial^2}{\partial\varphi^2}\left(e^{-i\phi}\right)\right]$$

$$-\hbar^2\sqrt{\frac{3}{8\pi}}\left[\frac{e^{-i\phi}}{\sin\vartheta}\frac{\partial}{\partial\theta}(\sin\vartheta\cos\vartheta) + \frac{(-i)^2}{\sin\vartheta}e^{-i\phi}\right]$$

$$-\hbar^2\sqrt{\frac{3}{8\pi}}\left[\frac{1}{2\sin\vartheta}\frac{\partial}{\partial\theta}(\sin 2\theta) - \frac{1}{\sin\vartheta}\right]e^{-i\phi}$$

$$-\hbar^2\sqrt{\frac{3}{8\pi}}\left[\frac{\cancel{2}\cos 2\theta}{\cancel{2}\sin\theta} - \frac{1}{\sin\vartheta}\right]e^{-i\phi}$$

$$-\hbar^2\sqrt{\frac{3}{\pi}}\left[\frac{(1-2\sin^2\theta)}{\sin\theta} - \frac{1}{\sin\theta}\right]e^{-i\phi}$$

$$= -\hbar^2\sqrt{\frac{3}{\pi}}\left[\cancel{\frac{1}{\sin\theta}} - \cancel{\frac{1}{\sin\vartheta}} - 2\sin\vartheta\right]e^{-i\phi}$$

$$= 2\hbar^2\underbrace{\sqrt{\frac{3}{\pi}}\sin\vartheta\, e^{-i\Phi}}_{Y_1^{-1}(\vartheta,\varphi)}$$

$$= 2\hbar^2 Y_1^{-1}$$

$$L^2 Y_1^{-1} = 2\hbar^2 Y_1^{-1}$$

eigenvalue

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8b $L_z Y_1^{-1}(\theta,\varphi) = -i\hbar \frac{\partial}{\partial\phi}\left(\sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\phi}\right)$

$\Rightarrow L_z Y_1^{-1}(\theta,\varphi) = -i\hbar\sqrt{\frac{3}{8\pi}}\sin\theta \frac{\partial}{\partial\varphi}(e^{-i\varphi})$

$\Rightarrow L_z Y_1^{-1}(\theta,\varphi) = -i\hbar(-i)\sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\varphi}$

$\Rightarrow L_z Y_1^{-1}(\theta,\varphi) = -\hbar \underbrace{\sqrt{\frac{3}{8\pi}}\sin\theta e^{-i\varphi}}_{Y_1^{-1}}$

$\Rightarrow$ So $L_z Y_1^{-1} = -\hbar Y_1^{-1}$

9. $|\psi\rangle = A|\psi_{100}\rangle + 2A|\psi_{211}\rangle + A|\psi_{3,2,-1}\rangle$

a) Normalization requires

$\langle\psi|\psi\rangle = 1 \Rightarrow$

$\left(A^*\langle\psi_{100}| + 2A^*\langle\psi_{211}| + A^*\langle\psi_{3,2,-1}|\right)\cdot$
$\left(A|\psi_{100}\rangle + 2A|\psi_{211}\rangle + A|\psi_{3,2,-1}\rangle\right) = 1$

$\Rightarrow |A|^2\langle\psi_{100}|\psi_{100}\rangle + 2|A|^2\langle\psi_{100}|\psi_{211}\rangle + |A|^2\langle\psi_{100}|\psi_{3,2,-1}\rangle$
$+ 2|A|^2\langle\psi_{211}|\psi_{100}\rangle + 4|A|^2\langle\psi_{211}|\psi_{211}\rangle + 2|A|^2\langle\psi_{211}|\psi_{3,2,-1}\rangle$
$+ |A|^2\langle\psi_{3,2,-1}|\psi_{100}\rangle + 2|A|^2\langle\psi_{3,2,-1}|\psi_{211}\rangle + |A|^2\langle\psi_{3,2,-1}|\psi_{3,2,-1}\rangle$
$= 1$

$\Rightarrow |A|^2 + 4|A|^2 + |A|^2 = 1 \Rightarrow 6|A|^2 = 1$

$\Rightarrow |A|^2 = 1/6 \Rightarrow \boxed{|A| = 1/\sqrt{6}}$

$|\psi\rangle = \frac{1}{\sqrt{6}}|\psi_{1,0,0}\rangle + \frac{2}{\sqrt{6}}|\psi_{2,1,1}\rangle + \frac{1}{\sqrt{6}}|\psi_{3,2,-1}\rangle$

The probabilities to find each state in a measurement are:

$P_{100} = \left(\frac{1}{\sqrt{6}}\right)^2 = \frac{1}{6}$

$P_{211} = \left(\frac{2}{\sqrt{6}}\right)^2 = \frac{4}{6}$

$P_{3,2,-1} = \left(\frac{1}{\sqrt{6}}\right) = \frac{1}{6}$

In each state the possible values of ℓ^2, ℓ_z and E are:

	ℓ^2	ℓ_z	E
$\lvert\psi_{100}\rangle$	0	0	E_1
$\lvert\psi_{211}\rangle$	$2\hbar^2$	$\hbar$	E_2
$\lvert\psi_{3,2-1}\rangle$	$6\hbar^2$	$-\hbar$	E_3

$\ell^2 =$

b) $\langle \ell^2 \rangle = 0 \cdot P_{100} + (2\hbar^2) \cdot P_{211} + 6\hbar^2 P_{3,2,-1}$

$\Rightarrow \langle \ell^2 \rangle = 2\hbar^2 \frac{4}{6} + 6\hbar^2 \cdot \frac{1}{6} = \frac{14}{6}\hbar^2 = \frac{7}{3}\hbar^2$

$\langle \ell_z \rangle = 0 \cdot P_{100} + \hbar P_{211} - \hbar P_{3,2,-1}$

$\Rightarrow \langle \ell_z \rangle = \hbar \frac{4}{6} - \hbar \frac{1}{6} = \frac{4\hbar - \hbar}{6} = \frac{3\hbar}{6} = \frac{\hbar}{2}$

$\langle E \rangle = E_1 P_{100} + E_2 P_{211} + E_3 P_{3,2-1}$

$= \frac{E_1}{6} + \frac{4E_2}{6} + \frac{E_3}{6} =$

$= \frac{E_1}{6} + \frac{\cancel{4}}{6} \frac{E_1}{\cancel{4}} + \frac{1}{6} \frac{E_1}{9} =$

$= \frac{E_1}{6} \left(1 + 1 + \frac{1}{9} \right) = \frac{E_1}{6} \frac{19}{9}$

$= \frac{19}{54} E_1 = (-13,6) \cdot \frac{19}{54}\ \mathrm{eV}$

$$\Delta l_z = \sqrt{\langle l_z^2 \rangle - \langle l_z \rangle^2}$$

but $\langle l_z^2 \rangle = 0 \cdot P_{100} + \hbar^2 P_{211} + (-\hbar)^2 P_{3,2,-1}$

$$\Rightarrow \langle l_z^2 \rangle = \frac{4\hbar^2}{6} + \frac{\hbar^2}{6} = \frac{5\hbar^2}{6}.$$

$$\Delta l_z = \sqrt{\frac{5\hbar^2}{6} - \left(\frac{\hbar}{2}\right)^2} = \sqrt{\frac{20\hbar^2 - 6\hbar^2}{24}} = \hbar\sqrt{\frac{14}{24}}$$

$$\Rightarrow \Delta l_z = \hbar\sqrt{\frac{7}{12}}.$$

For the time development of the state we have

$$|\psi(r,t)\rangle = \frac{1}{\sqrt{6}}|\psi_{100}\rangle e^{-iE_1 t/\hbar} + \frac{2}{\sqrt{6}}|\psi_{211}\rangle e^{-iE_2 t/\hbar} + \frac{1}{\sqrt{6}}|\psi_{3,2,-1}\rangle e^{-iE_3 t/\hbar}$$

$$= \frac{1}{\sqrt{6}}|\psi_{100}\rangle e^{-iE_1 t/\hbar} + \frac{2}{\sqrt{6}}|\psi_{211}\rangle e^{-iE_1 t/4\hbar} + \frac{1}{\sqrt{6}}|\psi_{3,2-1}\rangle e^{-iE_1 t/9\hbar}$$

Expres $E_1 = -13{,}6\,eV$ in Joules

$$E_1 = -13{,}6\times(1.6\times10^{-19})\,J = 21.76\times10^{-19}\,J$$

$$E_1/\hbar = \frac{21.76\times10^{-19}\cdot 2\pi}{6.63\times10^{-34}} = 20.6\times10^{15}\ r/s.$$

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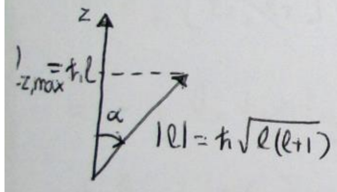
10) Given that

$|\underset{\sim}{l}| = \sqrt{l(l+1)}\,\hbar$ and $l_z = \hbar m$

$m = -l, \dots 0, \dots, +l$

It is obvious that the maximum value of the projection l_z is given by: $l_{z,max} = \hbar l$



This is always smaller than $|l|$

Now for the angle α we have

$$\cos\alpha = \frac{l_x}{|\underset{\sim}{l}|} = \frac{l}{\sqrt{l(l+1)}}$$

At the classical limit $(l \to \infty)$

$\cos\alpha = 1 \Rightarrow \alpha = 0$ (as expected)

12) a) Correct (as in classical mechanics)

b) Wrong (the central potential is responsible only for the conservation of angular momentum and thus we could use it for the description of the energy states of the particle)
Angular momentum in QM is always quantized.

c) Correct (operators p_x, p_y, p_z commute)

d) Wrong (operators l_x, l_y, l_z do not commute)

e) Correct. It is always $[l^2, l_i] = 0$ for any $i = 1, 2, 3 \equiv x, y, z$.

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13 a) $[\ell^2, \ell_z] = [\ell_x^2 + \ell_y^2 + \ell_z^2, \ell_z] =$

$= [\ell_x^2, \ell_z] + [\ell_y^2, \ell_z] + [\ell_z^2, \ell_z] =$

$= -[\ell_z, \ell_x^2] - [\ell_z, \ell_y^2] - [\ell_z, \ell_z^2] =$

$= -\{ \ell_x [\ell_z, \ell_x] + [\ell_z, \ell_x] \ell_x$

$+ \ell_y [\ell_z, \ell_y] + [\ell_z, \ell_y] \ell_y$

$+ \ell_z \overset{0}{\cancel{[\ell_z, \ell_z]}} + \overset{0}{\cancel{[\ell_z, \ell_z]}} \ell_z \}$

$= -\{ \ell_x (i\hbar \ell_y) + (i\hbar \ell_y) \ell_x$

$+ \ell_y (-i\hbar \ell_x) + (-i\hbar \ell_x) \ell_y \}$

$= -\{ i\hbar \cancel{\ell_x \ell_y} + i\hbar \cancel{\ell_y \ell_x} - i\hbar \cancel{\ell_y \ell_x} - i\hbar \cancel{\ell_x \ell_y} \}$

$= 0$

(b) $\underset{\sim}{\ell} \times \underset{\sim}{\ell} = \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ \ell_x & \ell_y & \ell_z \\ \ell_x & \ell_y & \ell_z \end{vmatrix} \Rightarrow$

$\Rightarrow (\underset{\sim}{\ell} \times \underset{\sim}{\ell}) = \underset{\sim}{i} \begin{vmatrix} \ell_y & \ell_z \\ \ell_y & \ell_z \end{vmatrix} - \underset{\sim}{j} \begin{vmatrix} \ell_x & \ell_z \\ \ell_x & \ell_z \end{vmatrix} + \underset{\sim}{k} \begin{vmatrix} \ell_x & \ell_y \\ \ell_x & \ell_y \end{vmatrix}$

$\Rightarrow (\underset{\sim}{\ell} \times \underset{\sim}{\ell}) = \underset{\sim}{i} (\ell_y \ell_z - \ell_z \ell_y) - \underset{\sim}{j} (\ell_x \ell_z - \ell_z \ell_x) + \underset{\sim}{k} (\ell_x \ell_y - \ell_y \ell_x)$

Thus

$(\underset{\sim}{\ell} \times \underset{\sim}{\ell})_x = \ell_y \ell_z - \ell_z \ell_y = [\ell_y, \ell_z] \overset{(2.4)}{=} i\hbar \ell_x$

$(\underset{\sim}{\ell} \times \underset{\sim}{\ell})_y = \ell_z \ell_x - \ell_x \ell_z = [\ell_z, \ell_x] = i\hbar \ell_y$

$(\underset{\sim}{\ell} \times \underset{\sim}{\ell})_z = \ell_x \ell_y - \ell_y \ell_x = [\ell_x, \ell_y] = i\hbar \ell_z$