PHYS 502

Solutions of Final Exam 21st May 2013

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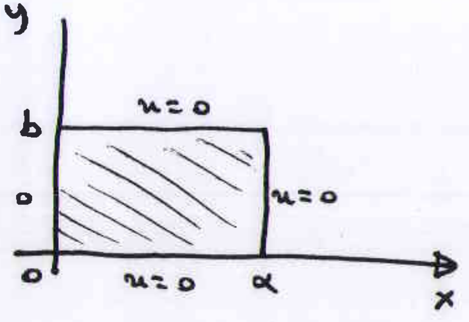


The wave eq. is given by $\bar{u}_{xx} + \bar{u}_{yy} - \frac{1}{c^2}\bar{u}_{tt} = 0$

Let $\bar{u} = u(x,y) \cdot T(t)$ (2). Then by inserting (2) → (1)

we get $T u_{xx} + T u_{yy} - \frac{1}{c^2} u \cdot \ddot{T} = 0$ (3)

Dividing (3) by uT we get: $\frac{u_{xx}}{u} + \frac{u_{yy}}{u} - \frac{1}{c^2}\frac{\ddot{T}}{T} = 0 \Rightarrow \frac{u_{xx}}{u} + \frac{u_{yy}}{u} = \frac{1}{c^2}\frac{\ddot{T}}{T} = -k^2$

$$\Rightarrow \begin{cases} \frac{u_{xx}}{u} + \frac{u_{yy}}{u} = -k^2 & (4) \\ \ddot{T} = -k^2 c^2 T & (5) \end{cases}$$

Solution of (5)

$$\ddot{T} = -k^2 c^2 T \underset{\omega = kc}{\Longrightarrow} T(t) = T_1 \cos\omega t + T_2 \sin\omega t \quad (6)$$

Solution of (4)

Let $u(x,y) = X(x)Y(y)$ (7). Insert (7) in (4) and get:

$$\frac{YX''}{YX} + \frac{XY''}{YX} = -k^2 \Rightarrow \frac{X''}{X} + \frac{Y''}{Y} = -k^2 \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} - k^2 = -k_x^2 \quad (8)$$

From 8 we get
$$\begin{cases} X'' + k_x^2 X = 0 & \text{(9a)} \\ \dfrac{Y''}{Y} + k^2 = k_x^2 & \text{(9b)} \end{cases}$$

Solution of 9a: $X(x) = X_1 \cos k_x x + X_2 \sin k_x x$ (10)

Solution of 9b: $\dfrac{Y''}{Y} + k^2 = k_x^2 \Rightarrow \dfrac{Y''}{Y} = k_x^2 - k^2 \Rightarrow \dfrac{Y''}{Y} = -k_y^2 \Rightarrow$

$\Rightarrow Y(y) = Y_1 \cos k_y y + Y_2 \sin k_y y$ (11)

Where $k_x^2 + k_y^2 = k^2$ (12)

Application of boundary conditions:

- $u(0,y) = 0 \Rightarrow X(0) = 0 \Rightarrow \boxed{X_1 = 0}$ (13a)
- $u(\alpha,y) = 0 \Rightarrow X(\alpha) = 0 \Rightarrow \sin k_x \alpha = 0$

$\Rightarrow k_x \alpha = n\pi \Rightarrow \boxed{k_x = \frac{n\pi}{\alpha}}$ (13b)

$n = 1, 2, \ldots$

- $u(x,0) = 0 \Rightarrow Y(0) = 0 \Rightarrow \boxed{Y_1 = 0}$ (13c)
- $u(x,b) = 0 \Rightarrow Y(b) = 0 \Rightarrow \sin k_y b = 0 \Rightarrow \boxed{k_y = \frac{m\pi}{b}}$ (13d)

Thus the partial solution is given by:

$$u_{n,m}(x,y) = \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right) \quad (14)$$

with $k_x^2 + k_y^2 = k^2 \Rightarrow \frac{\omega^2}{c^2} = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2 \Rightarrow \boxed{\omega_{m,n} = c\pi\sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2}}$ (15)

These are the normal mode frequencies of the system

- The initial speed is zero. This means $\left.\frac{\partial \tilde{u}}{\partial t}\right|_{t=0} = 0$. From eq. (6) this leads us to consider $\boxed{T_2 = 0}$ (16)

Thus the general solution of the drum motion is given by

$$\boxed{\tilde{u}(x,y,t) = \sum_{n,m} B_{nm} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\cos(\omega_{n,m} t)} \quad (17)$$

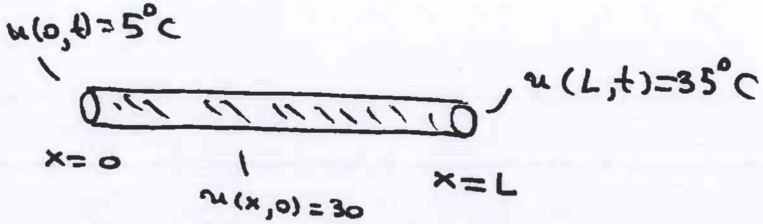
Since at $t=0$ $\tilde{u}(x,y) = f(x,y)$ we get :

$$f(x,y) = \sum_{n,m} B_{mn} \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right)$$

$$B_{nm} = \frac{\int_{x=0}^{\alpha} \int_{y=0}^{b} f(x,y) \sin\left(\frac{n\pi x}{\alpha}\right) \sin\left(\frac{m\pi y}{b}\right) dx\, dy}{\int_{x=0}^{\alpha} \sin^2\left(\frac{n\pi x}{\alpha}\right) dx \int_{y=0}^{b} \sin\left(\frac{m\pi y}{b}\right) dy} \qquad (18)$$

where $\int_{x=0}^{\alpha} \sin^2\left(\frac{n\pi x}{\alpha}\right) dx = \frac{\alpha}{2}$, $\int_{y=0}^{b} \sin^2\left(\frac{m\pi y}{b}\right) dy = \frac{b}{2}$

(2)



The solution comprises of two parts : one which corresponds to the homogeneous b.c and one to the non-homogeneous b.c

Thus $u(x,t) = v(x,t) + w(x,t)$ (1) with $v(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2 t}{L^2}}$ (2)

and $w(x) = A + \frac{(B - A)}{L} x$ (3)

By inserting the values of the relevant parametes we get .

$v(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{2}\right) e^{-\frac{n^2\pi^2 t}{4}}$ and $w(x) = 5 + \frac{(35-5)}{2} x \Rightarrow w(x) = 5 + 15x$

Thus the general solution is given by:

$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{2}\right) e^{-\frac{n^2\pi^2 t}{\bullet}} + 5 + 15x$ (4)

Applying the initial condition we have

$u(x,0) = 30 = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{2}\right) + 5 + 15x \Rightarrow$

$\Rightarrow 25 - 15x = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{2}\right)$

To find the coefficients C_n we apply the Fourier series formalism

$$C_n = \frac{\int_0^2 (25-15x)\sin\left(\frac{n\pi x}{2}\right) dx}{\int_0^2 \sin^2\left(\frac{n\pi x}{2}\right) dx} = \frac{10\left(\frac{5+\cos(n\pi)}{n\pi}\right)}{1}$$

$$\Rightarrow \boxed{C_n = \frac{10\left(5+\cos(n\pi)\right)}{n\pi}}$$

(3) $\nabla^2 u = 0 \Rightarrow \dfrac{\partial^2 u}{\partial \rho^2} + \dfrac{1}{\rho}\dfrac{\partial u}{\partial \rho} + \dfrac{1}{\rho^2}\dfrac{\partial^2 u}{\partial \theta^2} = 0 \Rightarrow \rho^2 \dfrac{\partial^2 u}{\partial \rho^2} + \rho \dfrac{\partial u}{\partial \rho} + \dfrac{\partial^2 u}{\partial \theta^2} = 0$

Let $u(\rho,\theta) = R(\rho)\,\Theta(\theta)$ then we get:

$$\rho^2 \Theta R'' + \rho \Theta R' + R\Theta'' = 0 \Rightarrow \rho^2 \frac{R''}{R} + \rho\frac{R'}{R} + \frac{\Theta''}{\Theta} = 0 \Rightarrow \rho^2\frac{R''}{R} + \rho\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda^2$$

$\Rightarrow \Theta(\theta) = \Theta_1 \cos\lambda\theta + \Theta_2 \sin\lambda\theta$ (1)

$R(\rho) = R_1 \rho^{\lambda} + R_2 \rho^{-\lambda}$ (2)

Since we need $\Theta(0) = \Theta(2\pi)$ $\Theta'(0) = \Theta'(2\pi)$ we get $\lambda = n = 0, 1, 2, 3, \dots,$

Also we need at $\rho \to 0$ our solution to remain finite, thus $\boxed{R_2 = 0}$

So the partial solution is: $u_n = \rho^n \left(A_n \cos(n\theta) + B_n \sin(n\theta)\right)$

and the general solution $u(\rho,\theta) = \sum_{n=0}^{\infty} \rho^n \left(A_n \cos(n\theta) + B_n \sin(n\theta)\right)$ (3)

Now we apply the boundary conditions $u(\alpha,\theta) = \begin{cases} T_1 & 0 < \theta < \pi \\ T_2 & \pi < \theta < 2\pi \end{cases}$ (4)

$$\alpha^n A_n = \frac{1}{\pi}\left\{\int_0^{\pi} T_1 \cos(n\theta)\,d\theta + \int_{\pi}^{2\pi} T_2 \cos(n\theta)\,d\theta\right\} = \dots = 0$$

$$\alpha^0 A_0 = \frac{1}{2\pi}\left\{\int_0^{\pi} T_1\,d\theta + \int_{\pi}^{2\pi} T_2\,d\theta\right\} \Rightarrow \boxed{A_0 = \frac{T_1 + T_2}{2}}$$ (5)

Also $B_n \alpha^n = \frac{1}{\pi}\left\{ \int_0^{\pi} T_1 \sin n\theta \, d\theta + \int_{\pi}^{2\pi} T_2 \sin n\theta \, d\theta \right\} \Rightarrow$

$$\Rightarrow B_n \alpha^n = \frac{(T_1 - T_2)}{n\pi}\left(1 - \cos(n\pi)\right) \Rightarrow B_n \alpha^n = \begin{cases} 0 & n = \text{even} \\ \frac{2(T_1 - T_2)}{n\pi} & \end{cases}$$

$$\Rightarrow \boxed{B_n = \begin{cases} 0 & n = \text{even} \\ \frac{2(T_1 - T_2)}{n\pi \alpha^n} & \end{cases}} \quad (6)$$

Now by substituting $\alpha = 1/2 \quad T_1 = 5, \; T_2 = 25$ we get

$$A_0 = 15 \qquad B_n = \begin{cases} 0 & n = \text{even} \\ -\frac{40}{n\pi} 2^n & \end{cases}$$

Thus
$$\boxed{u(\rho, \theta) = \sum_{n=\text{odd}}^{\infty} \frac{(-40)}{n\pi} (2\rho)^n \sin(n\theta) + 15} \quad (7)$$

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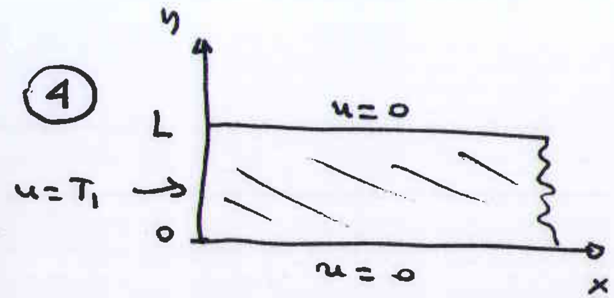

$\nabla^2 u(x,y)=0 \Rightarrow \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ with $u(0,y)=T_1$

$u(x,0)=0=u(x,L)$

Let $u(x,y)=X(x)Y(y)$ then we get

$YX''+XY''=0 \Rightarrow \frac{X''}{X}+\frac{Y''}{Y}=0 \Rightarrow \frac{Y''}{Y}=-\frac{X''}{X}=-\lambda \Rightarrow$

$\Rightarrow \begin{cases} Y(y)=Y_1\cos\sqrt{\lambda}y+Y_2\sin(\sqrt{\lambda}y) & (1) \\ X(x)=X_1 e^{\sqrt{\lambda}x}+X_2 e^{-\sqrt{\lambda}x} & (2) \end{cases}$

$u(x,0)=0 \Rightarrow Y(0)=0 \Rightarrow \boxed{Y_1=0}$ (3a)

$u(x,L)=0 \Rightarrow Y(L)=0 \Rightarrow \sin(\sqrt{\lambda}L)=0 \Rightarrow \sqrt{\lambda}L=n\pi \Rightarrow \boxed{\lambda=n^2\pi^2/L^2}$ (3b)

Also as $x\to\infty$ we need $u(x,y)<\infty$ thus $\boxed{X_1=0}$ (3c)

Thus the partial solution will be: $u_n(x,y)=B_n e^{-\frac{n^2\pi^2}{L^2}x}\sin\left(\frac{n\pi y}{L}\right)$

The general solution:

$$u(x,y)=\sum_{n=1}^{\infty} B_n e^{-\frac{n^2\pi^2}{L^2}x}\sin\left(\frac{n\pi y}{L}\right) \quad (4)$$

We apply the condition $u(0,y) = T_0$

$$T_0 = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi y}{L}\right) \qquad B_n = \frac{T_0 \int_0^L \sin\left(\frac{n\pi y}{L}\right) dy}{\int_0^L \sin^2\left(\frac{n\pi y}{L}\right) dy} = \frac{T_0 \int_0^L \sin\left(\frac{n\pi y}{L}\right) dy}{L/2}$$

$$\Rightarrow B_n = -\frac{T_0}{(L/2)}\left(\frac{L}{n\pi}\right)\left\{\cos(n\pi) - 1\right\} \Rightarrow B_n = \begin{cases} 0 & n = \text{even} \\ T_0/n\pi & n = \text{odd.} \end{cases}$$

$$u(x,y) = \frac{4T_0}{\pi} \sum_{n:\text{odd}}^{\infty} \frac{1}{n} e^{-\frac{n^2\pi^2}{L^2}\cancel{x}} \sin\left(\frac{n\pi y}{L}\right)$$